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*A Generalization of a Prime Number Thorem of Rodosskii-Tatsuzawa for an algebraic Field*

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## 1. Introduction

*The Prime Number Theorem in an Arithmetic Progressions is stated as follows ;*

*Theorem A .* If $(q, l) = 1$ and $q \leqq \exp(c\sqrt{\log x})$ then

$$\pi(x, q, l) = \frac{1}{\varphi(q)}\int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{\varphi(q)\log x}\right) + O\left(x e^{-c'\sqrt{\log x}}\right)$$

*where* $\beta_1$ *is a so-called Siegel zero.*

*For* $q$ *, which does not satisfy the above restriction , we have the following theorem ;*

*Theorem B ( Rodosskii - Tatsuzawa ) ( See , Prachar ( 8 ) Chap . 9 Satz 2 . 2 )*

*There exist constant* $c_1$ *and* $c_2$ *, such that , if*

$$c_1 \log q \log\log q \leqq \log x \leqq c_2 (\log q)^2$$

$$\pi(x, q, l) = \frac{1}{\varphi(q)}\int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{\varphi(q)\log x}\right) + O\left(\frac{x}{\varphi(q)\log x} e^{-c\frac{\log x}{\log q}}\right)$$

*The prime ideal theorem in an ideal classes of an algebraic field , which is a generalization of theorem A , has been already obtained under the similar restriction.(see, Mitsui (5) )*

*The purpose of this paper is to offer a generalization of theorem B for an algebraic field under the similar condition. We now write it down ;*

*Theorem K is an algebraic field of degree n and* $\mathfrak{f}$ *is an integral ideal in K. We denote a product of* $\mathfrak{f}$ *and some infinite prime ideals by* $\tilde{\mathfrak{f}}$ *h* $(\tilde{\mathfrak{f}})$ *is the number of ideal classes mod* $\tilde{\mathfrak{f}}$ *and* $\mathfrak{l}$ *is one of the ideal classes . Then we have*

*for all c > 0 , there exist* $c_1$ *and* $c_2$ *such that, under the condition*

$$c_1 \log N(\mathfrak{f}) \log\log N(\mathfrak{f}) \leqq \log x \leqq c\,(\log N(\mathfrak{f}))^2$$

$$\pi(x,\mathfrak{l}) = \sum_{\substack{N(\mathfrak{z})<x \\ \mathfrak{z}\in\mathfrak{l}}} 1 = \frac{1}{h(\tilde{\mathfrak{f}})}\int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{h(\tilde{\mathfrak{f}})\log x}\right) + O\left(\frac{x}{h(\tilde{\mathfrak{f}})\log x}\, e^{-c_2\frac{\log x}{\log N(\mathfrak{f})}}\right)$$

*( The constant in O depend only on K and c . )*

*The auther has already given another proof of theorem B in (2) , using Gallagher's mean value theorem ( see, Gallagher (1) Theorem 2 ) In this paper, we use its generalization for an algebraic field , again . The functions* $\mu(\mathfrak{n})$, $d(\mathfrak{n})$, $\Lambda(\mathfrak{n})$ *and* $\varphi(\mathfrak{a})$ *are that of generalization for an algebraic field of Möbius , divisor , von Mangoldt and Euler function , respectively . All constants of all estimations depend only on K and c .*

*The auther wishes to express his thanks to Prof. T . Tatszawa , who*

taught him Theorem 2 and 3 in this paper , and encouraged him during preparing this paper .

## 2. The number of ideals of an ideal class mod $\tilde{\mathfrak{f}}$ whose norm lie in a given interval

Let K be an algebraic field of degree n with $r_1$ real conjugates and 2 $r_2$ complex conjugates and $\mathfrak{f}$ an integral ideal of K . And we put $\tilde{\mathfrak{f}} = \mathfrak{f}\,\mathfrak{z}_\infty^{(1)} \cdots \mathfrak{z}_\infty^{(g)}$ and $h(\tilde{\mathfrak{f}})$ is the number of ideal classes mod $\tilde{\mathfrak{f}}$ .Then ,

<u>Theorem</u> 1

$$h(\tilde{\mathfrak{f}}) = \frac{h\,\varphi(\tilde{\mathfrak{f}})}{e(\tilde{\mathfrak{f}})} \qquad (1)$$

where h is the ideal class number of K , $e(\tilde{\mathfrak{f}})$ the number of residue classes mod $\tilde{\mathfrak{f}}$ of units of K , and $\varphi(\tilde{\mathfrak{f}}) = 2^{g}\varphi(\mathfrak{f})$

