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A New Kind of Boundary Layer over a Convex Solid Boundary in Rarefied Gas

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Abstract In a flow of a slightly rarefied gas over a solid boundary, a new kind of boundary layer the thickness of which is of order of *the mean free path squared divided by the radius of the curvature of the boundary* is proposed to exist at the bottom of the kinetic boundary layer (or Knudsen layer) when the boundary is *convex* in shape.

The new kind of boundary layer is demonstrated by investigating the asymptotic behavior for small mean-free-path limit of thermal creep flow around a circular cylinder which has a constant temperature gradient along its axis.

As is well known, in the flow of slightly rarefied gas over a solid boundary, a thin layer with thickness of order of the mean free path appears adjacent to the boundary. It is called kinetic boundary layer or Knudsen layer. In the present paper we propose that a new kind of boundary layer the thickness of which is of order of *the mean free path squared divided by the radius of the curvature of the boundary* exists at the bottom of the kinetic boundary layer when the boundary is *convex* in shape.

We demonstrate the new kind of boundary layer by investigating the thermal creep flow of a slightly rarefied gas around a circular cylinder which has a constant temperature gradient along its axis. We take the Cartesian coordinate $X=(x_1,x_2,x_3)$ with x_3 as the axis of symmetry of the circular cylinder (Fig.1). The temperature T_w of the cylinder is assumed to be $T_w= T_0(1+k(x_3/a))$ where a is the radius of the cylinder and k is a constant $(aT_0^{-1}dT_w/dx_3)$. We investigate the asymptotic behavior for small Knudsen number of the flow induced around the cylinder by the temperature gradient in the absence of the pressure gradient and derive the explicit expression for the new kind of boundary layer. We use the Boltzmann-Krook-Welander equation. Diffuse reflection is assumed as the boundary condition on the cylinder. Further we assume that the temperature gradient is so small that the fundamental equation and boundary condition may be linearized.

Proceeding in the same way as in the analysis of the thermal creep flow *through* a circular pipe[1] (referred to as Case I), we can show :

1) The temperature and density of the gas are functions of x_3 only and the pressure is uniform. Namely, the temperature T is given by

$$T = T_0(1+k(x_3/a)). \qquad (1)$$

2) The velocity has only the x_3-component u_3 and it is independent of x_3. The velocity (in its non-dimensional form ψ) is determined by the following integral equation.

$$\psi = \frac{A}{\pi}\iint_D \psi \frac{J_0(A|R-R_0|)}{|R-R_0|} dR_0 - \frac{1}{\pi}\iint_D \frac{J_2(A|R-R_0|)-J_0(A|R-R_0|)}{|R-R_0|} dR_0, \qquad (2)$$

where

$$J_n(t) = \int_0^\infty \zeta^n \exp(-\zeta^2 - \frac{t}{\zeta})d\zeta \ , \quad \psi = 2k^{-1}(2R_g T_0)^{-1/2}u_3 \ ,$$

$$R = (\frac{x_1}{a}, \frac{x_2}{a}) \ , \qquad A = \lambda a (2R_g T_0)^{-1/2} \ .$$

The R_g is the gas constant and λ is a constant (collision frequency) related to the mean free path l as $l = \pi^{-1/2}\lambda^{-1}(8R_g T_0)^{1/2}$. Thus, the A is a quantity of order of the inverse Knudsen number (a/l). The domain of integration D is the shaded region extending to infinity outside the circle R = 1 where R = $|R|$ in Fig.2.

We investigate the asymptotic behavior of the solution ψ of the integral equation for large value of A. Because the function $J_n(t)$ vanishes faster than any inverse power of t as $t \to \infty$, the integration over the domain outside the circle with its center at R and its radius of order of A^{-1} contributes little to the total integral over D and can be neglected when we evaluate the asymptotic behavior of the integrals in Eq.(2) for large A. The substantial domain of integration is classified into the three cases as shown by shaded domain in Fig.3a,b,c depending on the distance between R and the boundary (i.e. $R - 1 > O(A^{-1})$, $O(A^{-1}) \geq R - 1 > O(A^{-2})$, $O(A^{-2}) \geq R - 1 \geq 0$)*. As will be discussed briefly below, the solution behaves differently in these three ranges of R corresponding to this difference of the domain of integration. Namely,

a) In the region $[(R - 1) > O(A^{-1})]$, the state of the gas does not change appreciably over the mean free path and admits fluid-dynamic-like description. Thus, the region is called fluid dynamic region.

b) In the layer $[O(A^{-2}) < R - 1 \leq O(A^{-1})]$, the length scale for variation of the physical variable normal to the boundary is of order of the mean free path. The layer is called kinetic boundary layer or Knudsen layer.

c) In the much thinner layer at the bottom of the kinetic boundary layer $[(R - 1) \leq O(A^{-2})]$, the length scale is of order of the mean free path squared divided by the radius of the curvature. This new type of boundary layer is characteristic of the behavior of slightly rarefied gas over a *convex* boundary and is hereafter referred to S-layer for convenience.

