

On the number of moduli
of certain algebraic surfaces
of general type

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0. Introduction. Let P^3 denote the projective 3-space defined over the field of complex numbers, S an irreducible hypersurface of degree $n=2r$ in P^3 , defined by the equation

$$(1) \quad g^2 + Agh + Bh^2 = 0$$

where g , h , A , and B are homogeneous polynomials of degree r , s , $r-s$, and $2(r-s)$, respectively, with two positive integers $r > s$. Clearly, the curve Δ , defined by $g=h=0$, is contained in the singular locus of S .

We assume that S is generic in the following sense:

- 1) S has only ordinary singularities (see [4]) and is non-singular outside of Δ .
- 2) Δ is non-singular.
- 3) The normalization X of S (which is non-singular by 1)) is a surface of general type.

In [4], Kodaira studied families of surfaces with

ordinary singularities in P^3 . In particular, he proved that above S belongs to an effectively parametrized family \mathfrak{F} of surfaces S_t , $t \in M_1$, with ordinary singularities in P^3 whose characteristic system on each S_t is complete (see [4], Theorem 8 and § 5.4). In our case, the number $\mu(S)$ of effective parameters of the family \mathfrak{F} is given by

$$\mu(S) = C(r) + C(s) + C(2r-2s) - C(r-2s) - 2$$

where

$$C(m) = \begin{cases} (m+3)(m+2)(m+1)/6 & \text{for } m \geq 0 \\ 0 & \text{for } m < 0 \end{cases}$$

On the other hand, Kodaira-Spencer introduced the concept of the number of moduli $m(X)$ of a compact complex manifold X (see [5], Definition 11.1).

Main Theorem. Let S be a generic hypersurface in P^3 defined by the equation (1), X the normalization of S . Then, the number of moduli $m(X)$ is defined, and we have

$$m(X) = \dim H^1(X, \mathcal{O}_X) = \mu(S) - 15 - 4\delta_{r,s+1}$$

where \mathcal{O}_X denotes the sheaf of germs of holomorphic vector fields on X , and $\delta_{r,s+1}$ is Kronecker's delta.

Let $f: X \rightarrow P^3$ denote the composition of normalization and the embedding. Then the difference of $\mu(S)$ and

$m(X)$ is the contribution of the number of parameters on which the holomorphic map f depends.

For $(r, s) = (3, 1)$ or $(4, 3)$, S is one of the examples of M. Noether [6].

For $(r, s) = (3, 1)$, X is a minimal algebraic surface with $p_g = 4$, $q = 0$, and $c_1^2 = 6$, where p_g , q , and c_1^2 denote, respectively, the geometric genus, the irregularity, and the Chern number. We have

$$m(X) = 10(p_g - q + 1) - 2c_1^2 = 38,$$

$$H^2(X, \mathcal{O}_X) = 0$$

(cf. Kodaira [8]).

For $(r, s) = (4, 3)$, X is a complete intersection of two hypersurfaces of degree 2 and 4 in P^4 . We have also $H^2(X, \mathcal{O}_X) = 0$.

1. Preliminaries. Let E be a hyperplane section of S , $\tilde{E} = f^*E$, $\tilde{\Delta} = f^{-1}(\Delta)$. From the equation (1), we infer

Lemma 1. $\tilde{\Delta}$ is linearly equivalent to $s\tilde{E}$ on X .

We note that $(n-4)\tilde{E} - \tilde{\Delta}$ is a canonical divisor on X , and that

$$H^v(X, \mathcal{O}(m\tilde{E} - \tilde{\Delta})) \simeq H^v(S, \mathcal{O}(mE - \Delta)) \quad \text{for } v=0,1,2$$

(see [4]). By a standard computation (cf. [7]), we get

Lemma 2. $\dim H^0(X, \mathcal{O}(\tilde{E})) = 4 + \delta_{r,s+1}$, $H^1(X, \mathcal{O}(\tilde{E})) = 0$.

Lemma 3. 1) The canonical bundle K of X is ample.
In particular, X is minimal.

$$2) \quad p_g = \binom{2r-1}{3} - (1/2)rs(3r-s-4),$$

$$q = 0,$$

$$c_1^2 = 2r(2r-s-4)^2.$$

Remark. If $r=s+1$, then the complete linear system $|\tilde{E}|$ is very ample, and X is a complete intersection of two hypersurfaces of degree 2 and r in P^4 .

