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On solutions of initial-boundary problem for $u_t = u_{xx} + \frac{1}{1-u}$

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§1. Introduction and Theorem

Various works[1),2),3)] have been published on the blowing-up of solutions of the Cauchy problem and the initial-boundary value problem of nonlinear partial differential equations. Blowing-up means that the solutions of these problems become infinite in a finite time.

The objective of the present paper is to introduce the concept of quenching which has more general sense than blowing-up and to find some sufficient conditions for quenching of the solutions of the following initial-boundary value problem for $u = u(t,x)$, $t>0$, $x \in (0,\ell)$,

(1.1a) $\quad u_t = u_{xx} + \frac{1}{1-u}$, $\quad t>0, \quad x \in (0,\ell)$,

(1.1b) $\quad u(t,0) = u(t,\ell) = 0$, $\quad t>0$,

(1.1c) $\quad u(0,x) = 0$, $\quad x \in (0,\ell)$, where ℓ is a positive constant. The above initial-boundary value problem (1.1a$\sim$c) is denoted by IVP. Our study may be said to be more illustrative than general, since we restrict ourselves to one-space-dimensional mixed problems of semilinear heat equations. Nevertheless, we hope that our results will give an insight into a more general situation. The nonlinear perturbation $\frac{1}{1-u}$ $(u \neq 1)$ in (1.1a) is a locally Lipshitz continuous. Thus IVP has a unique solution which may be local in t .

The present problems came to our attention in connection with the diffusion equation generated by a polarization phenomena in ionic conductors[4)].

We shall define quenching for the solutions of the initial value problems.

Definition 1. Let $u = u(t,x)$ be the solution of the initial value problems which are defined in $t>0$, $x \in \Omega$. Ω means R^m which stands for the m-dimensional Euclidean space or the bounded domain in R^m .

We shall say that u quenches if $\| u_t \|_C$ becomes infinite in a finite time where $\| \cdot \|_C$ denotes the maximum norm over Ω .

In order to clarify the nature of quenching, let us take some examples.

Example 2. α being constant, the solution of the initial value problem for $u = u(t)$, $t>0$,

$$\begin{cases} \frac{du}{dt} = \frac{1}{1-u} , & t>0 \\ u(0) = \alpha , & \end{cases}$$

is $u = 1 + \sqrt{(1-\alpha)^2 - 2t}$, if $\alpha>1$ and $u = 1 - \sqrt{(1-\alpha)^2 - 2t}$, if $\alpha<1$. In both cases, we see quenching at $t = \frac{(1-\alpha)^2}{2}$.

Example 3. Let α be as above. The solution of the initial-boundary value problem for $u = u(t,x)$, $t>0$, $x\in(0,\ell)$,

$$\begin{cases} u_t = u_{xx} + \frac{1}{1-u} , & t>0 , \quad x\in(0,\ell) , \\ u_x(t,0) = u_x(t,\ell) = 0 , & t>0 \\ u(0,x) = \alpha , & x\in(0,\ell) \end{cases}$$

is the same as above.

Example 4. Blowing-up in the initial value problems means quenching. As our main result, we have

Theorem. In the IVP, suppose $\ell>2\sqrt{2}$. Then the solution of the IVP quenches.

The present paper has two sections apart from this section. In §2, we shall give a Lemma. §3 is devoted to the proof of our Theorem.

§2. Lemma

As a preparation for the proof of Theorem we state the following lemma. Henceforce, let $u = u(t,x)$ be the solution of IVP.

Lemma. In the IVP, suppose $\ell > 2\sqrt{2}$. Then u reaches 1 in a finite time at $x = \frac{\ell}{2}$.

Proof:

1st Step. We show that $u(t,x)$ is increasing in t for every x in $(0,\ell)$ as long as u exists. In fact, putting $v = u_t$, we have

$$(2.1) \qquad v_t = v_{xx} + \frac{1}{(1-u)^2} \cdot v , \quad x \in (0,\ell),$$

$$v(t,0) = v(t,\ell) = 0 ,$$

and

$$v(0,x) = 1 , \quad x \in (0,\ell) \text{ as long as } u \text{ exists.}$$

We notice that v is a solution of the linear parabolic equation (2.1) and is non-negative on the "parabolic boundary". Thus v is non-negative everywhere, which implies the required monotonicity of u .

2nd Step. The solution $u_1 = u_1(t,x)$ of the initial-boundary value problem for $u = u(t,x)$,

$$\begin{cases} u_t = u_{xx} + 1 , & t>0 , \quad x\in(0,\ell) , \\ u(t,0) = u(t,\ell) = 0 , & t>0 , \\ u(0,x) = 0 , & x\in(0,\ell) \end{cases}$$

converges its stationary solution $\psi(x) = \frac{1}{2}\ell(\ell-x)$ $(0<x<\ell)$ as $t \to +\infty$. Thus u_1 crosses 1 in a finite time if $\ell>2\sqrt{2}$.