Much has been done on the fluid dynamic region and the kinetic boundary layer in general (e.g.,Refs.2,3,4). It is the purpose of the present work to demonstrate the new boundary layer (S-layer).

Here we do not have enough space to give the complete description of the analysis of the problem. We briefly describe how the S layer makes its appearance in the solution of Eq.(2) for $A \to \infty$. Let us observe the inhomogeneous term in Eq.(2). The integral can be divided in the following two terms,i.e.

$$\iint_D = \iint_{D_1} - \iint_{D_2} ,$$

* Over the concave boundary the situation of Fig.3c never occurs and that of Fig.3b holds for the whole range of $0 \leq R - 1 \leq O(A^{-1})$.

where D_1 is the region given by $R \geq 1$ and $D_2 = D_1 - D$ in Fig.2. The asymptotic behavior (for $A \to \infty$) of the integral over D can be analyzed in the same way as in the Appendix A of Ref.1. It is sufficiently small (smaller than A^{-n} for any n) in the region $R - 1 > O(A^{-1})$ and appreciable only in the layer, $R - 1 \leq O(A^{-1})$, and shows the same behavior as kinetic boundary layer. Next the integral over D_2 is considered. When $R - 1 > O(A^{-2})$(Fig.3(a),(b)), the domain of integration, D_2, lies outside the circle of radius of order of A^{-1} with center at R so that the integrand is sufficiently small (smaller than A^{-n} for any n). So does the integral over D_2 there. When $R - 1 \leq O(A^{-2})$(Fig.3(c)), a part of the domain D_2 overlaps the circle of radius of order A^{-1} with center at R where the integrand is no longer small and of order of $|R_0 - R|^{-1}$ since the argument of the functions J_n is of order of unity. Thus, the integral over D_2 can easily be seen to be of order of A^{-2} in the thin layer, $R - 1 \leq A^{-2}$. The behavior of the inhomogeneous term suggests the behavior of solution of Eq.(2), especially the existence of the S layer and that it is of order of A^{-2}.

Here we give only the result of analysis. The solution is split into three parts i.e.

$$\psi = \psi_H(R) + \psi_K(\eta) + \psi_S(y) \ , \qquad (3)$$

where

$$\eta = A(R - 1) \ , \quad y = A\eta = A^2(R - 1) \ .$$

The ψ_K and ψ_S vanish as η and $y \to \infty$ respectively. The ψ_H is called the Hilbert part of the solution or the generalized slip flow and describes the behavior of the gas in the fluid dynamic region. The ψ_K is the correction to ψ_H in the kinetic boundary layer. The ψ_S which is the distinctive feature of the present study gives the correction to ψ_H and ψ_K in the S-layer at the bottom of the kinetic boundary layer.

Each part is expanded in power series of A^{-1} i.e.,

$$\begin{aligned}
\psi_H &= \psi_{H0} + \frac{\psi_{H1}}{A} + \frac{\psi_{H2}}{A^2} + \cdots , \\
\psi_K &= \qquad\;\; \frac{\psi_{K1}}{A} + \frac{\psi_{K2}}{A^2} + \cdots , \qquad (4) \\
\psi_S &= \qquad\qquad\qquad \frac{\psi_{S2}}{A^2} + \cdots .
\end{aligned}$$

ψ_{H0} is zero and ψ_{Hi} is some constant (say d_i) in the present problem. The numerical value of d_i is uniquely determined simultaneously with ψ_{Ki} by solving the following integral equations under the condition that $\psi_{Ki} \to 0$ as $\eta \to \infty$.

$$\sqrt{\pi}\psi_{K1} = J_2(\eta) - (d_1 + \tfrac{1}{2})J_0(\eta) + \int_0^\infty \psi_{K1} J_{-1}(|\eta - \eta_0|)d\eta_0 \quad ,(5)$$

$$\sqrt{\pi}\psi_{K2} = - \frac{\sqrt{\pi}}{2}\int_\infty^\eta \psi_{K1}(\eta_0)d\eta_0 + J_3(\eta) - d_2 J_0(\eta) + \int_0^\infty \psi_{K2} J_{-1}(|\eta - \eta_0|)d\eta_0 \quad . \qquad (6)$$

