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On the Fourier ultra-hyperfunctions, I

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This paper is a continuation of our previous work [7]. We study here the Fourier ultra-hyperfunctions representing them by means of holomorphic functions with some growth conditions. We will find a theory analogous to that of analytic functionals as discribed in [4].

§1. Definitions

Let V be a real Euclidean space of dimension n and E a compexification of V: $V = R^n$, $E = V \times iV \cong R^n \times iR^n = C^n$, $i = \sqrt{-1}$. Put $V' = \mathrm{Hom}_R(V, R)$ and $E' = \mathrm{Hom}_C(E, C)$, then E' is a complexification of V' : $E' = V' \times iV'$. We are going to denote generally the point of E by $z = (x, y) = x + iy$ and the point of E' by $\zeta = (\xi, \eta) = \xi + i\eta$. We denote the canonical inner product of $V \times V'$ and $E \times E'$ by the same notation $\langle , \rangle$ so that we have

$\langle x + iy, \xi + i\eta\rangle = \langle x, \xi\rangle - \langle y, \eta\rangle + i\{\langle y, \xi\rangle + \langle x, \eta\rangle\}$.

For a bounded set K' of V', we put

$h_{K'}(x) = \sup\{\langle x, \eta\rangle ; \eta \in K'\}$

and call it the indicator function of K'. For two convex compact sets K' and L' of V', the inclusion $K' \subset L'$ is equivalent to the relation " $h_{K'}(x) \leq h_{L'}(x)$ for all $x \in V$". If $V = R^n$, we identify V' with R^n by the inner product $\langle x, \xi\rangle = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$. If $K' = [-k', k']^n$

$= \left\{ x \in R^n ;\ |x_j| \leqslant k' \text{ for any } j = 1, 2, \ldots, n \right\}$, we have clearly $h_{K'}(x) = k'(|x_1| + \ldots + |x_n|) = k'|x|$, where we put $|x| = |x_1| + \ldots + |x_n|$.

Notation. Γ, Δ represent convex closed sets of V.

K, L (resp. K' and L') represent convex compact sets in V (resp. V').

Definition 1. Suppose Γ, K and K' have non-empty interiors $\mathring{\Gamma}$, $\mathring{K}$ and $\mathring{K}'$. Then we denote by $Q_b(\Gamma \times iK; K')$ — the space of all continuous functions f on $\Gamma \times iK$ which are holomorphic in the interior $\mathring{\Gamma} \times i\mathring{K}$ and satisfy the estimate:

$$\sup \left\{ \exp(h_{K'}(x)) |f(z)| ;\ z \in \Gamma \times iK \right\} < \infty, \quad x = \operatorname{Re} z. \tag{1}$$

It is clear that the space $Q_b(\Gamma \times iK'\ K')$ endowed with the norm (1) is a Banach space. If $\Delta \times iL \supset \Gamma \times iK$ and $L' \supset K'$, the restriction

$$Q_b(\Delta \times iL;\ L') \longrightarrow Q_b(\Gamma \times iK;\ K') \tag{2}$$

is continuous.

Definition 2. For general Γ, K and K', we put

$$Q(\Gamma \times iK;\ K') = \lim_{\substack{\Delta \times iL \Supset \Gamma \times iK \\ L' \Supset K'}} \operatorname{ind} Q_b(\Delta \times iL;\ L'), \tag{3}$$

where $A \Supset B$ means that A contains a neighborhood of B and the inductive limit is taken following the mappings (2).

Proposition 1. The space $Q(\Gamma \times iK;\ K')$ endowed with the locally convex inductive limit topology is a DFS space, namely the strong dual space of a Fréchet-Schwartz space.

In fact, if $\Delta \times iL \Supset \Gamma \times iK$ and $L' \Supset K'$, then the mapping (2) is compact. Hence, トル ~~by the definition~~ $Q(\Gamma \times iK;\ K')$ is a DFS space.

Suppose $V = R^n$ and put

$$B_\varepsilon = \left\{ x \in R^n;\ |x_j| < \varepsilon \text{ for any } j = 1, 2, \dots, n \right\}. \tag{4}$$

For Γ and K we put $\Gamma_\varepsilon = \Gamma + B_\varepsilon$, $K_\varepsilon = K + B_\varepsilon$. We will denote by $\mathcal{O}$ the sheaf of germs of holomorphic functions on E and $\mathcal{O}(\Omega)$ the space of all holomorphic functions on an open set Ω of E. A function $f \in Q(\Gamma \times iK; K')$ can be characterized as follows: There exist $\varepsilon_0 > 0$ and $\varepsilon_0' > 0$ such that $f \in \mathcal{O}(\Gamma_{\varepsilon_0} \times iK_{\varepsilon_0})$ and that

$$\sup \left\{ \exp(h_{K'}(x) + \varepsilon_0' |x|)\, |f(z)|;\ z \in \Gamma_{\varepsilon_0} \times iK_{\varepsilon_0} \right\} < \infty. \tag{5}$$

<u>Definition 3</u> For an open convex set ω of V and an open convex set ω' of V' we put

$$Q(\Gamma \times i\omega; K') = \lim_{\substack{\longleftarrow \\ K \subset\subset \omega}} \text{proj } Q(\Gamma \times iK; K'), \tag{6}$$

$$Q(\Gamma \times iK; \omega') = \lim_{\substack{\longleftarrow \\ K' \subset\subset \omega'}} \text{proj } Q(\Gamma \times iK; K'), \tag{7}$$

$$Q(\Gamma \times i\omega; \omega') = \lim_{\substack{K \subset\subset \omega \\ K' \subset\subset \omega'}} \text{proj } Q(\Gamma \times iK; K'), \tag{8}$$

where the projective limits are taken following the canonical mappings induced from the mappings (2).

In [7] we studied the space $Q(E) = Q(V \times iV; V')$. We established among others that the space Q(E) is a Frechet nuclear space and invariant under the Fourier transformation. A continuous linear functional on Q(E) is, by definition, a Fourier ultra-hyperfunction. On the other hand, Kawai [2] studied the space $Q(V \times i\omega; 0)$ which he denoted by $\tilde{\mathcal{O}}(D^n \times i\omega)$ and its dual space in his study on the Fourier hyperfunctions of M. Sato.

In this paper, we study generally the space $Q(\Gamma \times iK; K')$ and its dual space $Q'(\Gamma \times iK; K')$. In Section 2, we improve a result of [7] concerning the Fourier invariance of the space $Q(E)$. Namely we prove the Fourier transformation gives a topological isomorphism of $Q(V \times iK; K')$ onto $Q(V' \times iK'; -K)$ (Theorem 1). By duality we can define the Fourier transformation $\mathfrak{F}_d$ of $Q'(V \times iK; K')$ onto $Q'(V' \times i(-K'); K)$. In Section 3, we introduce some new spaces of holomorphic functions, by which the dual space $Q'(V \times iK; K')$ can be described. In Sections 4 and 5 we restrict ourselves to the case where $\dim V = 1$. First we represent $Q'(V \times iK; K')$ as the quotient space of the space introduced in Section 3 (Theorem 2), then we study again the Fourier transformation of $Q'(V \times iK; K')$ and give it another definition. We may generalize the results of Sections 4 and 5 to the n-dimensional case. This will be the subject of the forthcoming paper.

§2. Fourier transformation of $Q(V \times iK; K')$.

We improve a result of [7] concerning the Fourier invariance of the space $Q(E) = Q(V \times iV; V')$.

