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EXTREME VALUES OF AN IMPLICIT FUNCTION

NOZOMU MOCHIZUKI

Let $\varphi(x,y)$ be a C^2-function in a domain E of $\mathbb{R}^2$ and $\varphi_x(a,b) = \varphi_y(a,b) = 0$ for $(a,b) \in E$. Then, as is well known, the local behavior of $\varphi(x,y)$ is examined by the sigh of $\Delta_0(a,b) = \varphi_{xx}(a,b)\varphi_{yy}(a,b) - \varphi_{xy}(a,b)^2$; that is, if $\Delta_0(a,b) > 0$ $\varphi(a,b)$ is a local maximum or a local minimum according to $\varphi_{xx}(a,b) < 0$ or $\varphi_{xx}(a,b) > 0$.

Let now $f(x,y,z)$ be a C^2-function in a domain D of $\mathbb{R}^3$ and let $f(a,b,c) = 0$, $f_z(a,b,c) \neq 0$ for $(a,b,c) \in D$. Then a C^2-function $\varphi(x,y)$ is uniquely determined in a neighborhood of (a,b), for which $z = \varphi(x,y)$ if and only if $f(x,y,z) = 0$.

As for the local behavior of such $\varphi(x,y)$, we have the following simple criterion. Although straightforward computations are sufficient for the proof, we have not found such a result in the literature.

THEOREM 1. Let $f(x,y,z)$ and $\varphi(x,y)$ be as above, and let

$$f(a,b,c) = f_x(a,b,c) = f_y(a,b,c) = 0, \quad f_z(a,b,c) \neq 0.$$

Put $\Delta(x,y,z) = f_{xx}(x,y,z)f_{yy}(x,y,z) - f_{xy}(x,y,z)^2$.

(i) If $\Delta(a,b,c) > 0$, then $c = \varphi(a,b)$ is a local maximum or a local minimum of $\varphi(x,y)$ according to $\varepsilon f_{xx}(a,b,c) < 0$

or $\varepsilon f_{xx}(a,b,c) > 0$, where $\varepsilon = \operatorname{sgn}(-f_z(a,b,c))$.

(2) If $\Delta(a,b,c) < 0$, then $c = \varphi(a,b)$ is not an extreme value of $\varphi(x,y)$.

PROOF. From $f_x + f_z \varphi_x = 0$ follows that $f_{xx} + f_z \varphi_{xx} = 0$, hence

$$\varphi_{xx}(a,b) = -\frac{f_{xx}(a,b,c)}{f_z(a,b,c)} .$$

Similarly, we have

$$\varphi_{yy}(a,b,c) = -\frac{f_{yy}(a,b,c)}{f_z(a,b,c)}, \quad \varphi_{xy}(a,b,c) = -\frac{f_{xy}(a,b,c)}{f_z(a,b,c)} .$$

Thus we have

$$\varphi_{xx}(a,b)\varphi_{yy}(a,b) - \varphi_{xy}(a,b)^2 = \frac{\Delta(a,b,c)}{f_z(a,b,c)^2},$$

which completes the proof.

We remark that the generalization to the case of n variables is immediate.

Let $f(x,y) = f(x_1,x_2,\ldots,x_n,y)$ be a C^2- function in a domain D of R^{n+1} such that $f(a,b) = 0$ for a point $(a,b) \in D$. Let $f_y(a,b) \neq 0$. Then there exists a C^2- function $\varphi(x) = \varphi(x_1,\ldots,x_n)$ in a suitable neighborhood of $a = (a_1,\ldots,a_n)$ in R^n such that $y = \varphi(x)$ if and only if $f(x,y) = 0$.

Let $f_{x_j}(a,b) = 0$, $j = 1,2,..,n$. We have then

$$\varphi_{x_j}(a) = 0; \quad \varphi_{x_i x_j}(a) = -\frac{f_{x_i x_j}(a,b)}{f_y(a,b)}, \quad 1 \leq i,j \leq n.$$

It is well known that $\varphi(a)$ is a local maximum if the matrix $A = \left[\varphi_{x_i x_j}(a)\right]$ is negative-definite and $\varphi(a)$ is a local minimum if A is positive-definite and, moreover, $\varphi(a)$ is not an extreme value if A is indefinite. We have thus

THEOREM 2. Let $f(x,y)$ be as above and let $f_{x_j}(a,b) = 0$, $j = 1,2,..,n$. Then

(1) $b = \varphi(a)$ is a local maximum or a local minimum according as the matrix $A = \left[\varepsilon f_{x_i x_j}(a,b)\right]$ is negative-definite or positive-definite , where $\varepsilon = \mathrm{sgn}(-f_y(a,b))$.

(2) $\varphi(x)$ does not admit an extreme value at $x = a$ if A is indefinite,

EXAMPLE. Theorem 1 is sometimes useful in computing extreme values of $\varphi(x,y)$. For example, let $\varphi(x,y) = (x+1)/\sqrt{x^2+y^2+1}$. Then, by setting $f(x,y,z) = (x+1)^2 - (x^2+y^2+1)z^2 = 0$, one can easily see that $\sqrt{2} = \varphi(1,0)$ is a local maximum of $\varphi(x,y)$.