In comparison with Eqs.(15b),(16b) in Ref.1, we have

$$\begin{aligned}
d_1 &= d_{T1} \text{ (in Ref.1)} = 0.766 , \\
\psi_{K1}(\eta) &= Q_{T1}(\eta) \text{ (in Ref.1)} , \\
d_2 &= - d_{T2} \text{(in Ref.1)} = - 0.267 , \\
\psi_{K2} &= - Q_{T2}(\eta) \text{ (in Ref.1)} .
\end{aligned}$$

The Q_{T1} and Q_{T2} are plotted versus η in Ref.1. The first order slip flow d_1 and its kinetic boundary layer correction ψ_{K1} are the same as those in Case I. The second order slip flow d_2 and its boundary layer correction ψ_{K2} are opposite in sign to those in Case I.

The ψ_{S2} is given in the following form :

$$\psi_{S2} = -\frac{1}{\pi}\{(d_1 + \psi_{K1}(0) + 1)\int_{\sqrt{2y}}^{\infty}\frac{(t-\sqrt{2y})^2}{t}J_0(t)dt - \int_{\sqrt{2y}}^{\infty}\frac{(t-\sqrt{2y})^2}{t}J_2(t)dt\}$$

$$= \frac{1}{\pi}[(d_1 + \psi_{K1}(0) + 1)\{2J_4(\sqrt{2y})-4J_2(\sqrt{2y})-2y\int_{\sqrt{2y}}^{\infty}\frac{J_0(t)}{t}dt\} - 2J_6(\sqrt{2y})+6J_4(\sqrt{2y})+2y\int_{\sqrt{2y}}^{\infty}\frac{J_2(t)}{t}dt], \qquad (7)$$

$$d_1 + \psi_{K1}(0) + 1 = 1.220 .$$

For large values of y,

$$\psi_{S2} = \frac{1}{\pi}\{J_2(\sqrt{2y}) - (\frac{1}{2} + d_1 + \psi_{K1}(0))J_0(\sqrt{2y}) + \cdots\}. \qquad (8)$$

Since $J_n(t) \stackrel{\sim}{=} (\pi/3)^{1/2}(t/2)^{n/3}\exp[-3(t/2)^{2/3}]$ for $t>>1$, ψ tends to zero faster than any inverse power of y as $y \to \infty$. For small values of y

$$\psi_{S2} = \frac{1}{\pi}\{\frac{\sqrt{\pi}}{4}(\frac{1}{2} - d_1 - \psi_{K1}(0))+(d_1 + \psi_{K1}(0))\sqrt{2y} + \cdots\}. \qquad (9)$$

It can be shown that ψ_{S2} first increases monotonically to its maximum value and then decreases monotonically to zero as y goes from zero to infinity (thus, ψ_{S2} is always positive). The ψ_{S2} is plotted in Fig.4. The non-monotonicity of ψ_{S2} does not mean that the velocity ψ is not monotonic in the region $y \leq O(1)$. Due to the ψ_{K1} in (3), ψ is shown to be monotonic increasing in $y \leq O(1)$. Since $y = A^2(R - 1)$, the thickness of the layer is of order of $aA^{-2}(\sim l^2 a^{-1}$: the mean free path squared divided by the radius of the curvature). It is noted that the ψ_S(the S-layer) is not present in Case I where the boundary is *concave* in shape. It may be added that the S-layer is not a special character of the present example but a general feature of the behavior of s slightly rarefied gas over a convex boundary although the analysis for the general case is postponed.