If $0 \in K$ and $0 \in K'$, the restriction

$$Q(V \times iK; K') \ni f \longmapsto f|_V \in \mathcal{S}(V) \tag{9}$$

is a continuous injection, where $\mathcal{S}(V)$ denotes the space of all rapidly decreasing C^∞ functions on V. In this case we may consider the space $Q(V \times iK; K')$ as a subspace of $\mathcal{S}(V)$ by the injection (9). We recall the Fourier transform $\mathfrak{F}f = \tilde{f}$ of $f \in \mathcal{S}(V)$ is defined as follows:

$$\tilde{f}(\xi) = \int_V \cdots \int f(x) \exp(-i\langle x, \xi \rangle)\, dx_1 \ldots dx_n. \tag{10}$$

It is well known the Fourier transformation $\mathfrak{F}$ is a topological isomrphism:

$$\mathfrak{F} : \mathcal{S}(V) \xrightarrow{\sim} \mathcal{S}(V'). \tag{11}$$

The inverse Fourier transformation $\overline{\mathfrak{F}}$ is given by

$$\overline{\mathfrak{F}}\varphi(x) = (2\pi)^{-n} \int\!\cdots\!\int_{V'} \varphi(\xi)\exp(i\langle x,\xi\rangle)\, d\xi_1\ldots d\xi_n \tag{12}$$

for $\varphi \in \mathcal{S}(V')$.

Let $f \in Q(V \times iK;\, K')$ be given. There exist positive numbers ε_0 and ε_0' such that $f \in \mathcal{O}(V \times iK_{\varepsilon_0})$ and that

$$\sup\left\{\exp(h_{K'}(x) + \varepsilon_0'|x|)|f(z)|\ ;\ z \in V \times iK_{\varepsilon_0}\right\} < \infty. \tag{13}$$

Therefore, for any $y \in K_{\varepsilon_0}$ we have

$$\int\!\cdots\!\int_V f(x+iy)\exp(-i\langle x,\zeta\rangle)\, dx_1\ldots dx_n$$
$$\leq C\int\!\cdots\!\int_V \exp(-h_{K'}(x) - \varepsilon_0'|x| + \langle x,\eta\rangle)\, dx_1\ldots dx_n.$$

Hence, if $\langle x,\eta\rangle < h_{K'}(x) + \varepsilon'|x|$ for all $x \in V$ $(0 < \varepsilon' < \varepsilon_0')$, the integral

$$\tilde{f}(\zeta;\, y) = \int\!\cdots\!\int_V f(x+iy)\exp(-i\langle x,\zeta\rangle)\, dx_1\ldots dx_n \tag{14}$$

converges absolutely and uniformly in ζ. As ε' may be chosen arbitrarily close to ε_0', $\tilde{f}(\zeta;\, y)$ is holomorphic in $V' \times iK'_{\varepsilon_0'}$. It can be easily shown by the Cauchy integral theorem that $\tilde{f}(\zeta;\, y)$ is independent of $y \in K_{\varepsilon_0}$. We denote

$$\tilde{f}(\zeta) = \tilde{f}(\zeta;\, y), \qquad y \in K_{\varepsilon_0} \tag{15}$$

and call it the Fourier transformation of $f \in Q(V \times iK;\, K')$.
If $0 \in K$ and $0 \in K'$, this Fourier transformation is (nothing but) the restriction of the Fourier transformation of $\mathcal{S}(V)$. Therefore we use the same notation $\mathfrak{F}$ for the Fourier transformation of $Q(V \times iK;\, K')$.

<u>Theorem 1.</u> The Fourier transformation $\mathfrak{F}$ gives a topological isomorphism:

$$\mathfrak{F} : Q(V \times iK;\, K') \xrightarrow{\sim} Q(V' \times iK';\, -K). \tag{16}$$

Proof. Let $f \in Q(V \times iK; K')$. Suppose f satisfy (13). We estimate $|\tilde{f}(\zeta)|$. For $y \in K_{\varepsilon_0}$ we have

$$\tilde{f}(\zeta) = \tilde{f}(\zeta; y)$$
$$= \int \cdots \int_V f(x + iy) \exp(-i\langle x, \xi\rangle + \langle x, \eta\rangle)\, dx_1 \ldots dx_n \cdot$$
$$\cdot \exp(i\langle y, \eta\rangle + \langle y, \xi\rangle).$$

Hence we have for $y \in K_{\varepsilon_0}$ and $\eta \in K'_{\varepsilon'}$ $(0 < \varepsilon' < \varepsilon_0')$,

$$|\tilde{f}(\zeta)| \exp(-\langle y, \xi\rangle)$$
$$\leqslant C \int \cdots \int_V \exp(-h_{K'}(x) - \varepsilon_0'|x| + h_{K'}(x) + \varepsilon'|x|)\, dx_1 \ldots dx_n$$
$$= C \int \cdots \int_V \exp((\varepsilon' - \varepsilon_0')|x|)\, dx_1 \ldots dx_n < \infty .$$

This gives

$$\sup\left\{ \exp(h_K(-\xi) + \varepsilon_0|\xi|)|\tilde{f}(\zeta)| \;;\; \eta \in K'_{\varepsilon'} \right\} < \infty .$$

We have thus proved

$$\mathfrak{F}\, Q(V \times iK; K') \subset Q(V' \times iK'; -K). \tag{17}$$

As the transformation $f(z) \longmapsto f(-z)$ is a topological isomorphism of $Q(V \times iK; K')$ onto $Q(V \times i(-K); -K')$, we have

$$\bar{\mathfrak{F}}\, Q(V' \times iK'; -K) \subset Q(V \times iK; K'), \tag{18}$$

where, by definition, $\bar{\mathfrak{F}}\varphi(z) = (2\pi)^{-n}(\mathfrak{F}\varphi)(-z)$.

If $0 \in K$ and $0 \in K'$, the theorem results from the topological isomorphism (11) thanks to (17) and (18).

Consider the general case. If $y_0 \in K$, the transfation

$$T_{-iy_0} : \; f(z) \longmapsto f_{iy_0}(z) = f(z + iy_0)$$

gives a topological isomorphism:

$$Q(V \times iK; K') \longrightarrow Q(V \times i(K - y_0); K').$$

If $\eta_0 \in K'$, the multiplication $f(z) \longmapsto e^{\langle z, \eta_0\rangle} f(z)$ gives a topological isomorphism:

$$Q(V \times iK; K') \longrightarrow Q(V \times iK; (K' - \eta_0)).$$

Fix $y_0 \in K$ and $\eta_0 \in K'$. Then the Fourier transformation of $Q(V \times iK; K')$ decomposes as follows:

$$Q(V \times iK;\, K') \xrightarrow{T_{-iy_0}} Q(V \times i(K - y_0);\, K') \xrightarrow{e^{\langle z, \eta_0 \rangle}}$$
$$Q(V \times i(K - y_0)\,;\, K' - \eta_0) \xrightarrow{\mathfrak{F}} Q(V' \times i(K' - \eta_0);\, -(K - y_0))$$
$$\xrightarrow{T_{i\eta_0}} Q(V' \times iK';\, -K + y_0) \xrightarrow{e^{-\langle y_0, \zeta \rangle}} Q(V' \times iK';\, -K).$$

All arrows being topological isomorphisms, (16) is a topological isomorphism. q.e.d.

<u>Corollary.</u> For an open convex set ω of V and an open convex set ω' of V', the Fourier transformation $\mathfrak{F}$ gives a topological isomorphism:

$$\mathfrak{F}: Q(V \times i\,\omega\,;\, \omega') \xrightarrow{\sim} Q(V' \times i\,\omega';\, -\omega). \tag{19}$$

<u>Remark.</u> We established the isomorphism (19) in the case where $\omega = V$ and $\omega' = V'$ in [7]. Kawai [2] studied the isomorphism (16) for $\underset{\sim}{\mathcal{O}}(D^n) = Q(V \times i0;\, 0)$.

By duality we can define the Fourier transformation of $Q'(V \times iK;\, K')$ which we denote occasionally by $\mathfrak{F}_d$:

$$(\,\mathfrak{F}f,\, \mathfrak{F}_d \ell\,) = (f(-z),\, \ell_z), \tag{20}$$

for $f \in Q(V \times i(-K);\, -K')$ and $\ell \in Q'(V \times iK;\, K')$. $\mathfrak{F}_d$ gives a topological isomorphism:

$$\mathfrak{F}_d: Q'(V \times iK;\, K') \longrightarrow Q'(V' \times i(-K');\, K). \qquad (\cancel{20})\ 21$$

We will give another definition of the Fourier transformation of $Q'(V \times iK;\, K')$.

