数理解析研究所講究録
第179巻 1973年 153-167

HARMONIC ANALYSIS OF SWITCHING FUNCTIONS

SHUZO YAJIMA AND NORITAKA UJI
(FACULTY OF ENGINEERING, KYOTO UNIVERSITY)

1 INTRODUCTION

This note describes the extension of Harmonic Analysis of binary n-variable switching function ([1] ∿ [4]) to that of a function defined on a finite field GF(p), and a trial of its application to pattern recognition. A 2-variable switching function on GF(p) can be made to correspond to a two dimensional digital pattern on the p×p meshed plane. In this case, a pattern is not regarded as an input vector like ordinary case, but is regarded as a switching function itself.

Classification of switching functions into equivalence classes under the transformation group defined on the domain leads to the classification to patterns. We discuss mainly Affine Transformation Group and its subgroups.

There is a one-to-one correspondence between switching function f(x) and its Fourier Transform F(w) and so by the spectrum invariance and other properties of Fourier Transform coefficients we can examine the classification easily. We also describes the relation between the distances in the space of functions and in the space of their spectra.

2 FOURIER TRANSFORM OF SWITCHING FUNCTIONS

2-1 Notation and Definition

In this note GF(p) represents a finite field of integers modulo p (p; prime). $\{GF(p)\}^n$ represents n-dimensional vector space over GF(p), and F denotes the set of all n-input, single output functions: $\{GF(p)\}^n \to GF(q)$. Let X represent the set of all p-ary n-tuples x in $\{GF(p)\}^n$, that is, domain for the space F. f,g are used as the elements of F, which are called switching functions. $p^n(=N)$ dimensional output vector

$$f=(f(x(0)),\cdots,f(x(p^n-1)))=(f_1,\cdots,f_N)$$

is one representation of f. Another representation of f, where output is binary, is a level set (or on-set) representation $f^{-1}(1)=\{x|f(x)=1\}$. The number of its elements x of f is called the weight of the switching function.

At first we define the discrete Fourier Transform (DFT) [11] by which we will get another representation of switching functions.

Definition 2.1 Multi-dimensional modulo $(t_1,\cdots,t_n)$ DFT$[A_{\mathbf{s}}]$ of an array $[a_{\mathbf{r}}]$ is defined as follows. For the array $[a_{\mathbf{r}}]$ composed of real $t_1\times\cdots\times t_n$ elements $a_{\mathbf{r}}$'s (suffix $\mathbf{r}$ is a vector $(r_1,\cdots,r_n)$), its transformed array $[A_{\mathbf{s}}]$ has complex number elements $A_{\mathbf{s}}$ defined by the equations;

$$A_{\mathbf{s}}=\sum_{\mathbf{r}} a_{\mathbf{r}} W_1^{r_1 s_1}\cdots W_n^{r_n s_n} \quad (r_j,s_j=0,\cdots,t_{j-1};\ j=1,\cdots,n) \quad (2.1)$$

where $W_j=\exp(2\pi i/t_j)$ $(j=1,\cdots,n)$

When applying DFT to a p-ary n-variable switching function according to the definition above, let $t_1=\cdots=t_n=p$, $\mathbf{r}$ be input vector x, and array $[a_r]$ to be the output vector of f.

Let F(w) represent the Fourier Transform of a function f(x) whose domain is another vector space W (p-ary n-tuples) isomorphic to X.

Definition 2.2 [2] Abstract Fourier Transform F(w) for a switching function f(x) is defined as follows;

$$F(w) = \sum_{X} f(x)\phi_w(x) \qquad w \in \{GF(p)\}^n, \tag{2.2}$$

where $\phi_w(x) = \exp[\frac{2\pi i}{p}(xw^t)] = \exp[\frac{2\pi i}{p}(x_1w_1 + \cdots + x_nw_n)]$.

Theorem 2.3 There is a unique inverse transform from F(w) to f(x):

$$f(x) = p^{-n}\sum_{W} F(w)\phi_w^*(x), \tag{2.3}$$

where $\phi_w^*(x)$ means complex conjugate of $\phi_w(x)$.