Finally we consider the difference of the behavior of a slightly rarefied gas over a convex boundary from that over a plane or concave boundary physically (Fig.5). At a point P near a solid boundary an appreciable portion of the gas molecules there has come from the boundary directly without experience of any collision with other molecules. The contribution of the molecules to the state of the gas may be called the direct effect of the boundary. If we pay attention to the molecules which have directly come from a specified point S on the boundary, we may call their contribution the direct effect of the point S. Only a part of the boundary which lies within the distance of order of the mean free path from the point P gives a substantial contribution to the direct effect. The direct effect of a point which is far away from P on the scale of the mean free path is negligibly small. In case of a plane or concave boundary, any point of the boundary which lies within a distance of order of the mean free path from P gives its direct effect on P. In case of a convex boundary only the part between A and B of the boundary in Fig.5a gives the direct effect irrespective of the distance AP. If the distance between P and the boundary (distance PO) is at least of order of the mean free path, the distance AP is much longer than the mean free path and the boundary point which is within a distance of order of the mean free path from P lies inside the part between A and B. Thus, no essential difference can be seen between a convex boundary and plane (or concave) boundary. However, when the distance PO is of order of or smaller than the mean free path squared divided by the radius of the boundary (denoted by δ for shortness), the distance AP is of order of or smaller than the mean free path and the part of the boundary which lies within a distance of order of the mean free path from P extends outside the region between A and B. Thus, an appreciable portion of the boundary points which are within a distance of order of the mean free path from P, (the region outside AB), does not give the direct effect on P. The range which gives the direct effect on P shrinks to vanish as P approaches O over a convex boundary. In other words the direct effect of the boundary on P is obstructed only by molecular collisions in case of a plane or concave boundary. On the other hand, the direct effect is obstructed not only by molecular collisions but also by the boundary itself when the boundary is convex in shape and the P lies within a distance of order δ from O ($PO \leq O(\delta)$). When P is much further away than δ from O, the latter effect (obstruction by the boundary itself) is unimportant since the shadow region (the one outside AB) is too far away from P for its direct effect to be considered. These facts suggest that the behavior of a slightly rarefied gas over a convex boundary shows a qualitatively different character in a thin layer with thickness δ from that of the gas over a plane or concave boundary. This is the new kind of boundary layer over a convex boundary which we have demonstrated in a simple example.

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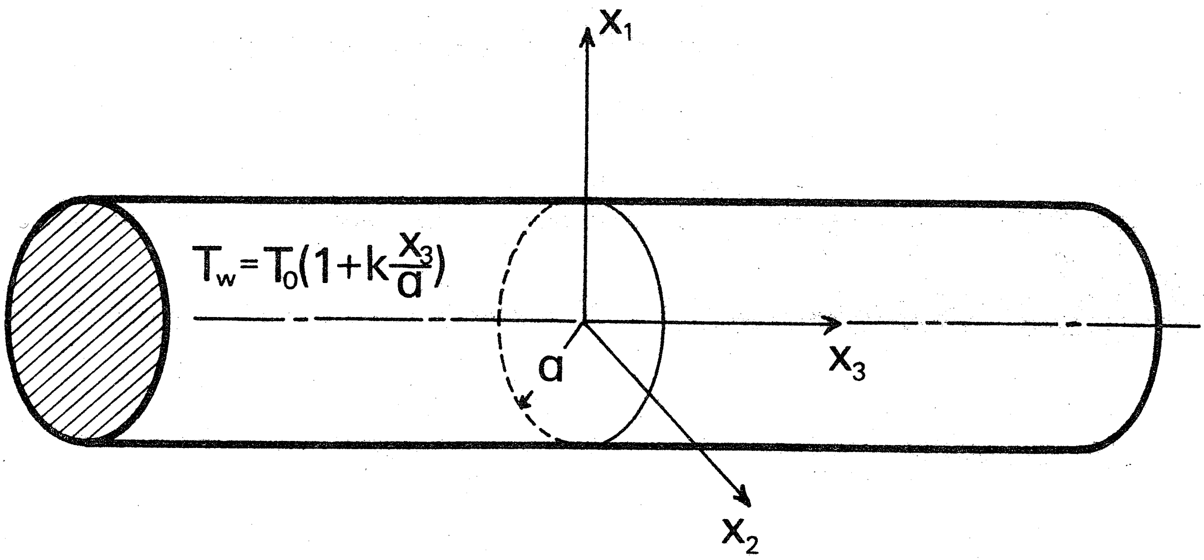