§3. Some new spaces of holomorphic functions.

Denote by $\overline{V}$ the spherical compactification of V: $\overline{V} = V \sqcup S^{n-1}$, where S^{n-1} is the sphere at the infinity. E is considered as a subset of V x iV. If a subset F of E is relatively compact in $\overline{V}$ x iV, F is said to be imaginary bounded.

<u>Definition 4.</u> For an imaginary bounded closed set in E and a convex compact set K' in V', we denote by $R_b(F;\ K')$ the space of all continuous functions f on F such that f is holomorphic in $\overset{\circ}{F}$ and that

$$\sup\left\{ |f(z)| \exp(-h_{K'}(x));\quad z \in F \cap V \right\} < \infty . \tag{21}$$

$R_b(F;\ K')$ endowed with the norm (21) is clearly a Banach space. If $F \subset G$ are two imaginary bounded closed sets of E and $K' \subset L'$, we have the following continuous mappings:

$$\begin{tikzcd} R_b(G;\ K') \arrow[r] \arrow[d] & R_b(G;\ L') \arrow[d] \\ R_b(F;\ K') \arrow[r] & R_b(F;\ L'). \end{tikzcd} \tag{22}$$

<u>Definition 5.</u> For an open set Ω of E we put

$$R(\Omega;\ K') = \varprojlim_{\substack{L' \supset\supset K' \\ F \subset\subset \Omega}} R_b(F;\ L'),$$

where F runs through the all imaginary bounded closed subsets of Ω and the projective limit is taken following the mappings (22).

<u>Proposition 2.</u> The space $R(\Omega;\ K')$ is an FS space.

In fact, if $F \subset\subset G$ and $K' \subset\subset L'$, the mappings in (22) are all compact.

We will see the dual space of $Q(\Gamma \times iK;\ K')$ can be represented by means of the spaces of type $R(\Omega;\ K')$. In the next section we study the case where V = R. The case where dim $V > 1$ will be treated in the forthcoming paper.

§4. Dual space of $Q(\Gamma \times iK;\ K')$. (one dimensional case)

In Sections 4 and 5 we assume V = R.

For a closed set D of C, we put $D_\varepsilon = D + \tilde{B}_\varepsilon$, where

$$\tilde{B}_\varepsilon = \left\{ z = x + iy \in C;\quad |x| < \varepsilon ,\ |y| < \varepsilon \right\}. \tag{23}$$

For a closed set D in C, the space $R(C \setminus D; K')$ coincides with the space of all holomorphic functions f on $C \setminus D$ such that, for any $\varepsilon > 0$ and $\varepsilon' > 0$ and for any compact set L in R,

$$\sup\{|f(z)|\exp(-h_{K'}(x) - \varepsilon'|x|);\ z \in (R \times iL) \cap (C \setminus D_\varepsilon)\} < \infty. \quad (24)$$

In the sequel, we assume for the simplicity $D = \Gamma \times iK$, where Γ is $\Gamma_\infty = R$, $\Gamma_+ = \{x \in R;\ x \geqslant 0\}$, $\Gamma_- = -\Gamma_+$ or $\Gamma_0 = \{0\}$, and $K = [k_1, k_2]$. We will denote $D_\infty = \Gamma_\infty \times iK$, $D_+ = \Gamma_+ \times iK$, $D_- = \Gamma_- \times iK$ and $D_0 = \Gamma_0 \times iK$. We will assume also $K' = [k_1', k_2']$, $L' = [\ell_1', \ell_2']$, etc.

Prepare a lemma.

Lemma 1. Let f be a holomorphic function in D_ε for some $\varepsilon > 0$ and satisfy the following two conditions:

(i) $\sup\{|f(z)|;\ z \in \partial D\} \leqslant M < \infty$;

(ii) $|f(z)| \leqslant C e^{c'|x|^n}$ on D with some integer n and $C \geqslant 0$ and $c' \geqslant 0$.

Then $|f|$ is bounded by M on D.

Proof. For $D = D_0$, Lemma is obvious by the maximum modulus principle. Suppose $D = D_+$. Put

$$w = r e^{i\theta} = \exp(\frac{\pi}{4k} z) = \exp(\frac{\pi}{4k} x)\exp(i\frac{\pi}{4k} y)$$

and

$$D' = \{w = r e^{i\theta}\ ;\ r \geqslant 1,\ |\theta| \leqslant \frac{\pi}{4}\}.$$

Then $f'(w) = f(z)$ is holomorphic in a neighborhood of D' and $|f'(w)|$ is bounded by M on $\partial D'$. On D', we have

$$|f'(w)| \leqslant C e^{c'|x|^n} = C\exp(c'(\frac{4k}{\pi}\log r)^n) = o(e^{\delta r^2})$$

for any $\delta > 0$. By the Phragmén-Lindelöf theorem, $|f'(w)|$ is bounded by M on D', which implies that $|f|$ is bounded by M on D. For $D = D_-$, the proof is similar. If $D = D_\infty$, applying the above argument two times we see that $|f|$ is bounded on D. Hence by a theorem of Lindelof, $|f|$ is bounded by M on D. q.e.d.

Corollary 1. The restriction mapping $R(C; K') \longrightarrow R(C \setminus D; K')$ being injective, $R(C; K')$ is considered as a subspace of $R(C \setminus D; K')$. Then $R(C; K')$ is a closed subspace of the space $R(C \setminus D; K')$.

Definition 6. We put

$$H^1_D(C; R(K')) = R(C \setminus D; K') / R(C; K'). \tag{25}$$

As a quotient space of an FS space by its closed subspace, the space $H^1_D(C; R(K'))$ is an FS space.

Lemma 2. If $K' \subset L'$, then the canonical mapping

$$H^1_D(C; R(K')) \longrightarrow H^1_D(C; R(L')) \tag{26}$$

is injective.

Proof. We have only to show that $f \in R(C \setminus D; K') \cap R(C; L')$ implies $f \in R(C; K')$. If $D = D_o$, $H^1_D(C; R(K'))$ is equal to $H^1_D(C; \mathcal{O})$. Hence it is independent of K'. Suppose $D = D_+$. If $f \in R(C \setminus D; K') \cap R(C; L')$, then for any $\varepsilon > 0$ and $\varepsilon' > 0$, $|e^{-(k' + \varepsilon')z} f(z)|$ is bounded on ∂D_ε and of exponential type on D_ε. Therefore, by Lemma 1, it is bounded on D_ε. The cases $D = D_-$ and $D = D_\infty$ can be treated similarly.

q.e.d.

Definition 7. We put

$$H^1_D(C; \tilde{R}(K')) = \lim_{\substack{\longleftarrow \\ L' \gg K'}} \text{proj}\ H^1_D(C; R(L')). \tag{27}$$

We will denote by $[\varphi_{\varepsilon'}]$ the element of $H^1_D(C; \tilde{R}(K'))$ defined by a system of functions $\varphi_{\varepsilon'} \in R(C \setminus D; \overline{K'_{\varepsilon'}})$, $\varepsilon' > 0$, such that for any pair $\varepsilon_1' < \varepsilon'$,

$$\varphi_{\varepsilon'} - \varphi_{\varepsilon_1'} \in R(C; \overline{K'_{\varepsilon_1'}}).$$

$\varphi_{\varepsilon'}$ will be said a representative of [$\varphi_{\varepsilon'}$] belonging to $R(C \setminus D;\ \overline{K'_{\varepsilon'}})$. Remark we have by Lemma 2,

$$H^1_D(C;\ R(K')) \subset H^1_D(C;\ \tilde{R}(K')).$$

An element of $H^1_D(C;\ \tilde{R}(K'))$ belongs to the subspace $H^1_D(C;\ R(K'))$ if and only if we can choose its representatives $\varphi_{\varepsilon'} \in R(C \setminus D;\ \overline{K'_{\varepsilon'}})$ such that for any $\varepsilon_1' < \varepsilon'$

$$\varphi_{\varepsilon_1'}(z) = \varphi_{\varepsilon'}(z).$$

<u>Proposition 3</u>. $H^1_D(C;\ \tilde{R}(K'))$ is an FS space.

