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On Hardy's Exponential Series II
Some Exponential Sums Involving Divisor Functions

Saburô UCHIYAMA
Department of Mathematics, Okayama University

In connexion with his study of the divisor problem of Dirichlet, G. H. Hardy *) discussed the exponential series of the form

$$S_N(t) = \sum_{1 \leqq n \leqq N} n^{-\frac{1}{2}} d(n) e^{-it\sqrt{n}} \qquad (t > 0),$$

where $d(n)$ denotes the number of positive divisors of the integer n, and proved that one has for $N \to \infty$

$$S_N(t) = o(N^{\varepsilon})$$

or

$$S_N(t) = \frac{2(1+i)d(q)}{q^{\frac{1}{4}}} N^{\frac{1}{4}} + o(N^{\varepsilon}),$$

according as t is not or is of the form $4\pi\sqrt{q}$ with q a positive integer, where ε is an arbitrary but fixed positive number. The aim of this note is to present some generalizations and analogues of that classical result of Hardy's.

Throughout in what follows we denote by k and ℓ fixed integers with $k \geqq 1$, $0 \leqq \ell < k$.

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*) G. H. Hardy: On Dirichlet's divisor problem. Proc. London Math. Soc. (2) 15 (1916), 1-25.

Denote by $d(n;\, k,\, \ell)$ the number of positive divisors d of n in the residue class $d \equiv \ell \pmod{k}$. We define for $x > 0$

$$S(x,\, N) = S(x,\, N;\, k,\, \ell) = \sum_{1 \leq n \leq N} n^{-\frac{1}{2}} d(n;\, k,\, \ell) e^{2\pi i x \sqrt{n}} .$$

We can show that for $N \to \infty$

(1) $$S(x,\, N) = O_x(\log N)$$

or

(2) $$S(x,\, N) = \frac{2(1 - i)\varepsilon(q;\, k,\, \ell)}{k^{\frac{3}{4}} q^{\frac{1}{4}}} N^{\frac{1}{4}} + O_x(\log N)$$

according as x is not or is of the form $2(q/k)^{\frac{1}{2}}$, q being a positive integer, provided that $x \geq 4k$. Here we set

$$\varepsilon(q;\, k,\, \ell) = \sum_{d | q} e^{2\pi i \frac{d\ell}{k}} ,$$

and the constants implied by the symbol O_x depend possibly on k, ℓ and x.

Next, we consider sums of the form

$$U(x,\, N) = U(x,\, N;\, k,\, \ell) = \sum_{\substack{1 \leq n \leq N \\ n \equiv \ell \pmod{k}}} n^{-\frac{1}{2}} d(n) e^{2\pi i x \sqrt{n}}$$

with $x > 0$. It is not quite difficult to show that, if $x \neq (2\sqrt{q})/k$ for any integer q, then one has for $N \to \infty$

(3) $$U(x,\, N) = O_x(\log N).$$

On the other hand, if $x = (2\sqrt{q})/k$ for some integer q, the situation becomes rather complicated, though it is in fact possible to find the corresponding result in its full generality. In the simplest case of $(k, \ell) = 1$ our result for $x = (2\sqrt{q})/k$ takes the following form:

$$U(x, N) = \frac{2(1 - i)\sigma(q; k, \ell)}{k^{\frac{3}{2}} q^{\frac{1}{4}}} N^{\frac{1}{4}} + O_x(\log N) \tag{4}$$

with

$$\sigma(q; k, \ell) = \sum_{m|q} S(k; \frac{q}{m}, m\ell),$$

where $S(k; u, v)$ is the Kloosterman sum,

$$S(k; u, v) = \sum_{\substack{a \bmod k \\ (a,k)=1}} \exp \frac{2\pi i}{k} (ua + v\bar{a})$$

$\bar{a}$ being defined (mod k) by $a\bar{a} \equiv 1 \pmod k$, $(a, k) = 1$.

In either case we have to assume in reality that x should be greater than a constant multiple of k^3.

We note that our results (1)-(2) and (3)-(4) are both the best possible in the sense that the O-terms therein contained cannot be improved to $o(\log N)$ (cf. the preceding article by T. Kano).

Fundamental is in our investigations the sum of the form

$$E(x, N) = E(x, N; k, \ell) \overset{\text{def}}{=} \sum_{\substack{1 \leq n \leq N \\ n \equiv \ell \pmod k}} e^{2\pi i x\sqrt{n}} .$$

By making use of the well-known Euler-Maclaurin sum formula we find

$$E(x, N) = \frac{1}{k}\left(\frac{e^{2\pi i x\sqrt{N}}}{\pi i x} N^{\frac{1}{2}} + \frac{e^{2\pi i x\sqrt{N}} - 1}{2\pi^2 x^2}\right) + R(x, N)$$

where

$$R(x, N) = P(x, N) - Q(x, N)$$

$$+ \mathcal{O}\left(x \sum_{m \neq \frac{kx}{2\sqrt{N}}} \frac{1}{|2mN^{\frac{1}{2}} - kx|^3}\right) + \mathcal{O}(xN^{-\frac{1}{2}} + 1)$$

with

$$P(x, N) = \frac{k^{\frac{1}{2}} x}{2^{\frac{1}{2}}} \sum^{*}_{m \geq \frac{kx}{2\sqrt{N}}} \frac{1}{m^{\frac{3}{2}}} \exp 2\pi i\left(\frac{kx^2}{4m} + \frac{m\ell}{k} - \frac{1}{8}\right)$$

and

$$Q(x, N) = \frac{kx}{2\pi i} \sum_{m \neq \frac{kx}{2\sqrt{N}}} \frac{1}{m(2mN^{\frac{1}{2}} - kx)} \exp 2\pi i\left(xN^{\frac{1}{2}} - \frac{m(N-\ell)}{k}\right).$$

Here, the $\mathcal{O}$-constants may depend at most on k and ℓ, and $\Sigma^{*}_{m \geq u}$ indicates that the summand corresponding to $m = u$ (if existent) should be added with the extra factor 1/2.