This shows that there is a one-to-one correspondence between a switching function f(x) and its Fourier Transform F(w). The value F(w) for each w is generally a complex number and is called a coefficient of the spectrum of Fourier Transform. As well known, for p=2, f(x) is an ordinary switching function, coordinates F(w) are integers (Lechner [1][2]). For p=2, there is another coordinates representation N(w) by Ninomiya that has a close relation to F(w) defined as follows ([1][3][4]),

$$N(w) = 2^{n-1}\delta_{wo} - F(w). \tag{2.4}$$

2-2 Some properties of Fourier Transform

By using previous definitions on Fourier Transform of a p-ary n-variable switching function, some general important properties are obtained. Here, for the convenience, the output range of the function is restricted to GF(2).

Lemma 2.4 The following special functions have unique Fourier Transforms.

$$f(x)=1 \quad \text{iff} \quad F(w)=p^n\delta_{wo} \tag{2.5}$$

$$f(x)=0 \quad \text{iff} \quad F(w)=0 \tag{2.6}$$

$$f(x)=\delta_{xo} \quad \text{iff} \quad F(w)=1 \tag{2.7}$$

Lemma 2.5 Sum of spectra over w has the unique value.

$$\sum_w F(w) = \begin{cases} 0 & \text{iff} \quad f(0)=0 \\ p^n & \text{iff} \quad f(0)=1 \end{cases} \tag{2.8}$$

Lemma 2.6 Fourier spectrum at w=0 gives the weight of corresponding function.

$$F(0)=\sum_x f(x)=m \text{ (weight)} \tag{2.9}$$

Definition 2.7 For two switching functions $f_1(x)$ and $f_2(x)$ from the same space F its convolution sum at x, written as $h(x)=f_1(x)*f_2(x)$, is defined as follows;

$$h(x)=\sum_t f_1(t)\cdot f_2(x-t). \tag{2.10}$$

Theorem 2.8 The Fourier Transform H(w) of the convolution sum $h(x)=f_1(x)*f_2(x)$ is equal to the componentwise product of two Fourier Transforms $F_1(w)$ and $F_2(w)$ of each function, i.e.,

$$H(w)=\sum_x\{f_1(t)*f_2(x)\}\phi_w(x)=F_1(w)\cdot F_2(w), \tag{2.11}$$

and its inverse relation is true, i.e.,

$$h(x)=f_1(x)*f_2(x)=p^{-n}\sum_w\{F_1(w)\cdot F_2(w)\}\phi_w^*(x). \tag{2.12}$$

Using above formula for $f(x)*f(-x)$, we find that Parseval equation holds for a switching function.

<u>Proposition 2.9</u> $$\sum_{w}|F(w)|^2=p^n\sum_{x}\{f(x)\}^2=p^n m \qquad (2.13)$$

From (2.13) we can see the relation between spectrum and function value and conclude that every switching function that has the same weight has the same $P=\sum|F(w)|^2$.

3 SPECTRUM INVARIANCE UNDER SOME TRANSFORMATIONS

3-1 Transformation Group on Switching Functions

In this section some transformation groups on the domain of switching functions and the classification of functions into the equivalence classes are described ([5]∿[9]).

<u>Definition 3.1</u> Let x be the input row vector over GF(p) of a p-ary n-variable switching function, so general linear transformation $T_l \in LG_n(GF(p))$ (the set of all such transformations) and affine transformation $T_a \in AG_n(GF(p))$ (the set of all such transformations) on the domain are

$$T_l f(x) = f(T_l x) = f(xA) \qquad (3.1)$$

$$T_a f(x) = f(T_a x) = f(xA+b) \qquad (3.2)$$

where A is a nonsingular matrix over GF(p) and b is a row vector in $V_n(GF(p))$.

<u>Definition 3.2</u> $f_1(x)$ and $f_2(x)$ are equivalent under the group G if and only if there exists some element $g \in G$ such that $f_1(x)=g \cdot f_2(x)$ for all $x \in \{GF(p)\}^n$.

In the case of affine group,

$$f_1(x) \simeq f_2(x) \quad \text{under } AG_n(GF(p))$$
$$\text{for } \forall x \in \{GF(p)\}^n \ \exists (A \text{ and } b) \quad \text{s.t. } f_1(x)=f_2(xA+b). \tag{3.3}$$

Similarly we can consider some subgroups of affine transformation group. The family of functions are partitioned in various ways according to the equivalence under respective transformation groups.

Proposition 3.3 All functions in the same equivalence class under the transformation group defined on the domain have the same weight m.