The proof of this theorem, for example , appears in Suetsuna ( 9) Chap . 2 Th . 4 ( p . 55 )

<u>Cor</u> .

$$h(\tilde{\mathfrak{f}}) \ll N(\mathfrak{f}) \qquad (2)$$

<u>Theorem</u> 2

$$\sum_{\substack{N(\mathfrak{a})<x \\ \mathfrak{a}\in\mathfrak{l}}} 1 = \frac{1}{h(\tilde{\mathfrak{f}})}\frac{\varphi(\mathfrak{f})}{N(\mathfrak{f})} c_K x + O\left(\frac{1}{h(\tilde{\mathfrak{f}})}\frac{\varphi(\mathfrak{f})}{N(\mathfrak{f})} N(\mathfrak{f})^{\frac{1}{n}} x^{1-\frac{1}{n}}\right) \qquad (3)$$

where $c_K$ is the constant only determined by K .

proof ) see, Tatsuzawa ( 10)

*Theorem 3* If $y \ll x$, then

$$\sum_{\substack{x \leqq N(\mathfrak{a}) \leqq x+y \\ \mathfrak{a} \in \mathfrak{l}}} 1 \ll \frac{y}{h(\tilde{\mathfrak{f}})} + \frac{N(\mathfrak{f})^{\frac{1}{n}}}{h(\tilde{\mathfrak{f}})} x^{1-\frac{1}{n}} + 1 \quad (4)$$

We now choose a ideal class $\mathfrak{l}$ mod $\mathfrak{f}$, and fix it.

## 3. A Generalization of Gallagher's Mean Value Theorem

*Lemma* If $z \ll y$, and $a(\mathfrak{a})$ are complex numbers, then

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \Big| \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} a(\mathfrak{a})\chi(\mathfrak{a}) \Big|^2 \ll (z + N(\mathfrak{f})^{\frac{1}{n}} y^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}})) \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} |a(\mathfrak{a})|^2 \quad (5)$$

proof ) By the orthogonal relation, Schwarz inequality and theorem 3,

we have
$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \Big| \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} a(\mathfrak{a})\chi(\mathfrak{a}) \Big|^2 \ll \sum_{\mathfrak{l} \bmod \tilde{\mathfrak{f}}} h(\tilde{\mathfrak{f}}) \Big| \sum_{\substack{y \leqq N(\mathfrak{a}) \leqq y+z \\ \mathfrak{a} \in \mathfrak{l}}} a(\mathfrak{a}) \Big|^2$$

$$\ll \sum_{\mathfrak{l} \bmod \tilde{\mathfrak{f}}} h(\tilde{\mathfrak{f}}) \Big( \sum_{\substack{y \leqq N(\mathfrak{a}) \leqq y+z \\ \mathfrak{a} \in \mathfrak{l}}} 1 \Big) \sum_{\substack{y \leqq N(\mathfrak{a}) \leqq y+z \\ \mathfrak{a} \in \mathfrak{l}}} |a(\mathfrak{a})|^2$$

$$\ll (z + N(\mathfrak{f})^{\frac{1}{n}} x^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}})) \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} |a(\mathfrak{a})|^2$$

*Theorem 4*. $\chi$ is a character mod $\tilde{\mathfrak{f}}$, and

$$S(\chi, t) = \sum a(\mathfrak{a}) \chi(\mathfrak{a}) N(\mathfrak{a})^{-it}$$

is an absolute convergent Dirichlet series. Then, if $T \geqq 1$, we have

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \int_{-T}^{T} |S(\chi,t)|^2 dt \ll \sum (h(\tilde{\mathfrak{f}})T + N(\mathfrak{f})^{\frac{1}{n}} T N(\mathfrak{a})^{1-\frac{1}{n}} + N(\mathfrak{a})) |a(\mathfrak{a})|^2 \quad (6)$$

proof ) We first refer the following inequality.

$S(t) = \sum a_n n^{it}$ is an absolute convergent Dirichlet series.

and $\tau = e^{1/T}$. Then

$$\int_{-T}^{T} |S(t)|^2 dt \ll T^2 \int_0^\infty \Big| \sum_{y}^{y\tau} a_n \Big|^2 \frac{dy}{y}$$

( See Gallagher (1) Theorem 1 ) Then we have

$$\sum_{\chi \bmod \mathfrak{f}} \int_{-T}^{T} |S(\chi,t)|^2 dt \ll T^2 \int_0^{\infty} \sum_{\chi \bmod \mathfrak{f}} \Big| \sum_{y \le N(\mathfrak{a}) \le y\tau} a_{\mathfrak{a}} \chi(\mathfrak{a}) \Big|^2 \frac{dy}{y}$$

