A classification of simple spinnable structures on S^{2n+1}

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§1. Introduction

The notion of a spinnable structure on a closed manifold has been introduced by I. Tamura [5] and independently by Winkelnkemper [6] ("open books" in his term), who obtained a necessary and sufficient condition for existence of it on at least a simply connected closed manifold.

The purpose of the paper is to classify "simple" spinnable structures on an odd dimensional sphere S^{2n+1} in terms of their Seifert matrices.

Definition. A closed manifold M is spinnable, if there is a compact manifold F, called generator, a diffeomorphism $h : F \longrightarrow F$, called characteristic diffeomorphism, such that $h \mid \partial F = \mathrm{id.}$, and a diffeomorphism $g : T(F, h) \longrightarrow M$, where $T(F, h)$ is a closed manifold obtained from $F \times [0, 1]$ by identifying $(x, 0)$ with $(h(x), 1)$ for all $x \in F$ and (x, t) with (x, t') for all $x \in \partial F$ and $t, t' \in [0, 1]$. A triple $\mathscr{S} = \{F, h, g\}$ will be called a spinnable structure on M. A second spinnable structure $\mathscr{S}' = \{F', h', g'\}$ on M is isomorphic with $\mathscr{S}$, if there is a diffeomorphism $f : M \longrightarrow M$ such that $f \circ g(F \times t) = g'(F' \times t)$ for all $t \in [0, 1]$. A spinnable structure $\mathscr{S} = \{F, h, g\}$ on M is simple if its generator is of the homotopy type of a finite CW-complex of dimension $\leq \left[\frac{\dim M}{2}\right]$.

We are interested in simple spinnable structures on S^{2n+1}.
In the case, a generator F is $(n-1)$-connected and a characteristic
diffeomorphism $h : F \longrightarrow F$ induces an isomorphism $h_* : H_n(F, \mathbb{Z})$
$\longrightarrow H_n(F, \mathbb{Z})$ of the integral n-dimensional homology group of F,
which will be called the monodromy of the spinnable structure. In
§ 2, we shall define a Seifert matrix $\Gamma(\mathcal{S})$ of a simple spinnable
structure $\mathcal{S}$ on S^{2n+1} so that it is unimodular and determines
the intersection matrix of F and the monodromy.

Theorem A. For a unimodular $m \times m$ matrix A, there is a
simple spinnable structure $\mathcal{S}$ on S^{2n+1} with $\Gamma(\mathcal{S}) = A$, provided
that $n \geqq 3$.

Theorem B. If $\mathcal{S}$ and $\mathcal{S}'$ are simple spinnable structures
on S^{2n+1} with congruent Seifert matrices, then $\mathcal{S}$ and $\mathcal{S}'$ are
isomorphic, provided that $n \geqq 3$.

These theorems imply that there is a one to one correspondence
of isomorphism classes of simple spinnable structures on S^{2n+1}
with congruence classes of unimodular matrices via the Seifert matrix.

§ 2. Seifert matrices of simple spinnable structures on an Alexander manifolds.

First of all we prove:

Proposition 2.1. If $\mathscr{S} = \{F, h, g\}$ is a simple spinnable structure on a closed orientable $(2n+1)$-manifold M, then $g \mid F \times t :$ $F \times t \longrightarrow M$ is n-connected, in particular, if $M = S^{2n+1}$, then F is $(n-1)$-connected and hence is of the homotopy type of a bouquet of n-spheres;

$$F \simeq \bigvee_{i=1}^{m} S_i^n .$$

Proof. For the proof, putting $F_t = g(F \times t)$, it suffices to show that (M, F_0) is n-connected. We put $W = g(F \times [0, \tfrac{1}{2}])$ and $W' = g(F \times [\tfrac{1}{2}, 1])$. Since $\mathscr{S}$ is simple, it follows from the general position that there is a PL embedding $f : K \longrightarrow \text{Int } W'$ from an n-dimensional compact polyhedron K into $\text{Int } W'$ which is a homotopy equivalence. Since $\partial W' = \partial W$ is a deformation retract of $W' - f(K)$, we have that

$$\pi_i(M, F_0) \cong \pi_i(M, W) = \pi_i(M, M-W')$$

$$\cong \pi_i(M, M-f(K))$$

$$= 0 \qquad \text{for } i \leqq n,$$

completing the proof.