Fig.1 Coordinate system

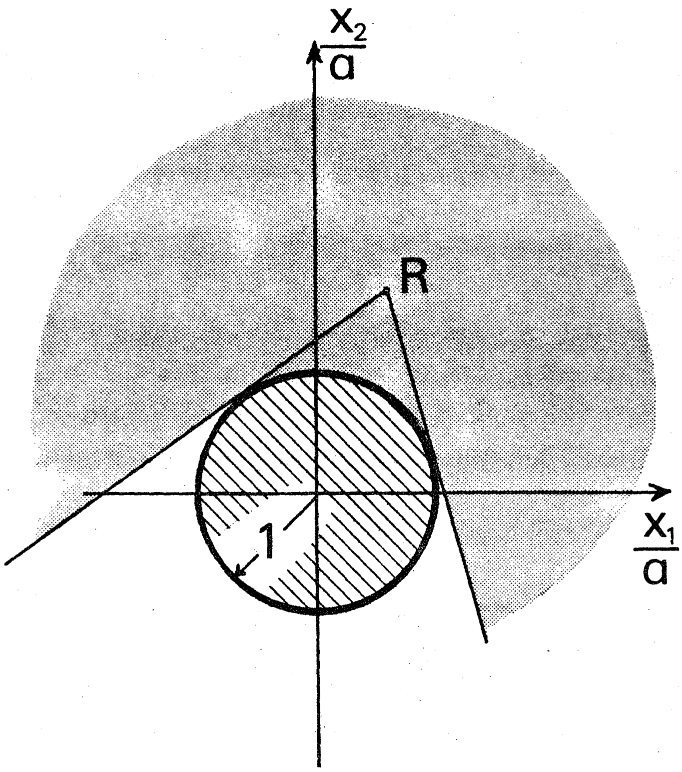
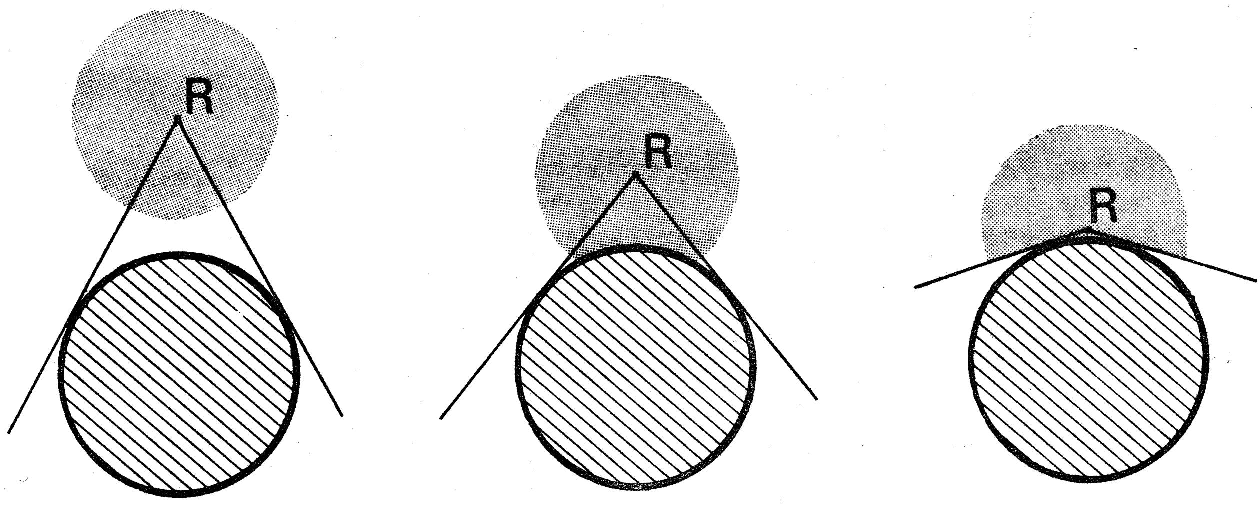