By the assumption $T \geqq 1$ , we can apply the above lemma to the inner sum .

$$\ll T^2 \int_0^{\infty} \left( y(\tau-1) + N(\mathfrak{f})^{\frac{1}{n}} y^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}}) \right) \sum_{y \le N(\mathfrak{a}) \le y\tau} |a_{\mathfrak{a}}|^2 \frac{dy}{y}$$

The coefficient of $|a_{\mathfrak{a}}|^2$ is ( $n>1$ )

$$T^2 \int_{N(\mathfrak{a})/\tau}^{N(\mathfrak{a})} \left( y(\tau-1) + N(\mathfrak{f})^{\frac{1}{n}} y^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}}) \right) \frac{dy}{y}$$

$$= h(\tilde{\mathfrak{f}})T + T^2 N(\mathfrak{a})(\tau-1)(1-\tau^{-1}) + \frac{1}{1-\frac{1}{n}} N(\mathfrak{f})^{\frac{1}{n}} N(\mathfrak{a})^{1-\frac{1}{n}} (1-\tau^{-1+\frac{1}{n}}) T^2$$

$$\ll h(\tilde{\mathfrak{f}})T + N(\mathfrak{a}) + N(\mathfrak{f})^{\frac{1}{n}} T \, N(\mathfrak{a})^{1-\frac{1}{n}}$$

In the case $n = 1$ , the proof is similarly done . Then we get the theorem .

## 4 . Some properties on $L(s, \chi)$

In this paragraph we pick up some properties of $L(s, \chi)$ . These proofs can be found in Mitsui ( 5) or be similarly done as the case of Dirichlet L-functions , using theorem 2 and 3 .

Theorem 5 . The following estimations are true uniformly on $\sigma \geqq 1-\frac{1}{2n}$

$$L(s,\chi) = \sum_{N(\mathfrak{a})<M} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} + E(\chi)\frac{N(\mathfrak{f})}{h(\tilde{\mathfrak{f}})\varphi(\mathfrak{f})} c_K \frac{M^{1-s}}{s-1}$$
$$+ O\left( |s| \frac{N(\mathfrak{f})^{\frac{1}{n}}}{M^{\sigma+\frac{1}{n}-1}} \right) \qquad (7)$$

*where* $E(\chi)$ *is 1 if* $\chi$ *is principal , 0 if not .*

*Theorem 6*

$$L(s,\chi)(s-1)^{E(\chi)} \ll N(\mathfrak{f})^2(|t|+2)^{\frac{2}{3}n(1-\sigma)+E(\chi)} \log((|t|+2)N(\mathfrak{f}))$$

( *uniformly* $-\frac{1}{2} \leq \sigma \leq 4$ ) $\qquad (8)$

*Theorem 7*

$$\frac{L'}{L}(s,\chi) = \sum_{|\gamma_0 - t| \leq 1} \frac{1}{s-\rho_0} - \frac{E(\chi)}{s-1} + O(\log(N(\mathfrak{f})(|t|+2))) \qquad (9)$$

*where* $\rho_0 = \beta_0 + i\gamma_0$ *is a zero of* $L(s,\chi)$

*Theorem 8* *There exists a constant* $A > 0$ *, such that , in the region*

$$\sigma \geq 1 - \frac{A}{\log(N(\mathfrak{f})(|t|+2))}, \quad |t|>0 \qquad (10)$$

*all* $L(s,\chi)$ *whose character are defined mod* $\mathfrak{f}$ *, are zero-free .*

*For* $t = 0$ *, all* $L(s,\chi)$ *except one* $L(s,\chi_1)$ *where* $\chi_1$ *is a real character mod* $\mathfrak{f}$*, are zero-free in the interval* $1 \geq \sigma \geq 1 - \frac{A}{\log N(\mathfrak{f})}$

*The number of the exceptional zeros is at most one with multiplicity .*

## 5 . Notations and Remarks

*Now we put the restriction*

$$c_1 \log N(\mathfrak{f}) \log\log N(\mathfrak{f}) \leqq \log x \leqq c_2 (\log N(\mathfrak{f}))^2 \qquad (11)$$

and $N(\mathfrak{f})$ is sufficiently large . We define the following functions .