We shall call a closed orientable $(2n+1)$-manifold M is an Alexander

manifold, if $H_n(M) = H_{n+1}(M) = 0$. By the Poincaré duality, then $H_{n-1}(M)$ is torsion free and hence if $\mathscr{S}$ is a simple spinnable structure on M, then $H_{n-1}(F)$ and $H_n(F)$ are torsion free. Then a bilinear form

$$\gamma : H_n(F) \otimes H_n(F) \longrightarrow Z$$

is defined by

$$\gamma(\alpha \otimes \beta) = L(g_\#(\alpha \times t_0), \ g_\#(\alpha \times t_1)),$$

where $0 \leqq t_0 < \frac{1}{2}$, $\frac{1}{2} \leqq t_1 < 1$, and $L(\xi, \eta)$ stands for the linking number of cycles ξ and η in M so that $L(\xi, \eta)$ = intersection number $\langle \lambda, \eta \rangle$ of chains λ and η in M for some λ with $\partial\lambda = \xi$.

For a basis $\alpha_1, \cdots, \alpha_m$ of a free abelian group $H_n(F)$, a square matrix $(\gamma(\alpha_i \otimes \alpha_j)) = (\gamma_{ij})$ will be called a Seifert matrix of $\mathscr{S}$ and denoted by $\Gamma(\mathscr{S})$. It is a routine work to make sure that the congruence class of $\Gamma(\mathscr{S})$ is invariant under the isomorphism class of $(M, \mathscr{S})$.

We have an alternative expression of $\Gamma(\mathscr{S})$ in terms of an isomorphism

$$a : H_n(W) \overset{\partial^{-1}}{\underset{\cong}{}} H_{n+1}(M, W) \overset{exc^{-1}}{\underset{\cong}{}} H_{n+1}(W', \partial W') \overset{\text{Poincaré}}{\cong} H^n(W') \overset{\text{dual space}}{\cong} H_n(W')$$

which will be called the Alexander isomorphism.

We have homomorphisms

$$\varphi : H_n(W) \overset{\partial^{-1}}{\cong} H_{n+1}(M, W) \overset{exc^{-1}}{\cong} H_{n+1}(W', \partial W) \overset{\partial}{\longrightarrow} H_n(\partial W)$$

and

$$\varphi': H_n(W') \cong H_{n+1}(M, W') \cong H_{n+1}(W, \partial W) \longrightarrow H_n(\partial W)$$

so that $i_* \circ \varphi = $ id. and $i'_* \circ \varphi'_* = $ id. and the following

sequences are exact:

$$0 \longrightarrow H_n(W') \xrightarrow{\varphi'} H_n(\partial W) \xrightarrow{i_*} H_n(W) \longrightarrow 0 ,$$

$$0 \longrightarrow H_n(W) \xrightarrow{\varphi} H_n(\partial W) \xrightarrow{i'_*} H_n(W') \longrightarrow 0 ,$$

where $i_* : H_n(\partial W) \longrightarrow H_n(W)$ and $i'_* : H_n(\partial W) \longrightarrow H_n(W')$ are

homomorphisms induced from the inclusion maps. Let $\alpha_1, \cdots, \alpha_m$

be a basis of $H_n(W)$. Then, putting $\beta_i = a(\alpha_i)$, $i = 1, \cdots, m$,

we have a basis $\beta_1, \cdots, \beta_m$ of $H_n(W')$. By the definition of the

Alexander isomorphism, if we put $\overline{\alpha}_i = \varphi(\alpha_i)$ and $\overline{\beta}_i = \varphi'(\beta_i)$,

$i = 1, \cdots, m,$ then we have that the intersection number in ∂W

$$\langle \overline{\alpha}_i, \overline{\beta}_j \rangle = \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases}$$