<u>Proof</u>. Put $T_n = \{z = x + iy;\ |y| \leq n\}$, $F_n = T_n \cap (C \setminus \overline{D_{1/n}})$. Put further $X_n = R_b(F_n;\ \overline{K'_{1/n}})$ and $Y_n = R_b(T_n;\ \overline{K'_{1/n}})$. Y_n is a closed subspace of a Banach space X_n. X_n and Y_n ~~are~~ form projective systems of Banach spaces and we have

$$R(C \setminus D;\ K') = \lim\ \text{prj}\ X_n,$$
$$R(C;\ K') = \lim\ \text{proj}\ Y_n \quad \text{and}$$
$$H^1_D(C;\ \tilde{R}(K')) = \lim\ \text{proj}\ X_n/Y_n.$$

The mapping $X_{n+1} \longrightarrow X_n$ and $Y_{n+1} \longrightarrow Y_n$ being compact, $X_{n+1}/Y_{n+1} \longrightarrow X_n/Y_n$ is compact. Hence $H^1_D(C;\ \tilde{R}(K'))$ is an FS space.

<u>Remark</u>. If $R(C \setminus D;\ K')$ is dense in any of X_n and if Y_n is the completion of $R(C;\ K')$ in the topology of X_n, then $H^1_D(C;\ \tilde{R}(K'))$ is equal to $H^1_D(C;\ R(K'))$. We see, by a different way, that $H^1_D(C;\ \tilde{R}(K')) = H^1_D(C;\ R(K'))$ for $D = R \times iK$.

With these terminologies, we can state our theorem.

<u>Theorem 2.</u> Put $D = \Gamma \times iK$. The dual space $Q'(D;\ K')$ of $Q(D;\ K')$ is topologically isomorphic to the space $H^1_D(C;\ \tilde{R}(K'))$. The duality is given by the following inner product:

$$\langle f,\ [\varphi] \rangle = -\int_{\partial D_\varepsilon} f(z)\, \varphi_{\varepsilon'}(z)\, dz \tag{29}$$

for $f \in Q(D;\ K')$ and $[\varphi] \in H^1_D(C;\ \tilde{R}(K'))$. In the right hand of (29) $\varepsilon > 0$, $\varepsilon' > 0$ and $\varphi_{\varepsilon'}$ is chosen as follows: for $f \in Q(D; K')$ we can find $\varepsilon_0 > 0$ and $\varepsilon_0' > 0$ such that $f \in \mathcal{O}(D_{\varepsilon_0})$ and the estimate (5) is satisfied. Choose ε and ε' such that $0 < \varepsilon < \varepsilon_0$ and $0 < \varepsilon' < \varepsilon_0'$. $\varphi_{\varepsilon'} \in R(C \setminus D; \overline{K'_{\varepsilon'}})$ is a representative of $[\varphi]$.

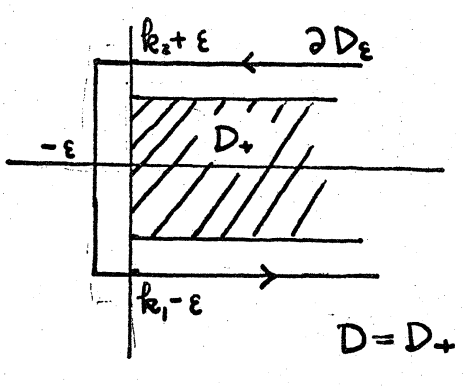
Remark. If $D = D_0 = \{0\} \times i[k_1, k_2]$, then the theorem reduces to the special case of the well known duality of $\mathcal{O}(D_0)$ and $H^1_{D_0}(C; \mathcal{O}) = \mathcal{O}(C \setminus D_0)/\mathcal{O}(C)$.

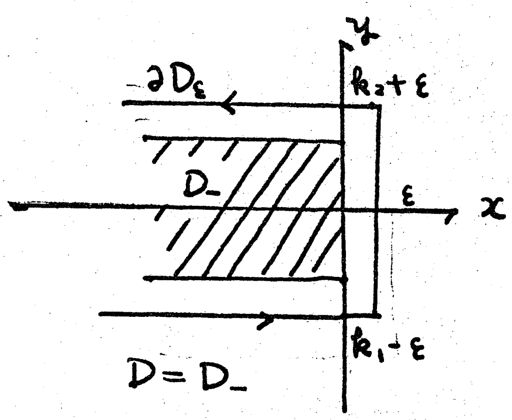
Proof. (i) Suppose first Γ is a propoerly convex closed cone, i.e. $\Gamma = \Gamma_+$ or Γ_-. If $f \in Q(D; K')$, we can find $\varepsilon_0 > 0$ and $\varepsilon_0' > 0$ such that $f \in \mathcal{O}(D_{\varepsilon_0})$ and that the estimate (5) is satisfied. Fix arbitrarily $0 < \varepsilon < \varepsilon_0$ and $0 < \varepsilon' < \varepsilon_0'$. By the Cauchy integral formula we have for $z \in D$

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{f(w)\, e^{(k_2' + \varepsilon')(w - z)}}{w - z}\, dw \quad \text{for } D = D_+ \qquad (30)$$

and

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{f(w)\, e^{(k_1' - \varepsilon')(w - z)}}{w - z}\, dw \quad \text{for } D = D_- \qquad (30')$$





Let $\ell \in Q'(D; K')$ be given. Then ℓ is continuous on $Q_b(\overline{D_\varepsilon}, \overline{K'_{\varepsilon'}})$ for any $\varepsilon > 0$ and $\varepsilon' > 0$. Therefore, if $0 < \varepsilon \leq \varepsilon_0$, $0 < \varepsilon' < \varepsilon_0'$ and $w \in C \setminus \overline{D_\varepsilon}$, we have

$$
\begin{aligned}
&\langle f(z),\, \ell_z\rangle \\
&= \left\langle \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{f(w)\,e^{(k_2'+\varepsilon')(w-z)}}{w-z}\,dw,\ \ell_z\right\rangle \\
&= -\int_{\partial D_\varepsilon} f(w)\ \check{\ell}_{\varepsilon'}(w)\,dw \qquad \text{for } D = D_+ \qquad (31)
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle f(z),\, \ell_z\rangle \\
&= \left\langle \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{f(w)\,e^{(k_1'-\varepsilon')(w-z)}}{w-z}\,dw,\ \ell_z\right\rangle \\
&= -\int_{\partial D_\varepsilon} f(w)\ \check{\ell}_{\varepsilon'}(w)\,dw \qquad \text{for } D = D_-, \qquad (31')
\end{aligned}
$$

where we put

$$\check{\ell}_{\varepsilon'}(w) = \frac{1}{2\pi i}\left\langle \frac{e^{-(k_2'+\varepsilon')(z-w)}}{z-w},\ \ell_z\right\rangle \qquad \text{for } D = D_+ \qquad (32)$$

and

$$\check{\ell}_{\varepsilon'}(w) = \frac{1}{2\pi i}\left\langle \frac{e^{-(k_1'-\varepsilon')(z-w)}}{z-w},\ \ell_z\right\rangle \qquad \text{for } D = D_-. \qquad (32')$$

$\check{\ell}_{\varepsilon'}$ is well defined for any $\varepsilon' > 0$, because the function

$$z \longmapsto \frac{e^{-(k_2'+\varepsilon')z}}{z-w}$$

belongs to $Q(D_+;K')$ for any fixed $w \in \mathbb{C}\setminus D_+$ and the function

$$z \longmapsto \frac{e^{-(k_1'-\varepsilon')z}}{z-w}$$

belongs to $Q(D_-;K')$ for any fixed $w \in \mathbb{C}\setminus D_-$.