$$\psi(x,\chi) = \sum_{N(\mathfrak{a})<x} \chi(\mathfrak{a})\Lambda(\mathfrak{a}) \qquad (12)$$

$$\psi_1(x,\chi) = \sum_{N(\mathfrak{a})<x} \chi(\mathfrak{a})\Lambda(\mathfrak{a}) \log\frac{x}{N(\mathfrak{a})} \qquad (13)$$

$$Q(s,\chi) = \sum_{N(\mathfrak{a})<N(\mathfrak{f})T^n} \frac{\mu(\mathfrak{a})\chi(\mathfrak{a})}{N(\mathfrak{a})^s} \qquad (14)$$

Furthermore , we assume that
$$\log T \ll \log N(\mathfrak{f}) \qquad (15)$$

$$\alpha = 1+\frac{1}{\log x}, \quad \beta = 1-\frac{1}{2n}+\frac{1}{\log x} \qquad (16)$$

$$\psi(x,\mathfrak{l}) = \sum_{\substack{N(\mathfrak{a})<x \\ \mathfrak{a}\in\mathfrak{l}}} \Lambda(\mathfrak{a}) \qquad (17)$$

and

$$\psi_1(x,\mathfrak{l}) = \sum_{\substack{N(\mathfrak{a})<x \\ \mathfrak{a}\in\mathfrak{l}}} \Lambda(\mathfrak{a}) \log\frac{x}{N(\mathfrak{a})} \qquad (18)$$

Then ,

$$\psi_1(x,\mathfrak{l}) = \frac{1}{h(\tilde{\mathfrak{f}})} \sum_{\chi \bmod \tilde{\mathfrak{f}}} \overline{\chi(\mathfrak{l})}\,\psi_1(x,\chi) \qquad (19)$$

We begin with the estimation of $\psi_1(x,\mathfrak{l})$

## 6 . The estimation of $\psi_1(x,\mathfrak{l})$

### 6 . 1 Transformation of $\psi_1(x,\mathfrak{l})$

<u>Lemma</u> $a>0$ , then

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{s^2}\,ds = \begin{cases} 0 & 0<x\leqq 1 \\ \log x & x\geqq 1 \end{cases} \qquad (20)$$

*The proof of this lemma is immediately done .*

*By the above lemma and the definition of* $L(s,\chi)$ *, we have*

$$\psi_1(x,\chi) = -\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{L'}{L}(s,\chi)\frac{x^s}{s^2}ds$$
$$= -\frac{1}{2\pi i}\left(\int_{\alpha-iT}^{\alpha+iT}+\int_{\alpha-i\infty}^{\alpha-iT}+\int_{\alpha+iT}^{\alpha+i\infty}\right)\frac{L'}{L}(s,\chi)\frac{x^s}{s^2}ds$$
$$= -\frac{1}{2\pi i}\left(I_1(\chi)+I_2(\chi)+I_3(\chi)\right) \qquad (21)$$

( *Say !* )

*By theorem 8 and the condition (15) , we may assume the following conditions .*

1 ) *All* $L(s,\chi)$ *except one real zero of* $L(s,\chi_1)$ *are zero- free in the region*

$$\sigma \geqq 1-\frac{A}{\log N(\mathfrak{f})}, \quad |t| \leqq T \qquad (A>0) \qquad (22)$$

2 ) *All zeros and poles of all* $L(s,\chi)$ *mod* $\mathfrak{f}$ *are far from the line*

$$\sigma = 1-\frac{A}{\log N(\mathfrak{f})}, \quad |t| \leqq T \qquad (23)$$

*by* $\gg \dfrac{1}{\log N(\mathfrak{f})}$ .

*We put* $\gamma = 1 - \dfrac{A}{\log N(\mathfrak{f})}$ (24)

*We change the way of integration ,and have*

$$I_1(\chi) = -2\pi i\, E(\chi)\, x + 2\pi i\, E_1(\chi)\frac{x^{\beta_1}}{\beta_1^2} + \int_{\gamma-iT}^{\gamma+iT}\frac{L'}{L}\frac{x^s}{s^2}ds$$
$$+\left(\int_{\gamma+iT}^{\alpha+iT}-\int_{\gamma-iT}^{\alpha-iT}\right)\frac{L'}{L}(s,\chi)\frac{x^s}{s^2}ds$$
$$= -2\pi i\, E(\chi)\, x + 2\pi i\, E_1(\chi)\frac{x^{\beta_1}}{\beta_1^2} + I_4(\chi) + I_5(\chi) \qquad (25)$$

( *Say!* )

$$E_1(\chi) = \begin{cases} 1 & \text{if there is a zero in the region} \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

We put $E_1^* = \sum_{\chi \bmod f} E_1(\chi)\overline{\chi(l)}$ (27)