Let $g_t : F \longrightarrow M$ be an embedding defined by

$$g_t(x) = g(x, t) \quad \text{for all } x \in F, \ t \in [0, 1].$$

For a subspace X of M with $g_t(F) \subset X$, we denote the range

restriction of g_t to X by $X \mid g_t : F \longrightarrow X$;

$$X \mid g_t(x) = g_t(x) \quad \text{for all } x \in F.$$

We identify a basis $\alpha_1, \cdots, \alpha_m$ of $H_n(W)$ with that of $H_n(F)$

via $(W \mid g_{1/3})_*$ and a basis $\beta_1, \cdots, \beta_m$ of $H_n(W)$ with that of

$H_n(F)$ via $(W \mid g_{2/3})_*$.

Again by the definition of the Alexander isomorphism, we have

that

$$L(\alpha_i, \beta_j) = \delta_{ij} \quad \text{for } i, j = 1, \cdots, m.$$

Since $W \mid g_{1/3}$ and $W \mid g_{1/2} = i \circ (\partial W \mid g_{1/2})$ are homotopic in W and

$W' \mid g_{2/3}$ and $W' \mid g_{1/2} = i' \circ (\partial W \mid g_{1/2})$ are homotopic in W', it follows

that $(\partial W \mid g_{1/2})_*(\alpha_i)$ is of a form

$$(\partial W \mid g_{1/2})_*(\alpha_i) = \overline{\alpha}_i + \sum_{j=1}^{m} a_{ij} \overline{\beta}_j .$$

and hence that $(W' \mid g_{\frac{2}{3}})_*(\alpha_i) = \sum_{j=1}^{m} a_{ij}\beta_j = \sum_{j=1}^{m} a_{ij}a(\alpha_j)$.

Therefore, we have that $\gamma_{ij} = L((g_{\frac{1}{3}})_\# \alpha_i, (g_{\frac{2}{3}})_\# \alpha_j) =$

$L(\alpha_i, \sum a_{jk}\beta_k) = a_{ji}$ for $i, j = 1, \cdots, m$. Thus we conclude as follows:

Proposition 2.2. For a basis $\alpha_1, \cdots, \alpha_m$ of $H_n(F) \overset{(W \mid g_{\frac{1}{3}})_*}{\cong} H_n(W)$, the following (1), (2) and (3) are equivalent.

(1) $(\partial W \mid g_{\frac{1}{2}})_*(\alpha_i) = \overline{\alpha}_i + \sum_{j=1}^{m} a_{ij}\overline{\beta}_j$,

(2) $a^{-1} \circ (W' \mid g_{\frac{2}{3}})_*(\alpha_i) = \sum_{j=1}^{m} a_{ij}\alpha_j$

and

(3) $\Gamma^t = (a_{ij})$.

In particular, the Seifert matrix Γ is unimodular.

Now we determine algebraic structures of simple spinnable structures on an Alexander manifold.

Theorem 2.3. Let $\mathscr{S} = \{F, h, g\}$ be a simple spinnable structure on an Alexander manifold M^{2n+1}.

(1) The intersection matrix $I = I(F)$ of F and the Seifert matrix $\Gamma = \Gamma(\mathscr{S})$ of $\mathscr{S}$ are related in a formula:

$$- I = \Gamma + (-1)^n \Gamma^t ,$$

where Γ^t is the transposed matrix of Γ.

(2) The n-th monodromy $h_* : H_n(F) \longrightarrow H_n(F)$ is given by a formula:

$$h_* = (-1)^{n+1}\Gamma^t \cdot \Gamma^{-1} ,$$

or

$$h_* - E = I \cdot \Gamma^{-1} .$$

Proof. For the proof of (1), we follow Levine [3], p.542. We

take chains $d = g_\#(\alpha_i \times [\frac{1}{3}, \frac{2}{3}])$, e_1 and e_2 in M such that

$$\partial d = g_\#(\alpha_i \times \tfrac{2}{3}) - g_\#(\alpha_i \times \tfrac{1}{3}) = (g_{\frac{2}{3}})_\#(\alpha_i) - (g_{\frac{1}{3}})_\#(\alpha_i),$$