3-2 Spectrum Invariance Property

This section describes several fundamental theorems on invariance on multivalued switching function. From (3.3), if two functions f and g are affine equivalent, then

$$g(x) = f(xA+b). \tag{3.4}$$

Consider their Fourier Transforms using (2.2),

$$\begin{aligned} G(w) &= \sum_x g(x)\phi_w(x) \\ &= \sum_y f(y)\phi_w\{(y-b)A^{-1}\} \\ &= \sum f(y)\alpha^{y(wA^{-t})^t}\alpha^{-bA^{-1}w^t} \qquad \alpha=\exp(2\pi i/p) \\ &= F(wA^{-t})\exp[-\tfrac{2\pi i}{p}(bA^{-1}w^t)]. \end{aligned} \tag{3.5}$$

From this formula we can derive the following useful theorems as in the Lechner's case of binary functions.

Theorem 3.4 (Affine Trans.) Every switching function belonging to the same equivalence class with respect to the affine transformation($x \to xA+b$) has the same value $S=\sum|F(w)|$ (the sum of the absolute values of the Fourier spectrum).

Theorem 3.5 (Linear Trans.) Every function belonging to the same equivalence class with respect to the linear transformation ($x \to xA$) has the same spectrum set $\{F(w)\}$, that is, Fourier transform coefficients are the same under permutation of w.

Theorem 3.6 (Translation) Any two functions that are in the same equivalence class with respect to the translation ($x \to x+b$) have the same absolute value of the spectrum $|F(w)|$ at each w.

Lemma 3.7 $g(x) = f(p-x)$ $G(w) = F^*(w)$ (* conjugate)

Lemma 3.8 $g(x) = \overline{f(x)}$ $G(w) = p^n \delta_{wo} - F(w)$ (Inversion)

From these theorems we can get the important information whether two functions are equivalent under such transformations or not. Unfortunately their converses are not necessarily true. In some cases it may happen that two or more equivalence classes have the same invariant parameters.

Proposition 3.9 The class closed under the same set $\{|F(w)|\}$ includes the equivalence class for affine transformation.

3-3 Some Other Properties of Spectrum

So far we have described various kinds of values with respect to the spectrum such as $F(w)$, $|F(w)|$, $\sum F(w)$, $\sum|F(w)|$, $\sum|F(w)|^2$, $\{F(w)\}$, $\{|F(w)|\}$. These values have some important messages and reflect some properties of original switching function f(x) and their equivalence classes.

Now we introduce some distance in the family of F(w).

<u>Definition 3.10</u> In the complex vector space $V_N(C)$, let F and G be spectrum vectors for f and g with coordinates $F=(F_1,\cdots,F_N)$ and $G=(G_1,\cdots,G_N)$ respectively and introduce the inner product:

$$(F,G) = F_1G_1^*+\cdots+F_NG_N^* , \tag{3.6}$$

where dimension $N=p^n$ for p-ary n-variable functions.

<u>Definition 3.11</u> The norm $\| F \|$ for the spectrum vector F is defined as follows,

$$\| F \| = (F,F)^{1/2} = (F_1F_1^*+\cdots+F_NF_N^*)^{1/2} . \tag{3.7}$$

By using this the relation among norm, spectrum and function itself is derived.

$$\| F \|^2 = \sum F(w)F^*(w) = \sum |F(w)|^2 = p^n\sum\{f(x)\}^2 = mp^n \tag{3.8}$$

<u>Proposition 3.12</u> The norm takes the discrete values of $(mp^n)^{1/2}$ depending on the function weight.

<u>Lemma 3.13</u> $\| F \| = 0$ iff $f(x) = 0$

$\| F \| = p^n$ iff $f(x) = 1$

$$0 \leq \| F \| \leq p^n \tag{3.9}$$

We see also functions whose weights are equal have the same norms of spectrum.

Next we examine the meaning of the inner product of F and G.

$$\begin{aligned}(F,G) &= \sum_w F(w)G^*(w) \\ &= \sum_w\{\sum_{xy} f(x)g(y)\phi_w(x-y)\} \\ &= \sum_{xc} f(x)g(x-c)\sum_w \phi_w(c) \quad = p^n\sum_x f(x)g(x)\end{aligned} \tag{3.10}$$

Proposition 3.14 Inner product (F,G) has the integral value which is multiples of p^n and has the relation (F,G) = (G,F).

The value of (F,G) corresponds to the number of points where f(x)=g(x)=1 is satisfied, that is, the overlapping of two functions.

Theorem 3.15 (F,G)=0 iff corresponding patterns are perfectly separated. $(F,G)=\|F\|^2=\|G\|^2$ iff corresponding patterns are completely equal.