Suppose that u does not reach 1 in a finite time if $\ell>2\sqrt{2}$. Then IVP has a global solution, i.e., u satisfies $0\leqq u\leqq 1$ in $(0,\ell)\times[0,+\infty)$ by virtue of the monotonicity of u . Comparing u with u_1 , we get $u \geqq u_1$ in $(0,\ell) \times [0,+\infty)$ since $\frac{1}{1-\lambda} \geqq 1$ in $0\leqq\lambda\leqq 1$. This contradicts the assumption. We shall denote the time when u reaches 1 by $t = T_0$.

3rd Step. u satisfies (i) $u_x(t,0) > 0$ by virtue of positivity of u ; (ii) $u_x(t,\frac{\ell}{2}) = 0$ since u is an even function with respect to $x = \frac{\ell}{2}$. Putting $\pi = u_x$, we have

$$\pi_t = \pi_{xx} + \frac{1}{(1-u)^2}\cdot\pi , \quad t\in[0,T_0) , \quad x\in(0,\tfrac{\ell}{2}) ,$$

$$\pi(t,0) > 0 , \quad \pi(t,\tfrac{\ell}{2}) = 0 , \quad t\in[0,T_0) ,$$

and

$$\pi(0,x) = 0 \ , \quad x\in(0,\tfrac{\ell}{2}) \ .$$

Repeating the same argument as in 1st Step, we see that

$$(2.2) \qquad \pi = u_x(t,x) > 0, \quad t\in[0,T_0) \ , \quad x\in(0,\tfrac{\ell}{2}) \ .$$

Combining (2.2) and (ii), we get that u takes its maximum at $x = \frac{\ell}{2}$ for any $t\in[0,T_0)$. This completes the proof.

§3. Proof of Theorem

1st Step.

1.a) Put $\mu = \mu(t) = u(t,\frac{\ell}{2})$ in $[0,T_0)$. μ satisfies

$$(3.1) \qquad \frac{d\mu}{dt} \leqq \frac{1}{1-\mu} \quad \text{in } [T_0-\varepsilon,T_0)$$

for sufficiently small $\varepsilon(>0)$ since $u_{xx}(t,\frac{\ell}{2}) \leqq 0$ in $[0,T_0)$. Put $T_1 = T_0- \varepsilon$ and $\Omega_\varepsilon = (0,\ell) \times [T_1,T_0)$. Comparing $\mu(t)$ with $\nu = \nu(t) = 1 - \sqrt{2}\sqrt{T_0-t}$ in $[T_1,T_0)$, we get

$$(3.2) \qquad \mu \geqq \nu \ , \quad \text{in } [T_1,T_0)$$

since ν satisfies (see Example 2)

$$\frac{d\nu}{dt} = \frac{1}{1-\nu} \ , \quad t\in[T_1,T_0)$$

and

$$\lim_{t\to T_0} \nu(t) = 1 \ .$$

(3.2) implies that there exists the domain D_ε in which u satisfies

$u(t,x) \geqq \nu(t)$.

Denote the compliment of D_ε by E_ε and put $E_\varepsilon^{(1)}=E_\varepsilon \cap \{(0,\frac{\ell}{2}) \times [T_1,T_0)\}$ and $E_\varepsilon^{(2)}=E_\varepsilon \cap \{(\frac{\ell}{2},\ell) \times [T_1,T_0)\}$.

For D_ε , there may be two cases:

Case (a) D_ε has no interior points; i.e., there holds

$$u_{xx}(t,\tfrac{\ell}{2})=0 \quad \text{in } [T_1,T_0).$$

Case (b) D_ε has interior points.

For the case (a), u quenches obviously. Henceforce we consider only the case (b).

1.b) Denote the boundary between D_ε and $E_\varepsilon^{(i)}$ by $x=s^{(i)}(t)$ $(t\in[T_1,T_0))$ for i=1,2. Then $x=s^{(i)}(t)$ satisfies

(i) $\lim_{t\to T_0} s^{(i)}(t) = \frac{\ell}{2}$;

(ii) $u_x(t,s^{(i)}(t))\cdot\dot{s}^{(i)}(t) = -u_{xx}(t,s^{(i)}(t))$, $t\in[T_1,T_0)$

where $\dot{s}^{(i)}(t)$ means $\frac{ds^{(i)}(t)}{dt}$ for i=1,2. In fact, there holds

(3.3) $u = \nu$ on $x = s^{(i)}(t)$, $t\in[T_1,T_0)$.

Differentiating both sides of (3.3) and using (3.3), we get

(3.4) $u_t(t,s^{(i)}(t)) + u_x(t,s^{(i)}(t))\cdot\dot{s}^{(i)}(t) = \dfrac{1}{1-u(t,s^{(i)}(t))}$.