Now estimate $|\check{\ell}_{\varepsilon'}(w)|$. By the continuity of ℓ, for any $\varepsilon > 0$ and $\varepsilon' > 0$, we can find $C \geq 0$ such that for any $f \in Q_b(\overline{D_\varepsilon}\ ;\ \overline{K'_{\varepsilon'}})$

$$|\langle f, \ell\rangle| \leq C \sup\left\{|f(z)|\, e^{(k_2'+\varepsilon')x}\ ;\ z \in D_\varepsilon\right\} \qquad \text{for } D = D_+ \qquad (34)$$

and

$$|\langle f, \ell\rangle| \leq C \sup\left\{|f(z)|\, e^{(k_1'-\varepsilon')x}\ ;\ z \in D_\varepsilon\right\} \qquad \text{for } D = D_-. \qquad (34')$$

Suppose $D = D_+$. For $w \in \mathbb{C}\setminus \overline{D_\varepsilon}$, we have

$$
\begin{aligned}
&\sup\left\{\frac{|e^{-(k_2'+\varepsilon')z}|\, e^{(k_2'+\varepsilon'/2)x}}{|z-w|}\ ;\ z \in D_{\varepsilon/2}\right\} \\
&\leq \sup\left\{\frac{\exp(-(\varepsilon'/2)x)}{|z-w|}\ ;\ x > -\varepsilon/2,\ k_1-\varepsilon < y < k_2+\varepsilon\right\} \\
&= C\,\mathrm{dist}(w, D_{\varepsilon/2})^{-1}
\end{aligned}
$$

with some constant C. Therefore, for any $\varepsilon > 0$ there exists a constant $C = C_{\varepsilon,\varepsilon'} > 0$ such that

$$|\check{\ell}_{\varepsilon'}(w)| \leq C_{\varepsilon,\varepsilon'}\, \mathrm{dist}(w, D)^{-1} e^{(k_2'+\varepsilon')u} \tag{35}$$

for any $w \in C \setminus D_{\varepsilon}$. In particular we get

$$\check{\ell}_{\varepsilon'}(w) \in R(C \setminus D;\ \overline{K'}_{\varepsilon'}). \tag{36}$$

For $D = D_-$, we can argue similarly and conclude (36).

We investigate now the ε'-dependency of $\check{\ell}_{\varepsilon'}$. We may suppose $D = D_+$ without loss of generality. Let $\varepsilon' > \varepsilon_1' > 0$. Put

$$F(w) = \check{\ell}_{\varepsilon'}(w) - \check{\ell}_{\varepsilon_1'}(w).$$

Then we have

$$\begin{aligned} F(w) &= \frac{1}{2\pi i} \left\langle \frac{e^{-(k_2'+\varepsilon')(z-w)} - e^{-(k_2'+\varepsilon_1')(z-w)}}{z-w},\ \ell_z \right\rangle \\ &= \frac{1}{2\pi i} \langle \exp(-(k_2'+\varepsilon_1')(z-w))\, g(z-w),\ \ell_z \rangle \end{aligned}$$

where $g(z) = (\exp(-(\varepsilon'-\varepsilon_1')z) - 1)/z$ is an entire function of z. We have clearly

$$\sup\{|g(z-w)|\ ;\ z \in D_{\varepsilon/2}\} \leq C\{\exp((\varepsilon' - \varepsilon_1')u) + 1\},$$

where C is a constant. Therefore $F(w)$ is an entire function of w and satisfies the following inequality for any $w \in C$.

$$\begin{aligned} |F(w)| &\leq C \exp((k_2' + \varepsilon_1')u) \sup\{\exp((k_2' + \varepsilon_1'/2)x)\,|\exp(-(k_2' + \varepsilon_1')z) \\ &\qquad g(z-w)|\ ;\ z \in D_{\varepsilon'/2}\} \\ &\leq C' \exp((k_2' + \varepsilon_1')u)\{\exp((\varepsilon' - \varepsilon_1')u) + 1\} \\ &= C'\{\exp((k_2' + \varepsilon')u) + \exp((k_2' + \varepsilon_1')u)\}. \end{aligned}$$

This means

$$\check{\ell}_{\varepsilon'}(w) - \check{\ell}_{\varepsilon_1'}(w) \in R(C;\ \overline{K'}_{\varepsilon_1'}). \tag{47}$$

Hence $\{\check{\ell}_{\varepsilon'}\}$ defines an element of $H^1_D(C;\ \tilde{R}(K'))$ which we denote by $[\check{\ell}]$. $[\check{\ell}]$ will be called the Cauchy transform of $\ell \in Q'(D; K')$. By the formula(22), the injectivity of the Cauchy transformation is clear.

Let us show its surjectivity. Let $[\varphi] \in H^1_D(C; \tilde{R}(K'))$. $\varphi_{\varepsilon'}$ denotes the representative of $[\varphi]$ belonging to $R(C \setminus D; \overline{K'_{\varepsilon'}})$. Then $\ell_{[\varphi]} : f \longmapsto \langle f, [\varphi]\rangle$ defines a continuous linear functional on $Q(D; K')$. By the definition, we have

$$\langle f, \ \varphi_{\varepsilon'} - (\check{\ell}_{\varphi})_{\varepsilon'} \rangle = 0 \quad \text{for any } f \in Q_b(\overline{D}_\varepsilon ; \overline{K'_{\varepsilon'}}). \tag{38}$$

Therefore we have only to show that $\psi_{\varepsilon'} = \varphi_{\varepsilon'} - (\check{\ell}_{[\varphi]})_{\varepsilon'}$ is an entire function and of exponential type in any horizontal band $R \times iL$ with compact base L. Fix $\varepsilon_1' > \varepsilon'$ and put

$$G(z) = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{e^{-(k_2'+\varepsilon_1')(\omega - z)}\psi_{\varepsilon'}(\omega)}{\omega - z}\, d\omega \tag{39}$$

for $z \in D_\varepsilon$. Let $\varepsilon_1 < \varepsilon$. If $z \in D_\varepsilon \setminus D_{\varepsilon_1}$ we have thanks to (38)

$$\begin{aligned} G(z) &= \frac{1}{2\pi i}\left\{ \int_{\partial D_\varepsilon - \partial D_{\varepsilon_1}} + \int_{\partial D_{\varepsilon_1}} \right\} \frac{e^{-(k_2'+\varepsilon_1')(\omega - z)}\psi_{\varepsilon'}(\omega)}{\omega - z}\, d\omega \\ &= \psi_{\varepsilon'}(z) + 0. \end{aligned}$$

Therefore $\psi_{\varepsilon'}$ can be extended to an entire function of exponential grouth.

(ii) Now we study the case $\Gamma = \Gamma_\infty = R$. Up to the end of this section we put

$$D = R \times iK, \quad D_+ = \Gamma_+ \times iK, \quad D_- = \Gamma_- \times iK, \quad D_0 = \Gamma_0 \times iK,$$
$$K = [k_1, k_2] \text{ and } K' = [k_1', k_2'].$$

<u>Lemma 3</u>. The following sequence is exact:

$$0 \longrightarrow Q(D; K') \xrightarrow{h_1} Q(D_+; K') \oplus Q(D_-; K') \xrightarrow{h_2} Q(D_0; K') \longrightarrow 0, \tag{40}$$

where, for $f \in Q(D; K')$, $h_1(f) = (f_1, f_2)$, f_1 (resp. f_2) being the restriction of f to D_+ (resp. to D_-); for $(f_1, f_2) \in Q(D_+; K') \oplus Q(D_-; K')$, $h_2(f_1, f_2) = f_1 - f_2$.