In the next place, for two spectrum vectors F and G, the distance between these two vectors is naturally introduced using the above norm as follows;

$$\|F-G\|^2=(F-G,F-G)=\|F\|^2+\|G\|^2-2(F,G) \qquad (3.11)$$

$$\|F-G\|^2=\sum_w |F(w)-G(w)|^2= p^n\sum_x \{f(x)-g(x)\}^2 \qquad (3.12)$$

Proposition 3.16 $0 \leqq \|F-G\|^2 \leqq \|F\|^2+\|G\|^2$ (3.13)

When f(x) and g(x) are two-valued (0,1) functions, (3.12) shows that the value of $\|F-G\|$ corresponds to the Boolean difference of f(x) and g(x).

Proposition 3.17 Let $\langle d\rangle=\sum_x\{f(x)\oplus g(x)\}$ be the Boolean difference of f(x) and g(x), then $\langle d\rangle= \frac{1}{p^n}\|F-G\|^2$. (3.14)

In this consideration the norm of spectrum has a close connection with the weight of corresponding function . It does not however show directly the functional equivalence under transformation groups. Lastly we will discuss something about the absolute value of spectrum.

<u>Definition 3.18</u> For the spectrum vectors F and G, we define <DF> and <DFA> as

$$\langle DF\rangle = \sum\{|F(w)|-|G(w)|\} \tag{3.15}$$

$$\langle DFA\rangle = \sum||F(w)|-|G(w)||. \tag{3.16}$$

For two functions f(x) and g(x), Boolean difference <Gd> under transformation group G is defined as the minimal value among all the Boolean differences of f(x) and any transformed functions g(tx) ($t \in G$) from g(x).

<u>Definition 3.19</u> $$\langle Gd\rangle = \min_{t\in G}\sum\{f(x)\oplus g(tx)\} \tag{3.17}$$

For example we can write,

$$\langle \text{Affine } d\rangle = \min_{\forall A \forall b}\sum\{f(x)\oplus g(xA+b)\}. \tag{3.18}$$

<u>Lemma 3.20</u> <Gd>=0 iff two functions are equivalent under that transformation group.

<u>Theorem 3.21</u> If two functions are equivalent under affine transformation, namely <Affine d>=0, then $\langle DF\rangle = \sum\{|F(w)|-|G(w)|\}=0$.

This is of course true for all subgroups of affine group.

<u>Theorem 3.22</u> If two functions are equivalent under translation, namely <Trans d>=0, then $\langle DFA\rangle = \sum||F(w)|-|G(w)||=0$.

4 Switching Functions of Two-Variables Defined on GF(p) and its Relation to Pattern

4-1 <u>Two-Variable Switching Functions and Pattern</u>

In this last chapter a switching function is restricted to two variables, that is, $f(x_1,x_2):\{GF(p)\}^2 \rightarrow GF(2)$ is considered. We regard a two-dimensional digital pattern not as one input vector of ordinary switching function but as one function itself.

There is a one-to-one correspondence among $f(x_1,x_2)$, two-valued pattern on a p×p meshed plane and a p×p matrix over GF(2).

In this case two variables x_1 and x_2 may have p integral values(0,1,···,p-1) and the function output vector of p^2-dimension is defined as

$$(f_{00}, f_{01}, \cdots, f_{0p-1}, \cdots, f_{p-1p-1}),$$

where the element f_{ij} corresponds to the output of switching function $f(x_1=i, x_2=j)$. This vector is made to correspond to the digital two-dimensional pattern on the p×p plane as Fig.4.1. By using DFT(2.2) we obtain the Fourier spectrum vector,

$$(F_{00}, F_{01}, \cdots, F_{0p-1}, \cdots, F_{p-1p-1}),$$

where $F_{ij}=F(w_1=i, w_2=j)$, which is usually a complex number.

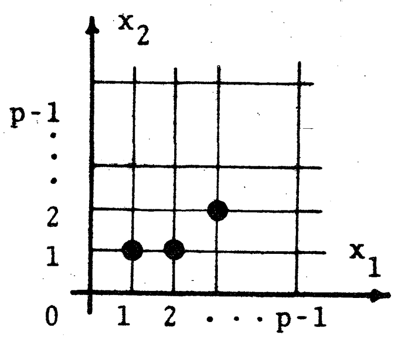


Fig.4.1

For example in the case of p=3,

$$(F_{00}, F_{01}, \cdots, F_{22}) = (f_{00}, f_{01}, \cdots, f_{22})(T), \tag{4.1}$$

where (T) is a $p^n \times p^n (=9\times 9)$ transformation matrix as below. In the matrix α means , $\alpha=\exp(2\pi i/3)$, the 3rd root of unity of this field.