$$\partial e_1 = -(g_{\frac{2}{3}})_\#(\alpha_i)$$

and

$$\partial e_2 = (g_{\frac{1}{3}})_\#(\alpha_i).$$

Since $d + e_1 + e_2$ is a cycle, we have that

$$0 = \langle d + e_1 + e_2, (g_{\frac{1}{2}})_\#(\alpha_j)\rangle$$

$$= \langle d, (g_{\frac{1}{2}})_\#(\alpha_j)\rangle + \langle e_1, (g_{\frac{1}{2}})_\#(\alpha_j)\rangle + \langle e_2, (g_{\frac{1}{2}})_\#(\alpha_j)\rangle$$

$$= \langle \alpha_i, \alpha_j\rangle + (-1)L((g_{\frac{2}{3}})_\#(\alpha_i), (g_{\frac{1}{2}})_\# \alpha_j) + L((g_{\frac{1}{3}})_\#(\alpha_i), (g_{\frac{1}{2}})_\# \alpha_j)$$

Since

$$L((g_{\frac{2}{3}})_\#(\alpha_i), (g_{\frac{1}{2}})_\#(\alpha_j)) = (-1)^{n+1} L((g_{\frac{1}{2}})_\#(\alpha_j), (g_{\frac{2}{3}})_\#(\alpha_i))$$

$$= (-1)^{n+1} \gamma(\alpha_j \otimes \alpha_i)$$

and

$$L((g_{\frac{1}{3}})_\#(\alpha_i), (g_{\frac{1}{2}})_\#(\alpha_j)) = \gamma(\alpha_i \otimes \alpha_j),$$

we have that

$$-I = \Gamma + (-1)^n \Gamma^t,$$

completing the proof of (1). To prove (2), we take chains $d = g_\#(\alpha_i \times [0, 1])$, e_0 and e_1 in M so that $\partial d = g_{1\#}(\alpha_i) - g_{0\#}(\alpha_i)$, $\partial e_0 = g_{0\#}(\alpha_i)$ and $\partial e_1 = -g_{1\#}(\alpha_i) = -g_{0\#}(h_*(\alpha_i))$. Since $d + e_0 + e_1$ is an (n+1)-cycle in M, we have that

$$0 = \langle d + e_0 + e_1, (g_{\frac{1}{2}})_\#(\alpha_j)\rangle$$

$$= \langle d, (g_{\frac{1}{2}})_\#(\alpha_j)\rangle + \langle e_0, (g_{\frac{1}{2}})_\#(\alpha_j)\rangle + \langle e_1, (g_{\frac{1}{2}})_\#(\alpha_j)\rangle$$

$$= \langle \alpha_i, \alpha_j\rangle + L(g_{0\#}(\alpha_i), (g_{\frac{1}{2}})_\#(\alpha_j)) + (-1)L(g_{0\#}(h_*(\alpha_i)), (g_{\frac{1}{2}})_\#(\alpha_j))$$

$$= \langle \alpha_i, \alpha_j\rangle + \gamma(\alpha_i \otimes \alpha_j) - \gamma(h_*(\alpha_i) \otimes \alpha_j)$$

$$= \langle \alpha_i, \alpha_j\rangle + \gamma((id - h_*)(\alpha_i) \otimes \alpha_j)$$

and hence that

$$- I = (E - h_*) \cdot \Gamma ,$$

where E is the identity matrix (δ_{ij}). Therefore, by making use of (1), we have that

$$(h_* - E) = I \cdot \Gamma^{-1}$$

$$= - E + (-1)^{n+1} \Gamma^{t} \Gamma^{-1} ,$$

or

$$h_* = (-1)^{n+1} \Gamma^{t} \Gamma^{-1} ,$$

completing the proof.

§ 3. Proof of Theorem A.