<u>Proof</u>. We abbreviate the sequence (40) as follows:

$$0 \longrightarrow Q \longrightarrow Q_+ \oplus Q_- \longrightarrow Q_0 \longrightarrow 0. \tag{40'}$$

The exactness at Q results from the unique continuation property of holomorphic functions. The exactness at the middle term is clear by the definition of the mappings. To show the exactness at Q_o, we first remark $Q_o = Q(D_o; K') = \mathcal{O}(D_o)$. Take any $f \in \mathcal{O}(D_o)$, then there exist bounded holomorphic functions $f_1' \in \mathcal{O}((D_+)_\varepsilon)$ and $f_2' \in \mathcal{O}((D_-)_\varepsilon)$ for some $\varepsilon > 0$ such that

$$\exp(z^2)\ f(z) = f_1'(z) - f_2'(z)$$

in a neighborhood of D_o. Putting

$$f_1(z) = \exp(-z^2)\ f_1'(z) \quad \text{and} \quad f_2(z) = \exp(-z^2)\ f_2'(z),$$

we have $f_1 \in Q_+$ and $f_2 \in Q_-$ such that $f = f_1 - f_2$. q.e.d.

The exact sequence of Lemma 3 is composed of DFS spaces. Therefore, taking the strong dual spaces, we get the exact sequence of FS spaces.

Corollary. The following sequence is exact:

$$0 \longleftarrow Q'(D;\ K') \xleftarrow{h_1'} Q'(D_+;\ K') \oplus Q'(D_-;\ K') \xleftarrow{h_2'} Q'(D_o;\ K') \longleftarrow 0. \quad (41)$$

We recall the definition of h_1' and h_2': for $\ell^o \in Q'(D_o;\ K')$ and $(f_1, f_2) \in Q_+ \oplus Q_-$, $h_2'(\ell^o)(f_1, f_2) = \ell^o(f_1 - f_2)$; for $(\ell^+, \ell^-) \in (Q_+)' \oplus (Q_-)'$ and $f \in Q(D;\ K')$, $h_1'(\ell^+, \ell^-)(f) = \ell^+(f) + \ell^-(f)$.

Now finish the proof of Theorem 2. By Corollary, for any $\ell \in Q'(D;\ K')$, there exist $\ell^+ \in Q'(D_+;\ K')$ and $\ell^- \in Q'(D_-;\ K')$ such that $\ell = \ell^+ + \ell^-$. For any $\varepsilon' > 0$, we put $\check{\ell}_{\varepsilon'} = \check{\ell}^+_{\varepsilon'} + \check{\ell}^-_{\varepsilon'}$. Then clearly $\check{\ell}_{\varepsilon'} \in R(C \setminus D;\ \overline{K'_{\varepsilon'}})$, and $\check{\ell}_{\varepsilon'}$ is determined uniquely by $\ell \in Q'(D;\ K')$. We have also

$$\langle f, \ell \rangle = \langle f, [\check{\ell}_{\varepsilon'}] \rangle. \quad (42)$$

Hence the mapping

$$Q'(D;\ K') \ni \ell \longmapsto [\check{\ell}] \in H^1_D(C;\ \tilde{R}(K')) \quad (43)$$

is injective. We call this mapping the Cauchy transformation.

Now we will prove the surjectivity of the Cauchy trnasformation (43). Suppose $\varphi \in R(C \setminus D; \overline{K'_{\varepsilon'}})$ satisfies

$\langle f, \varphi \rangle = 0$ for any $f \in Q(D; K')$.

Then put

$$G(z) = \frac{1}{2\pi i} \int_{\partial D_\varepsilon} \frac{e^{-(w^2 - z^2)} \varphi(w)}{w - z} dw, \quad z \in D_\varepsilon. \qquad (43)$$

The function G is holomorphic on D_ε and of order 2 on D_ε.

Let $\varepsilon > \varepsilon_1 > 0$ and fix $z \in D_\varepsilon \setminus D_{\varepsilon_1}$. Then

$$G(z) = \frac{1}{2\pi i} \left\{ \int_{\partial D_\varepsilon - \partial D_{\varepsilon_1}} + \int_{\partial D_{\varepsilon_1}} \right\} \frac{e^{-(w^2 - z^2)} \varphi(w)}{w - z} dw$$
$$= \varphi(z) + 0,$$

which implies φ can be analytically extended onto D_ε and define an entire function. As this entire function φ is of order finte in any horizontal bands with compact base, we can conclude that $\varphi \in R(C; \overline{K'_{\varepsilon'}})$ by Lemma 1. This implies the surjectivity of the Cauchy transformation as in the proof of the case (i). q.e.d.

§5. Fourier transformation of $Q'(\Gamma \times iK; K')$.

We restrict ourselves to the case of dimension 1. We proved in § 1 the (dual) Fourier transformation $\mathfrak{F}_d$ is a topological isomorphism of $Q'(R \times iK; K')$ onto $Q'(R \times i(-K'); K)$. We give here another definition of the Fourier transformation of $Q'(R \times iK; K')$.

Put $D = \Gamma \times iK$. Let $\ell \in Q'(D; K')$ be given.

$$\tilde{\ell}(\zeta) = \langle \exp(-iz\zeta), \ell_z \rangle$$

is defined for ζ such that the function $z \longmapsto \exp(-iz\zeta)$ belongs to $Q(D; K')$. The function $\tilde{\ell}$ of ζ is called the Fourier transformation of $\ell \in Q'(D; K')$. If $D = D_0 = 0 \times iK$,

$\tilde{\ell}$ is an entire function of ζ. Suppose $D = D_+$ (resp. D_-). As we have

$$\exp(-iz\zeta) = \exp(-i(x\xi - y\eta)) \exp(x\eta + y\xi),$$

the function $z \longmapsto \exp(-iz\zeta)$ belongs to $Q(D_+;\ K')$ if $\eta < -k_2'$ (resp. to $Q(D_-;\ K')$ if $\eta > -k_1'$). It is easy to see $\tilde{\ell}(\zeta)$ is a holomorphic function on $\{\zeta \in C;\ \eta = \operatorname{Im} \zeta < -k_2'\}$ (resp. $\{\zeta;\ \eta > -k_1'\}$). If $D = D_\infty$, there is no such ζ but we may define the Fourier tranformation properly as we will see soon later.

We first consider the case where $D = D_0$, D_+ or D_-. In order to state explicitely the image of $Q'(D;\ K')$ under the Fourier transformation, we introduce some new spaces of holomorphic functions.

Let V be a real vector space of dimension n and V' its dual. For an open set Ω' in $E' = V' \times iV'$, we denote by $R_{\exp}(\Omega';\ K)$ the space of all holomorphic functions f on Ω' which satisfy the following estimate:

$$\forall \varepsilon > 0,\quad \forall \varepsilon' > 0,\qquad \exists\, C \geqslant 0 \quad \text{such that}$$

$$\sup\{|f(\zeta)|\exp(-(h_K(\xi) + \varepsilon|\xi| + \varepsilon|\eta|));\ \zeta \in \Omega'_{-\varepsilon'}\} < \infty \qquad (\quad)$$

where $\Omega'_{-\varepsilon'} = \Omega' - \tilde{B}_\varepsilon \equiv \{\zeta \in E';\ \zeta + \tilde{B}_\varepsilon \subset \Omega'\}$. (Let A and B be subsets of a vector space X. We put $A - B = \{x \in X;\ x + B \subset A\}$. Remark that $A - B \neq A + (-B)$ in general.) We endow $R_{\exp}(\Omega';\ K)$ with the topology defined by the seminorm ( ). $R_{\exp}(\Omega';\ K)$ becomes an FS space. It is clear that $R_{\exp}(\Omega';\ K)$ is a subspace of $R(\Omega';\ K)$ introduced in §3.