2. Relation between deformations of S and X . Let $\{S_t\}_{t \in M}$ be a family of surfaces of degree n in P^3 with ordinary singularities, $S = S_0$, $0 \in M$. Letting $T_0(M)$ denote the tangent space of M at 0 , we have the characteristic map

$$\sigma : T_0(M) \longrightarrow H^0(S, \mathcal{N}_S^0)$$

where \mathcal{N}_S^0 denotes the sheaf $\mathcal{O}(nE - \Delta - \sum \tau_i^1)$ in the notation of [4]. We note that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{P^3|S} \longrightarrow \mathcal{N}_S^0 \longrightarrow 0.$$

On the other hand, the normalization X_t of S_t form a family $\mathcal{X} = \{X_t\}_{t \in M}$ of deformations of $X = X_0$ and the holomorphic map $f: X \rightarrow P^3$ extends to a holomorphic map $\Phi: \mathcal{X} \rightarrow P^3 \times M$ over M . Let \mathcal{O}_X and \mathcal{O}_{P^3} denote the sheaves of germs of holomorphic vector fields on X and P^3 respectively, and let \mathcal{S}_{X/P^3} denote the cokernel

of the canonical homomorphism $F: \mathbb{O}_X \rightarrow f^* \mathbb{O}_{P^3}$. Then we have the characteristic map

$$\tau: T_0(M) \longrightarrow H^0(X, \mathcal{I}_{X/P^3})$$

(see [3], §1).

Lemma 4. There is a canonical isomorphism

$$f: \mathcal{N}_S^0 \longrightarrow f_* \mathcal{I}_{X/P^3}$$

which induces an isomorphism

$$f: H^0(S, \mathcal{N}_S^0) \longrightarrow H^0(X, \mathcal{I}_{X/P^3})$$

such that $-\tau = f \circ \sigma$.

We have two exact sequences

$$0 \rightarrow \mathbb{O}_S \rightarrow \mathbb{O}_{P^3|_S} \rightarrow \mathcal{N}_S^0 \rightarrow 0,$$

$$0 \rightarrow f_* \mathbb{O}_X \rightarrow f_* f^* \mathbb{O}_{P^3} \rightarrow f_* \mathcal{I}_{X/P^3} \rightarrow 0.$$

Moreover, there exists a canonical homomorphism

$$f^*: \mathbb{O}_{P^3|_S} \longrightarrow f_* f^* \mathbb{O}_{P^3}.$$

One can easily see that f^* induces a desired isomorphism.

3. Vanishing of obstructions.

Lemma 5. The coboundary map

$$\delta: H^0(X, \mathcal{I}_{X/P^3}) \longrightarrow H^1(X, \mathbb{O}_X)$$

is surjective.

Proof. Let $\rho \in H^1(X, \mathcal{O}_X)$. Then ρ corresponds to a deformation X_ρ of X over $I = \text{Spec } \mathbb{C}[t]/(t^2)$.

By Lemma 1, we have $K = (n-s-4)[\tilde{E}]$. It follows that the line bundle $[\tilde{E}]$ extends to a line bundle on X_ρ . Then, by Lemma 2, the holomorphic map f extends to a holomorphic map $X_\rho \rightarrow P^3 \times I$ over I . This means that δ is surjective.

By the result of Kodaira cited in Introduction, and by Lemma 4, we obtain a family $\mathcal{X}_1 = \{X_t\}_{t \in M_1}$ of deformations of $X = X_0$ such that

$$\tau: T_0(M_1) \longrightarrow H^0(X, \mathcal{S}_{X/P^3})$$

is surjective. By Lemma 5 and [3], Proposition 1.4, the infinitesimal deformation map

$$\rho: T_0(M_1) \longrightarrow H^1(X, \mathcal{O}_X)$$

is surjective.

This implies the existence of an effectively parametrized complete family of deformations of X and the equality $m(X) = \dim H^1(X, \mathcal{O}_X)$.

On the other hand, we have

$$\dim H^0(X, f^* \mathcal{O}_{P^3}) = 15 + 4 \delta_{r,s+1}$$

by Lemmas 2 and 3. Finally it follows that

$$\dim H^1(X, \mathcal{O}_X) = \mu(S) - 15 - 4 \delta_{r, s+1}$$

by Lemma 5.

References

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