Suppose that we are given an $m \times m$ unimodular matrix $A = (a_{ij})$. Let K denote a bouquet of m n-dimensional spheres; $K = \bigvee_{i=1}^{m} S_i^n$. We have a PL embedding $f : K \longrightarrow S^{2n+1}$. Let W be a smooth regular neighborhood of $f(K)$ in $S^{2n+1} = S$ and $W' = S - \text{Int } W$. We denote the Alexander isomorphism

$$H_n(W) \cong H^n(S - \text{Int } W) = H^n(W') = \text{Hom}(H_n(W')) \cong H_n(W')$$

by $a : H_n(W) \cong H_n(W')$. Thus we have that W, W' and ∂W are $(n-1)$-connected, and there are splittings

$$\varphi : H_n(W) \cong H_{n+1}(S, W) \cong H_{n+1}(W', \partial W) \longrightarrow H_n(\partial W) ,$$

$$\varphi' : H_n(W) \cong H_{n+1}(S, W') \cong H_{n+1}(W, \partial W) \longrightarrow H_n(\partial W)$$

of $i_* : H_n(\partial W) \longrightarrow H_n(W)$ and $i'_* : H_n(\partial W) \longrightarrow H_n(W')$, respectively. Note that the following sequences are exact.

$$0 \longrightarrow H_n(W) \overset{\varphi}{\longrightarrow} H_n(\partial W) \overset{i'_*}{\longrightarrow} H_n(W') \longrightarrow 0$$

and

$$0 \longrightarrow H_n(W') \overset{\varphi}{\longrightarrow} H_n(\partial W) \longrightarrow H_n(W) \longrightarrow 0 .$$

If $\alpha_1, \cdots, \alpha_m$ is a basis of $H_n(K) \cong H_n(W)$ and we put $a'(\alpha_i)$

$= \beta_i$, $\varphi(\alpha_i) = \bar{\alpha}_i$, and $\varphi(\beta_i) = \bar{\beta}_i$, $i = 1, \cdots, m$, then we

have that the intersection numbers in ∂W $\langle \bar{\alpha}_i, \bar{\alpha}_j \rangle = 0$,

$\langle \bar{\beta}_i, \bar{\beta}_j \rangle = 0$ and $\langle \bar{\alpha}_i, \bar{\beta}_j \rangle = \delta_{ij}$ for $i, j = 1, \cdots, m$, and

the linking numbers in S $L(\alpha_i, \beta_j) = \delta_{ij}$, $i, j = 1, \cdots, m$.

A splitting $s : H_n(W) \longrightarrow H_n(\partial W)$ of $i_* : H_n(\partial W) \longrightarrow H_n(W)$

will be called a _non-singular section_, if $i'_* \circ s : H_n(W) \longrightarrow H_n(W')$

is an isomorphism. Indeed, a section $s : H_n(W) \longrightarrow H_n(\partial W)$ has

to be of a form

$$s(\alpha_i) = \bar{\alpha}_i + \sum_{j=1}^{m} a_{ij} \bar{\beta}_j$$

and hence $i'_* \circ s(\alpha_i) = \sum_{j=1}^{m} a_{ij} \beta_j$. Thus the correspondence $s \longmapsto (a_{ij})$

gives rise to a one to one correspondence of non-singular sections

$H_n(W) \longrightarrow H_n(\partial W)$ with unimodular $m \times m$ matrices (a_{ij}). As is

found by Winkelnkemper [6] and also Tamura [4] for a non-singular

section $s : H_n(W) \longrightarrow H_n(\partial W)$, there is a PL embedding f' :

$K^n \longrightarrow \partial W$, provided that $n \geq 3$, which is homotopic to f :

$K \longrightarrow W$ and $f'_*(\alpha_i) = s(\alpha_i)$ in ∂W. Moreover, if F is a

regular neighborhood of $f'(K)$ in ∂W and $F' = \partial W - \text{Int } F$, then

$(W; F, F')$ and $(W'; F', F)$ are relative h-cobordisms, since

$s(\alpha_1), \cdots, s(\alpha_m)$ is a basis of $H_n(F)$ as a subgroup of $H_n(\partial W)$

and the inclusion maps induce isomorphisms

$$j_* : H_n(F) \cong H_n(W) \; ; \; j_*(s(\alpha_i)) = \alpha_i$$

and

$$j_* : H_n(F) \cong H_n(W'); \; j'_*(s(\alpha_i)) = i'_* \circ s(\alpha_i) = \sum_{j=1}^{m} a_{ij} \beta_j$$

and W, W', F, F' are 1-connected.