Define for $x > 0$

$$H(x, N) = H(x, N; k, \ell) = \sum_{1 \leq n \leq N} d(n; k, \ell) e^{2\pi i x\sqrt{n}} .$$

THEOREM 1. *We have for* $N \to \infty$

$$H(x, N) = \frac{e^{2\pi i x\sqrt{N}}}{\pi i k x} N^{\frac{1}{2}} \log N + O_x(N^{\frac{1}{2}})$$

if $x \neq 2(q/k)^{\frac{1}{2}}$ *for any integer* q, *and*

$$H(x, N) = \frac{2(1 - i)\varepsilon(q;\, k,\, \ell)}{3k^{\frac{3}{4}} q^{\frac{1}{4}}} N^{\frac{3}{4}}$$

$$+ \frac{e^{2\pi i x\sqrt{N}}}{\pi i k x} N^{\frac{1}{2}} \log N + O_x(N^{\frac{1}{2}})$$

if $x = 2(q/k)^{\frac{1}{2}}$ *for some integer* q, *provided that* $x \geqq 4k$.

Proof can be carried out by noticing that

$$H(x, N) = \sum_{1 \leqq a \leqq \sqrt{N}} \left\{ E\left(xa^{\frac{1}{2}}, \frac{N}{a}; k, \ell\right) - E(xa^{\frac{1}{2}}, a; k, \ell) \right\}$$

$$+ \sum_{\substack{1 \leqq a \leqq \sqrt{N} \\ a \equiv \ell \ (\mathrm{mod}\ k)}} \left\{ E\left(xa^{\frac{1}{2}}, \frac{N}{a}; 1, 0\right) - E(xa^{\frac{1}{2}}, a; 1, 0) \right\}$$

$$+ \sum_{\substack{1 \leqq a \leqq \sqrt{N} \\ a \equiv \ell \ (\mathrm{mod}\ k)}} e^{2\pi i x a},$$

and by appealing to the result on $E(x, N)$ mentioned above.

The result (1)-(2) will follow from Theorem 1 by partial summation.

Write for $x > 0$

$$V(x, N) = V(x, N; k, \ell) = \sum_{\substack{1 \leq n \leq N \\ n \equiv \ell \pmod{k}}} d(n) e^{2\pi i x\sqrt{n}} .$$

THEOREM 2. *We have for* $N \to \infty$

$$V(x, N) = \frac{\Phi_k(\ell)}{k^2} \frac{e^{2\pi i x\sqrt{N}}}{\pi i x} N^{\frac{1}{2}} \log N + O_x(N^{\frac{1}{2}})$$

if $x \neq (2\sqrt{q})/k$ *for any integer* q, *and*

$$V(x, N) = \frac{2(1 - i)\sigma(q; k, \ell)}{3k^{\frac{3}{2}} q^{\frac{1}{4}}} N^{\frac{3}{4}}$$

$$+ \frac{\Phi_k(\ell)}{k^2} \frac{e^{2\pi i x\sqrt{N}}}{\pi i x} N^{\frac{1}{2}} \log N + O_x(N^{\frac{1}{2}})$$

if $x = (2\sqrt{q})/k$ *for some integer* q, *provided that* $x \geq 4k^3$, *where we set*

$$\Phi_k(\ell) = \sum_{d|(k,\ell)} d\phi\left(\frac{k}{d}\right)$$

and

$$\sigma(q; k, \ell) = 2 \sum_{d|D} d \sum^{*}_{\substack{0 < m \leq \sqrt{q}/d \\ m|q/d}} S\left(\frac{k}{d}; \frac{q}{dm}, \frac{m\ell}{d}\right)$$

with $D = (q, k, \ell)$.

Proof is to rewrite $V(x, N)$ in the following manner:

$$V(x, N) = 2 \sum_{d \mid (k,\ell)} \sum_{\substack{1 \leq a \leq \sqrt{N}/d \\ (a,\, k/d)=1}} \left\{ E\left(x(ad)^{\frac{1}{2}}, \frac{N}{ad}; \frac{k}{d}, \frac{\ell\bar{a}}{d} \right) - E\left(x(ad)^{\frac{1}{2}}, ad; \frac{k}{d}, \frac{\ell\bar{a}}{d} \right) \right\}$$

$$+ \sum_{\substack{1 \leq a \leq \sqrt{N} \\ a^2 \equiv \ell \pmod{k}}} e^{2\pi i x a},$$

where $\bar{a}$ is defined modulo k/d by $a\bar{a} \equiv 1 \pmod{k/d}$, $(a, k/d) = 1$.

Our result (3)-(4) (with $\sigma(q; k, \ell)$ defined above for the general case) is an immediate consequence of Theorem 2.

The detailed proofs of Theorems 1 and 2 with some comments and applications will be published elsewhere.