It is well known (Martineau [ ]) that the Fourier transformation establishes a topological isomorphism:

$$Q'(D_0;\, K') \xrightarrow{\sim} R_{\exp}(C;\, K).$$

We study now the case where $D = D_+$ (resp. D_-). Let $\ell \in Q'(D_+;\, K')$ (resp. $Q'(D_-;\, K')$). As we have seen, $\tilde{\ell}(\zeta)$ is defined and holomorphic for $\eta < -k_2'$ (resp. $\eta > -k_1'$). By the continuity of ℓ, for any $\varepsilon > 0$ and $\varepsilon' > 0$ there exists a constant $C \geq 0$ such that

$$|\langle f, \ell \rangle| \leq C \sup\left\{ e^{(k_2' + \varepsilon')x} |f(z)|\,;\ z \in D_\varepsilon \right\}$$

(resp.

$$|\langle f, \ell \rangle| \leq C \sup\left\{ e^{(k_1' - \varepsilon')x} |f(z)|\,;\ z \in D_\varepsilon \right\})$$

for any $f \in Q_b(\overline{D_\varepsilon}\,;\ \overline{K^*_{\varepsilon'}})$. Therefore, for any $\eta < -k_2' - \varepsilon'$ (resp. $\eta > -k_1' + \varepsilon'$), we have

$$\begin{aligned}
&|\tilde{\ell}(\zeta)| \\
&\leq C \sup\left\{ e^{(k_2' + \varepsilon')x} \exp(x\eta + y\xi)\,;\ z \in D_\varepsilon \right\} \\
&\leq C \sup\left\{ e^{(k_2' + \varepsilon' + \eta)x} \exp(y\xi);\ x > -\varepsilon,\ y \in K_\varepsilon \right\} \\
&= C \exp(-\varepsilon(k_2' + \varepsilon' + \eta)) \exp(h_K(\xi) + \varepsilon|\xi|)
\end{aligned}$$

(resp.

$$\begin{aligned}
&|\tilde{\ell}(\zeta)| \\
&\leq C \sup\left\{ e^{(k_1' - \varepsilon')x} \exp(x\eta + y\xi);\ z \in D_\varepsilon \right\} \\
&\leq C \sup\left\{ e^{(k_1' - \varepsilon' + \eta)x} \exp(y\xi);\ x < \varepsilon,\ y \in K_\varepsilon \right\} \\
&= C \exp(\varepsilon(k_1' - \varepsilon' + \eta)) \exp(h_K(\xi) + \varepsilon|\xi|)).
\end{aligned}$$

Hence $\tilde{\ell}$ belongs to $R_{\exp}(\{\zeta\,;\ \eta < -k_2'\};\ K)$ (resp. $R_{\exp}(\{\zeta;\ \eta > -k_1'\};\ K)$).

Let us denote by Γ^* the dual cone of a cone Γ of V. Namely

$$\Gamma^* = \{\xi \in V'\,;\ \langle x, \xi \rangle \leq 0 \text{ for any } x \in \Gamma\}.$$

Remark that

$$\Gamma_0^* = V',\quad \Gamma_+^* = \{\xi \leq 0\}\quad \Gamma_-^* = \{\xi \geq 0\},\quad \Gamma_\infty^* = \{0\}.$$

We have clearly

$$\mathring{\Gamma}_0^* - K' = V', \quad \mathring{\Gamma}_+^* - K' = \{\xi < -k_2'\}, \quad \mathring{\Gamma}_-^* - K' = \{\xi > -k_1'\}.$$

Therefore the Fourier transformation $\mathfrak{F}$ maps $Q'(D; K')$ into $R_{\exp}(V' \times i(\mathring{\Gamma}^* - K'); K)$ for $D = D_0$, D_+ or D_-.

Theorem 3. Let V be a real vector space of dimension 1. Suppose $D = \Gamma \times iK = D_0$, D_+ or D_-. Then the Fourier transformation $\mathfrak{F}$ gives a topological isomorphism

$$Q'(D; K') \xrightarrow{\sim} R_{\exp}(V' \times i(\mathring{\Gamma}^* - K'); K).$$

As the case $D = D_0$ was treated by A. Martineau in much more general situation, we will suppose $D = D_+$ or D_-. We construct a mapping of $R_{\exp}(V' \times i(\mathring{\Gamma}^* - K'); K)$ into $Q'(D; K')$ which will be called the Laplace transformation.

Choose $\eta_0 \in \mathring{\Gamma}^* - K'$ and put $\zeta_0 = i\eta_0$. Let ζ' be the unit vector such that $\operatorname{Im} \zeta' = \eta' \in \Gamma^*$. (If $D = D_+$, then $\eta_0 = -k_2' - \varepsilon'$, $\eta' \leqslant 0$. If $D = D_-$, then $\eta_0 = -k_1' + \varepsilon'$, $\eta' \geqslant 0$.) It is clear the real half line $\zeta_0 + R^+\zeta'$ lies in $V' \times i(\mathring{\Gamma}^* - K')$. Let $F \in R_{\exp}(V' \times i(\mathring{\Gamma}^* - K'); K)$ be given. Put

$$\hat{F}(z; \zeta_0, \zeta') = \frac{1}{2\pi}\int_{\zeta_0 + \mathbb{R}^+\zeta'} F(\tau) e^{iz\tau}\, d\tau = \frac{1}{2\pi}\int_0^\infty F(t\zeta' + \zeta_0) e^{iz(t\zeta' + \zeta_0)} \zeta' dt$$

where $\mathbb{R}^+ = \{t > 0\}$ is oriented from 0 to ∞. As we have

$$|F(\zeta)| \leqslant C e^{\varepsilon|\xi| + \varepsilon|\eta| + h_K(\xi)}$$

for $\zeta \in \Gamma^* - \overline{K'_{\varepsilon'}}$, we have

$$|F(t\zeta' + \zeta_0)| \leqslant C \exp(\varepsilon t|\xi'| + \varepsilon|t\eta' + \eta_0| + h_K(t\xi')) \leqslant C e^{\varepsilon\eta_0} \exp(t(\varepsilon|\xi'| + h_K(\xi') + \varepsilon|\eta'|)).$$

Therefore if

$$- \operatorname{Im} z\zeta' + \varepsilon|\xi'| + h_K(\xi') + \varepsilon|\eta'| < 0,$$

the integral $\hat{F}(z;\ \zeta_0,\ \zeta')$ converges absolutely. Put

$$W_\varepsilon(\zeta') = \{z;\ -\operatorname{Im} z\zeta' + \varepsilon|\xi'| + h_K(\xi') + \varepsilon|\eta'| < 0\}$$

and

$$W(\zeta') = \{z;\ -\operatorname{Im} z\zeta' + h_K(\zeta') < 0\}.$$

(If $D = D_+$, we may put $\zeta' = -i$, 1, or -1 and we have $W(-i) = \{z;\ x < 0\}$, $W(1) = \{z;\ y > k_2\}$ and $W(-1) = \{z;\ y < k_1\}$.)

We can easily prove the following three points:

1) $\hat{F}(z;\ \zeta_0, \zeta')$ is holomorphic in $W(\zeta')$.

2) $\hat{F}(z;\ \zeta_0, \zeta') = \hat{F}(z;\ \zeta_0, \zeta'')$ in $W(\zeta') \cap W(\zeta'')$.

Hence $\hat{F}(z,\ \zeta_0, \zeta')$'s define a holomorphic function $F(z;\ \zeta_0)$ $\in R(C \setminus D;\ \overline{K'_{\varepsilon'}})$, where $\operatorname{Im} \zeta_0 = -k_2' - \varepsilon'$ if $D = D_+$ and $\operatorname{Im} \zeta_0 = -k_1' + \varepsilon'$ if $D = D_-$.

3) Put $\hat{F}_{\varepsilon'}(z) = F(z;\ \zeta_0)$.

Then $\hat{F}_{\varepsilon'}(z) - \hat{F}_{\varepsilon_1'}(z) \in R(C;\ K'_{\varepsilon_1'})$, $\varepsilon' > \varepsilon_1' > 0$.