Fig.2 Domain of integration



(a) $R - 1 > O(A^{-1})$

(b) $O(A^{-1}) \geq R - 1 > O(A^{-2})$

(c) $O(A^{-2}) \geq R - 1 \geq 0$

Fig.3 Substantial range of integration

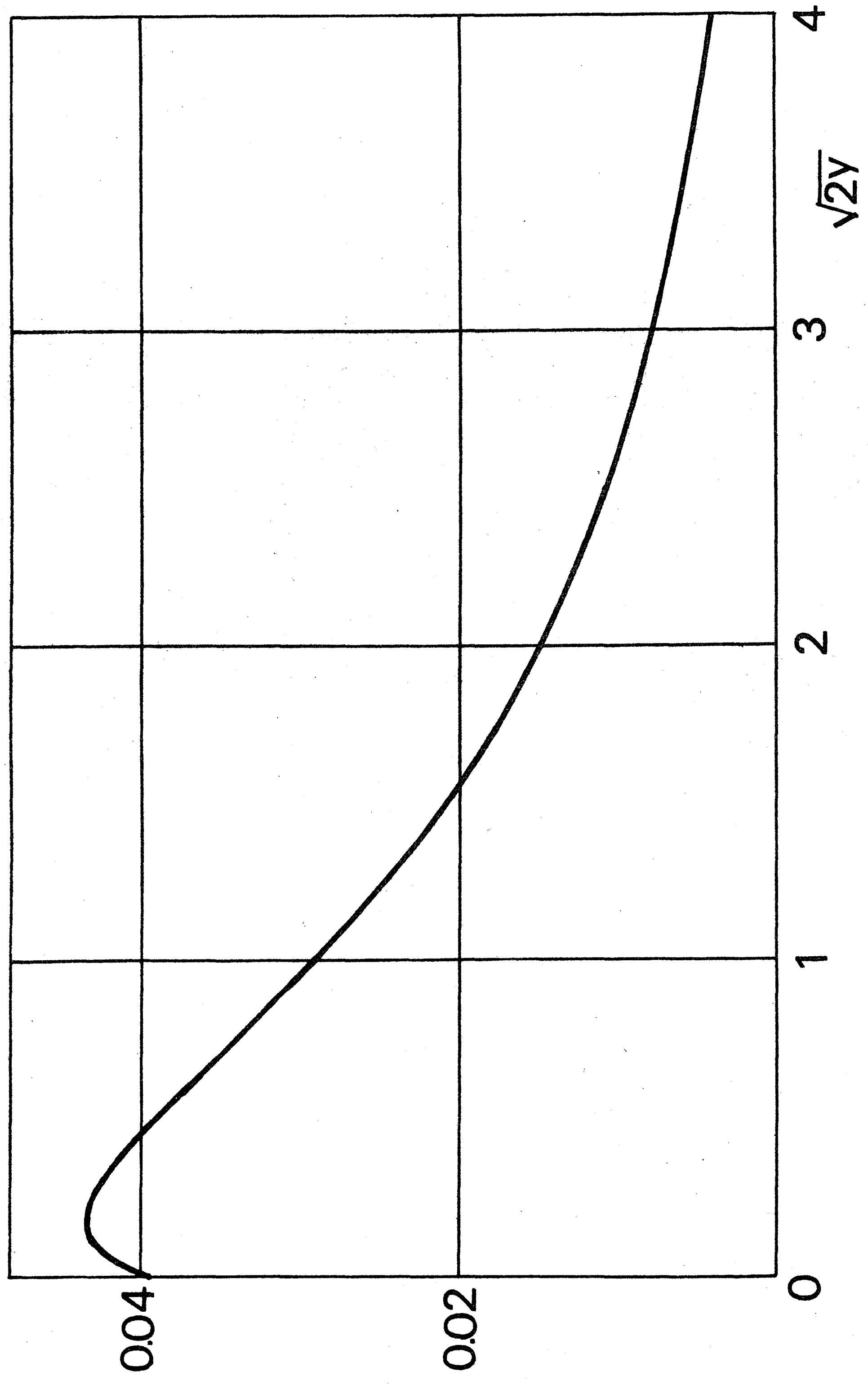
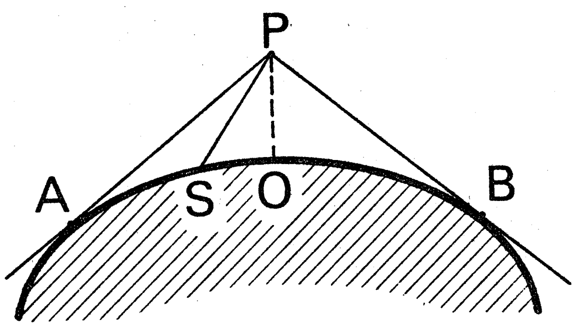
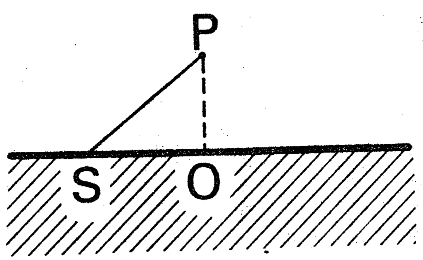


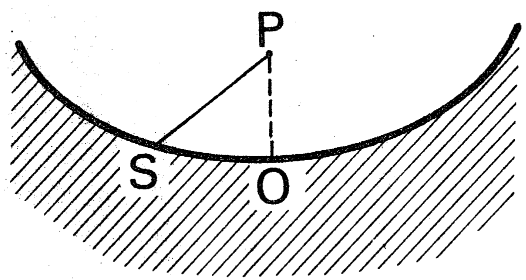
Fig.4 ψ_{S2}



(a)
convex



(b)
plane



(c)
concave

Fig.5 Direct effect of solid boundary