A basis function $f_{bx_0}(x)$, or a weight 1 function, where $f_{bx_0}(x)=1$ only at a point x_0 in the pattern and $f_{bx_0}(x)=0$ otherwise, has the spectrum which is indicated as the row vector of the matrix (T) corresponding to the point f_{x_0}.

. 64

For example, one of the basis functions,

$$f_{b02} = (0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0)$$

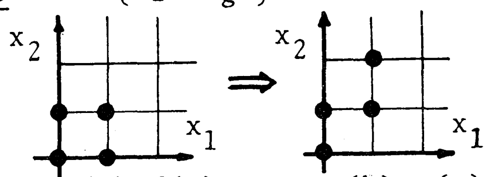
$$\Longleftrightarrow\ F = (1\ \alpha^2\ \alpha\ 1\ \alpha^2\ \alpha\ 1\ \alpha^2\ \alpha).$$

The spectrum of any switching function ($m \geqq 2$) can be obtained as the sum of the spectra of corresponding basis functions using the linearity property.

$$(T) = \begin{array}{c} \\ 00 \\ 01 \\ 02 \\ 10 \\ 11 \\ 12 \\ 20 \\ 21 \\ 22 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & 1 & \alpha & \alpha^2 & 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha & 1 & \alpha^2 & \alpha & 1 & \alpha^2 & \alpha \\ 1 & 1 & 1 & \alpha & \alpha & \alpha & \alpha^2 & \alpha^2 & \alpha^2 \\ 1 & \alpha & \alpha^2 & \alpha & \alpha^2 & 1 & \alpha^2 & 1 & \alpha \\ 1 & \alpha^2 & \alpha & \alpha & 1 & \alpha^2 & \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 & \alpha^2 & \alpha^2 & \alpha^2 & \alpha & \alpha & \alpha \\ 1 & \alpha & \alpha^2 & \alpha^2 & 1 & \alpha & \alpha & \alpha^2 & 1 \\ 1 & \alpha^2 & \alpha & \alpha^2 & \alpha & 1 & \alpha & 1 & \alpha^2 \end{pmatrix} \end{array}$$

[Transformation Matrix]

<u>Example</u> ($f \simeq g$)



(a) $f(x)$ (b) $g(x)$

Fig. 4.2

$$\begin{array}{c} f(x)=1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \end{array} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{array}{c} g(x)=1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \end{array} \qquad (4.2)$$

$f = (110110000)$ $\qquad F = (4, -2\alpha^2, -2\alpha, -2\alpha^2, \alpha, 1, -2\alpha, 1, \alpha^2)$

$g = (110011000)$ $\qquad G = (4, \alpha, \alpha^2, -2\alpha^2, 1, -2\alpha, -2\alpha, -2\alpha^2, 1)$

There are two patterns on the 3×3 meshed plane as in Fig.4.2, which correspond to two functions f and g. The on-set of f can be transformed to be the on-set of g by the nonsingular matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, as shown in (4.2). So f and g are linearly equivalent. Spectra for these two functions show that F and G are equivalent under permutation of w according to Theorem 3.5. There is a relation as to the affine transform by Theorem 3.4, that is, $\{|F(w)|\} = \{|G(w)|\} = 1^4 2^4 4^1$.

Using these invariant parameters of spectrum, patterns which are transformed linearly are classified without the normalization of size of patterns. The property of spectral invariance under translation of patterns would be also useful in various applications. The concept of distance, <DF> and <DFA> could be effectively utilized in the classification of patterns.

4-2 Some Results of Computation

As one example we have examined the classification for all the functions of 3-ary, 2-variables, that is, all the patterns on the 3×3 plane, whose total number is $2^{3^2}=512$. As to the affine transformation, we have obtained following results.

weight	$S=\sum_w \lvert F(w) \rvert$	$\{\lvert F(w) \rvert\}$	weight	$S=\sum_w \lvert F(w) \rvert$	$\{\lvert F(w) \rvert\}$
0	0.0	0^9	9	9.0	$0^8 9^1$
1	9.0	1^9	8	16.0	$1^8 8^1$
2	12.0	$1^6 2^3$	7	17.0	$1^6 2^2 7^1$
3	9.0	$0^6 3^3$	6	12.0	$0^6 3^2 6^1$
	13.4	$0^2 \sqrt{3}^6 3^1$		16.4	$0^2 \sqrt{3}^6 6^1$
4	15.3	$1^6 \sqrt{7}^2 4^1$	5	16.3	$1^6 \sqrt{7}^2 5^1$
	16.0	$1^4 2^4 4^1$		17.0	$1^4 2^4 5^1$

Table. 4.1.