Hence $\{F_{\varepsilon'}\}$ defines an element of $H^1_D(C;\ \tilde{R}(K'))$ which is identified with $Q'(D;\ K')$ by Theorem 2. The mapping $F \longmapsto \{\hat{F}_{\varepsilon'}\}$ is said to be the Laplace transformation $\mathcal{L}$.

We will show that the Fourier transformation and the Laplace transformation are inverse to each other.

Proof of $\mathcal{L} \circ \mathcal{F} =$ identity.

Let $[\varphi_{\varepsilon'}] \in H^1_D(C;\ R(K'))$ and $\mathcal{l}$ be the element of $Q'(D;\ K')$ which corresponds to $[\varphi_{\varepsilon'}]$. Let $0 < \varepsilon_1' < \varepsilon'$. We have

$$F(\zeta) = \tilde{\mathcal{l}}(\zeta)$$

$$= - \int_{\partial D_\varepsilon} e^{-iw\zeta} \varphi_{\varepsilon_1'}(w)\, dw$$

for $\zeta < -k_2' - \varepsilon_1'$ (resp. $\zeta > -k_1' + \varepsilon_1'$). If $\zeta_0 = -k'_2 - \varepsilon'$ (resp. $\zeta_0 = -k_1' + \varepsilon'$), we have

$$
\begin{aligned}
&F(z;\ \zeta_0,\ \zeta') \qquad\qquad (\zeta' = -i)\\
&= \frac{1}{2\pi}\int_{\zeta_0-\mathbb{R}^+ i} F(\zeta)e^{iz\zeta}\,d\zeta\\
&= -\frac{1}{2\pi}\int_{\zeta_0-\mathbb{R}^+ i} e^{iz\zeta}d\zeta\int_{\partial D_\varepsilon} e^{-i\omega\zeta}\varphi_{\varepsilon_1'}(\omega)\,d\omega\\
&= \frac{1}{2\pi}\int_{\partial D_\varepsilon}\varphi_{\varepsilon_1'}(\omega)\,d\omega\int_{\zeta_0-\mathbb{R}^+ i} e^{i(z-\omega)\zeta}\,d\zeta\\
&= \frac{1}{2\pi i}\int_{\partial D_\varepsilon}\varphi_{\varepsilon_1'}(\omega)e^{-i(z-\omega)(k_i'+\varepsilon')}/(\omega-z)\,d\omega\\
&= \varphi_{\varepsilon_1'}(w). \qquad\qquad \text{q.e.d.}
\end{aligned}
$$

Proof of $\mathfrak{F}\cdot\mathfrak{L}$ = identity.

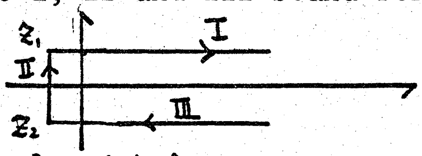
We have to show for any $F \in R(V' \times i(\mathring{\Gamma}^* - K');\ K)$

$$F(\zeta) = \int_{-\partial D_\varepsilon} e^{-iz\zeta}\hat{F}_{\varepsilon'}(z)\,dz$$

We decompose the right hand side into three integrals:

$$\int_{\mathrm{I}} + \int_{\mathrm{II}} + \int_{\mathrm{III}}$$

where I, II and III stand for the following integration path:



$$
\begin{aligned}
&\int_{\mathrm{I}} e^{-iz\zeta}\hat{F}_{\varepsilon'}(z;\ 1)\,dz\\
&= \frac{1}{2\pi}\int_{\mathrm{I}} e^{-iz\zeta}\int_{\zeta_0+\mathbb{R}^+} F(\tau)e^{iz\tau}\,d\tau\\
&= \frac{1}{2\pi}\int_{\mathrm{I}} e^{-iz(\zeta-\tau)}\int_{\zeta_0+\mathbb{R}^+} F(\tau)\,d\tau\\
&= \frac{1}{2\pi}\int_{\zeta_0+\mathbb{R}^+} -e^{-iz_1(\zeta-\tau)}/(-i(\zeta-\tau))\cdot F(\tau)\,d\tau
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
&\int_{\mathrm{II}} e^{-iz\zeta}\hat{F}_{\varepsilon'}(z;-i)\,dz\\
&= \frac{1}{2\pi}\int_{\zeta_0-i\mathbb{R}^+}\{\exp(-iz_1(\zeta-\tau)) - \exp(-iz_2(\zeta-\tau))\}/(-i(\zeta-\tau))\cdot F(\tau)\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathrm{III}} e^{-iz\zeta}\hat{F}_{\varepsilon'}(z)\,dz\\
&= \frac{1}{2\pi}\int_{\zeta_0-\mathbb{R}^+}\frac{e^{-iz_2(\zeta-\tau)}}{-i(\zeta-\tau)}F(\tau)\,d\tau
\end{aligned}
$$

Therefore we conclude

$$
\begin{aligned}
&\int_{-\partial D_\varepsilon} e^{-iz\zeta}\hat{F}_{\varepsilon'}(z)\,dz\\
&= \frac{1}{2\pi i}\int\frac{e^{iz_1(\tau-\zeta)}}{\tau-\zeta}F(\tau)\,d\tau + \frac{1}{2\pi i}\int\frac{e^{iz_2(\tau-\zeta)}}{\tau-\zeta}F(\tau)\,d\tau.
\end{aligned}
$$

Hence we have

$$\tilde{\mathcal{L}}(\zeta) = \int_{-\partial D_\varepsilon} e^{-iz\zeta}\,\hat{F}_{\varepsilon'}(z)\,dz = F(\zeta)$$

for ζ such that $\eta < -k' - \varepsilon'$ and $\xi \neq 0$. As ε' is arbitrary, we have $\tilde{\mathcal{L}} = F$ by the unique continuation property of holomorphic functions. q.e.d.

Now we proceed to the case $Q(R \times iK; K')$. We have constructed the exact sequence (39) and the topological isomorphisms:

$$Q'(D_+; K') \oplus Q'(D_-; K')$$

$$\xrightarrow{\sim} R_{exp}(\{\eta < -k_2'\}; K) \oplus R_{exp}(\{\eta > -k_1'\}; K) = R_{exp}(C \setminus (-D'); K)$$

and

$$Q'(D_0; K) = \mathcal{O}(D_0)' \cong R_{exp}(C; K).$$

Hence we have a topological isomorphism, which we name the Fourier transformation:

$$\mathfrak{F}: Q'(D; K') \longrightarrow H^1_{-D'}(C; R_{exp}(K)) \cong R_{exp}(C \setminus (-D'); K)/R_{exp}(C; K)$$

such that the diagram

$$\begin{tikzcd}
0 & Q'(D; K') \arrow[l] \arrow[d, "\wr\,\mathfrak{F}"] & Q'(D_+; K') \oplus Q'(D_-; K') \arrow[l] \arrow[d, "\wr\,\mathfrak{F}"] & Q'(D_0; K') \arrow[l] \arrow[d, "\wr\,\mathfrak{F}"] & 0 \arrow[l] \\
0 & H^1_{-D'}(C; R_{exp}(K)) \arrow[l] & R_{exp}(C \setminus (-D'); K) \arrow[l] & R_{exp}(C; K) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

is commutative.

<u>Theorem 4</u>. Put $F = R \times iK'$. The Fourier transformation $\mathfrak{F}$: $Q'(D; K') \longrightarrow H^1_{-D'}(C; R_{exp}(K))$ is a topological isomorphism and the composed mapping

$$Q'(D; K') \longrightarrow H^1_{-D'}(C; R_{exp}(K)) \longrightarrow H^1_{-D'}(C; \tilde{R}(K)) = Q'(-D'; K)$$

is equal to the (dual) Fourier transformation:

$$\mathfrak{F}_d: Q'(D; K') \longrightarrow Q'(-D'; K).$$

<u>Corollary</u>. We have the isomorphism:

$$H^1_{D'}(C; R_{exp}(K)) = H^1_{D'}(C; \tilde{R}(K)),$$

for $D' = R \times iK'$.

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