It follows that by the h-cobordism theorem, S^{2n+1} admits a

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spinnable structure $\mathscr{S}_A = \{F, h, g\}$ for a given unimodular matrix

$A = (a_{ij})$ such that

$$g(F \times [0, \tfrac{1}{2}]) = W ,$$

$$g(F \times [\tfrac{1}{2}, 1]) = W'$$

and $\qquad\qquad g(x, \tfrac{1}{2}) \qquad\qquad$ for all $x \in F$.

We would like to show that $\Gamma(\mathscr{S}_A) = A^t$. We have seen that

$(\partial W \mid g_{\frac{1}{2}})_*(\alpha_i) = s(\alpha_i) = \overline{\alpha}_i + \sum\limits_{j=1}^{m} a_{ij} \beta_j$. It follows from Prop-

osition 2.2 that $\Gamma(\mathscr{S}_A) = A^t$. Therefore, for a given unimodular

matrix A, $\mathscr{S}_{A^t}$ is the required spinnable structure on S^{2n+1},

completing the proof.

§ 4. <u>Proof of Theorem</u> B.

The crux of the proof of Theorem B is due to J. Levine [2], who

proved essentially the following:

<u>Proposition</u> 4.1 (Levine). Let $\mathscr{S} = \{F, h, g\}$ and $\mathscr{S}' = \{F', h', g'\}$ be spinnable strcutures on S^{2n+1}. Suppose that $n \geq 3$.

Then two generators F_0 and F_0' are ambient isotopic in S^{2n+2} if

$\Gamma(\mathscr{S})$ and $\Gamma(\mathscr{S}')$ are congruent.

<u>Proof</u>. By a suitable change of bases, we may assume that

$\Gamma(\mathscr{S}) = \Gamma(\mathscr{S}')$. The rest of the proof is what Levine has done in

his classification of simple knots (Lemma 3, [2], §14- §16, pp.191-

192). His arguments work equally well in our case, completing the

proof.

Thus we have a diffeomorphism $f : S^{2n+1} \longrightarrow S^{2n+1}$ such that

$f(F_0) = F_0'$, and f is diffeotopic to the identity. By opening

out

the spinnable structure, we have a diffeomorphism $H : F \times [0, 1] \longrightarrow$

$F' \times [0, 1]$ such that

$$H(x, 0) = (k(x), t) \quad \text{for all} \quad (x, t) \in \partial F \times [0, 1]$$

$$H(x, 0) = (k(x), 0) \quad \text{for all} \quad x \in F \quad \text{and}$$

$$H(x, 1) = (h'^{-1} \circ k \circ h(x), 1) \quad \text{for all} \quad x \in F,$$

where

$$(k(x), 0) = (g')^{-1} \circ f \circ g(x, 0) \quad \text{for all} \quad x \in F.$$

This implies that $(k^{-1} \times \text{id}) \ H : F \times [0, 1] \longrightarrow F \times [0, 1]$ is an

pseudo-diffeotopy from id to $k^{-1} \circ j'^{-1} \circ k \circ h$ keeping ∂F fixed.

Since $n \geq 3$, F and ∂F are 1-connected, it follows from Cerf [1]

that the pseudo-diffeotopy is diffeotopic to a diffeotopy $G : F \times I$

$\longrightarrow F \times I$ keeping $\partial (F \times I)$ fixed. This implies that f is dif-

feotopic to an isomorphism $(S^{2n+1}, \mathscr{S}) \longrightarrow (S^{2n+1}, \mathscr{S}')$ keeping F_0

fixed. Therefore, $\mathscr{S}$ and $\mathscr{S}'$ are isomorphic, completing the

proof.

Remark. As is known from the proof, $\mathscr{S}$ and $\mathscr{S}'$ are iso-

morphic by an ambient diffeotopy.

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