This shows that in this space of functions there are at least fourteen affine equivalence classes and the classes that have the same value $S=\sum_w |F(w)|$ are distinguished only by the affine parameter $\{|F(w)|\}$, and their weight.

Next we show some patterns in different classes at weight 3∿6.

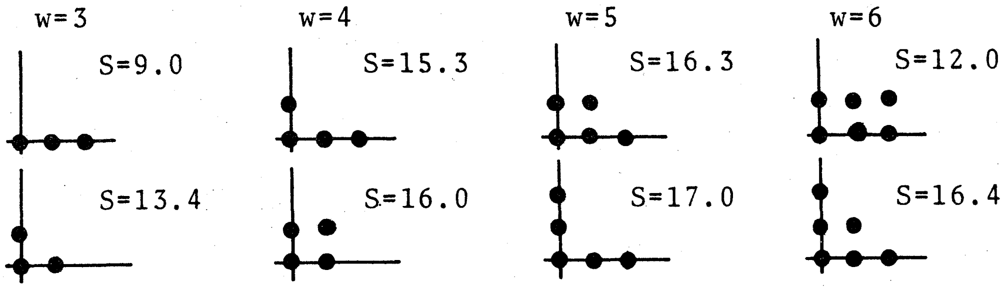


Fig. 4.3.

Now we perform the computer simulation for practical patterns. The Fast Fourier Transform (FFT) can be applied in the computation. The application of this method to a body in three dimension space would be also interesting.

The authors would like to express their thanks to the members of Yajima's Laboratory for their valuable discussion.

References

1. R.J. Lechner, "A Transform Approach to Logic Design", IEEE, Trans. Comput. vol.C-19, No.7, pp.627-640, July 1970.
2. ———, "Harmonic Analysis of Switching Functions", Ch.5 in *Recent Development in Switching Theory*, Academic Press, 1970.
3. I. Ninomiya, "A Study of the Structures of Boolean Functions and its Application to the Synthesis of Switching Circuits", Mem. Fac. Eng. Nagoya Univ. vol.13, No.2, Nov.1961.
4. ———, "A Theory of the Coordinate Representation of Switching Functions", Mem. Fac. Eng. Nagoya Univ. vol.10, Nov.1958.
5. M.A. Harrison, "On the Classification of Boolean Functions by

the General Linear and Affine Groups", J.Soc.Appl.Math., vol.12, pp.285-299, 1964.

6. S.W. Golomb, "On the Classification of Boolean Functions", IRE, Trans. Circuit Theory, vol.CT-6, Special Suppl., pp.176-186, May 1959.
7. H.S. Stone and C.L. Jackson, "Structures of Affine Families of Switching Functions", IEEE, Trans. Comput, vol.C-18, No.3, pp.252-257, March 1969.
8. W.S. Mathson, "PCN Equivalence Class Invariants and Information Quantities", IEEE, Trans. Comput, vol.C-20,pp.691-694, June 1971.
9. C.L. Jackson and R.A. Ankerlin, "A Rapid Method for Identification of the Type of a four-variable Boolean Function", IEEE, Trans. Comput, vol.EC-16, pp.370-371, Dec.1967.
10. J.W. Cooley and T.W. Tarkey, "An Algolithm for the Machine Calculation of Complex Fourier Series",Math. Comp. vol.19 pp.297-301, 1965.
11. I.J. Good, "The Relationship between two Fast Fourier Transformations", IEEE, Trans. Comput, vol.C-20, No.2, pp.310-317, March 1971.
12. G.H. Granlund, "Fourier Processing for Hand Print Character Recognition", IEEE, Trans. Comput, vol.C-21, No.2, pp.195-201, Feb.1972.
13. M.A. Harrison, *Introduction to Switching and Automata Theory*, McGraw-Hill, 1965.
14. S. MacLane and G. Birkhoff, *Algebra*, Macmillan, 1968.