By virtue of (1.1a) on $x = s^{(i)}(t)$ and (3.3) we have (ii).

1.c) Obviously we have the following inequalities

(3.5a) $\dfrac{1}{1-u} \geqq \dfrac{1}{\sqrt{2}\sqrt{T_0-t}}$ in D_ε , and

(3.5b) $\frac{1}{1-u} < \frac{1}{\sqrt{2}\sqrt{T_0-t}}$ in E_ε .

2nd Step.

2.a) Let $p = p(t,x)$ be $\frac{1}{2(T_0-t)}$ in D_ε and $\frac{1}{(1-u)^2}$ in E_ε. Then the solution $v_1 = v_1(t,x)$ of the initial-boundary value problem for $v = v(t,x)$ in Ω_ε ,

$$\begin{cases} v_t = v_{xx} + p\cdot v & \text{in } \Omega_\varepsilon \\ v(t,0) = v(t,\ell) = 0, & t\in[T_1,T_0) , \\ v(T_1,x) = \beta(x) = u_t(T_1,x) , & x\in(0,\ell) , \end{cases}$$

exists and satisfies $v_1 \leqq v$ in Ω_ε by virtue of (3.5a).

2.b) Put $W = W(t,x) = \sqrt{T_0-t}\cdot v_1$. Denoting W in D_ε by $W^{(1)}$, we have $W^{(1)}_t = W^{(1)}_{xx}$ in D_ε .

3rd Step.

3.a) We shall deal with the following initial-boundary value problem for $V = V(t,x)$ in $(-\infty,+\infty) \times [T_1,T_0)$.

(3.6a) $V_t = V_{xx}$ in $(-\infty,+\infty) \times [T_1,T_0)$

(3.6b) $V = W^{(1)}$ in D_ε

(3.6c) $V = \sqrt{\varepsilon}\cdot\beta(x)$, $x\in[0,s^{(1)}(T_1)) \cup (s^{(2)}(T_1),\ell]$

(3.6d) $\quad V = 0 , \qquad x \in (-\infty, 0) \cup (\ell, +\infty)$.

In what follows we impose on the solution $V(t,x)$ the following conditions at infinity: $V(t,x)$ and $V_x(t,x)$ are bounded as $x \to \pm\infty$ uniformly with respect to t in $[T_1, T_0)$. We see the solution $\hat{W} = \hat{W}(t,x)$ of (3.6) uniquely exists. Uniqueness of $\hat{W}$ is shown by Holmgren's theorem. Using the Green's function

$$K(t,x;\tau,\xi) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\{-\frac{(x-\xi)^2}{4(t-\tau)}\} ,$$

$\hat{W}$ is represented by

$$\begin{aligned}
\hat{W}(t,x) = & \int_{T_1}^{t} [K(t,x;\tau,s^{(1)}(\tau))W_\xi^{(1)}(\tau,s^{(1)}(\tau)) \\
& - W^{(1)}(\tau,s^{(1)}(\tau))K_\xi(t,x;\tau,s^{(1)}(\tau))]d\tau \\
& + \int_0^{s^{(1)}(T_1)} K(t,x;T_1,\xi)\sqrt{\varepsilon}\cdot\beta(\xi)d\xi \qquad (3.7) \\
& + \int_{T_1}^{t} K(t,0;\tau,s^{(1)}(\tau))W^{(1)}(\tau,s^{(1)}(\tau))\cdot\dot{s}^{(1)}(\tau)d\tau , \\
& -\infty < x < s^{(1)}(t) , \quad t\in[T_1,T_0) .
\end{aligned}$$

Also in $s^{(2)}(t) < x < +\infty$, $t \in [T_1, T_0)$, we have the similar expression as (3.7).

3.b) Using the positivity of β , W and maximum principle, we have

$$\hat{W}(t,x) \geqq 0 \quad \text{in } (-\infty,+\infty) \times [T_1,T_0).$$

Thus from (3.6) and (3.5b) we see

$$\hat{W}(t,x) \geqq W(t,x) \qquad \text{in } \Omega_\varepsilon .$$

4th Step. We claim that

$$\lim_{t\to T_0} \hat{W}(t,\tfrac{\ell}{2}) > 0 .$$

On the contrary, we suppose that

$$\lim_{t\to T_0} \hat{W}(t,\tfrac{\ell}{2}) = 0 ,$$

which implies that $0 \equiv \hat{W}(t,x) \geqq W(t,x) \geqq 0$ in Ω_ε by the strong maximum principle[5]. This is a contradiction. Thus we get that

$$\lim_{t\to T_0} \frac{d\mu(t)}{dt} = \lim_{t\to T_0} v(t,x) \geqq \lim_{t\to T_0} v_1(t,\tfrac{\ell}{2}) = \lim_{t\to T_0} \frac{\hat{W}(t,\frac{\ell}{2})}{\sqrt{t-T_0}} = +\infty$$

This completes the proof.

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