Using ( 14 ) , we transform $I_4(\chi)$ as follows ;

$$\begin{aligned} I_4(\chi) &= \int_{\gamma-iT}^{\gamma+iT} \frac{L'}{L}(s,\chi)(1-L(s,\chi)Q(s,\chi))^2 \frac{x^s}{s^2}ds \\ &+ \left(\int_{\beta-iT}^{\beta+iT} + \int_{\beta+iT}^{\gamma+iT} - \int_{\beta-iT}^{\gamma-iT}\right)(2L'Q - L'LQ^2)\frac{x^s}{s^2}ds \\ &= I_7(\chi) + I_8(\chi) + I_9(\chi) + I_{10}(\chi) \quad (\text{say}) \end{aligned} \tag{28}$$

6 . 2 Estimation of $I_2(\chi)$ and $I_3(\chi)$

By theorem 7 , we have the following estimations on the line $\sigma = \alpha$

$$\frac{L'}{L}(s,\chi) \ll \log(N(f)(|t|+2)) \log x \tag{29}$$

Hence ,

$$I_3(\chi) + I_4(\chi) \ll x \log x \int_T^\infty \frac{\log N(f)(|t|+2)}{\alpha^2+T^2}dt \ll \frac{x\log^2 x}{T} \tag{30}$$

6 . 3 Estimation of $I_5(\chi)$ and $I_6(\chi)$

As the above paragraph , we have

$$I_5(\chi) + I_6(\chi) \ll \frac{\log^2 x}{T^2}\int_\gamma^\alpha x^\sigma d\sigma \ll \frac{x\log^2 x}{T^2} \tag{31}$$

6 . 4 Estimation of $I_9(\chi)$ and $I_{10}(\chi)$

By using Cauchy's integration formula , we have

$$L'(s,\chi) \ll N(f)^2 (|t|+2)^{\frac{2}{3}n(1-\sigma)} \log^3 N(f)(|t|+2) \tag{32}$$

*On the other hand , the following estimations are obtained by using convexity argument .*

$$Q(s,\chi) \ll N(f)\, T^{n(1-\sigma)} \log x$$
$$\text{for} \quad \beta \leq \sigma \leq \alpha . \tag{33}$$

*Therefore we have*

$$\begin{aligned} I_9(x) + I_{10}(x) &\ll N(f)^6 \frac{x}{T^2} \log^5 x \int_\beta^\gamma \left(\frac{T^{\frac{10}{3}n}}{x}\right)^{1-\sigma} d\sigma \\ &\ll N(f)^6 \frac{x^\gamma}{T^2} \log^5 x \\ &\left( \text{if} \quad T^{\frac{10}{3}n} \leq x \right) \end{aligned} \tag{34}$$

*6 . 5 Estimation of $I_8(\chi)$*

*By theorem 6 , ( 32) ,and ( 33) , we have*

$$\begin{aligned} I_8(x) &\ll N(f)^6\, T^{\frac{10}{3}n(1-\beta)}\, x^\beta \log^6 x \\ &= N(f)^6\, T^{5/3}\, x^\beta \log^6 x \end{aligned} \tag{35}$$

*6 . 6 Estimations of $I_7(\chi)$*

*Putting $M = N(f)\, T^n$ in theorem 5 , we have on the line* $\sigma = \alpha$

$$L(s,\chi)Q(s,\chi) - 1 = \sum_{N(f)T^n \leq N(\mathfrak{a}) \leq (N(f)T^n)^2} b(\mathfrak{a})\chi(\mathfrak{a})N(\mathfrak{a})^{-s} + O\left(|Q(s,\chi)|\left(\frac{|s|}{T} + E(x)\log x\right)\right) \tag{36}$$

*where* $$b(\mathfrak{a}) = \sum_{\mathfrak{z} | \mathfrak{a},\, N(\mathfrak{z}) < N(f)T^n} \mu(\mathfrak{z}) \tag{37}$$

*Therefore* ,

$$\sum_{\chi \bmod \tilde{f}} |I_\gamma(x)| \ll x^\gamma \log^2 x \left\{ \int_{-T}^{T} \frac{\sum_{\chi \bmod \tilde{f}} \left| \sum_{N(f)T^n \le N(\mathfrak{r}) \le (N(f)T^n)^2} b(\mathfrak{r})\chi(\mathfrak{r})N(\mathfrak{r})^{-\gamma - it} \right|^2}{|\gamma + it|^2} dt \right.$$

$$\left. + \frac{1}{T^2}\int_{-T}^{T} \sum_{\chi \bmod \tilde{f}} |Q(\gamma+it,\chi)|^2 dt + \log^2 x \int_{-T}^{T} \frac{|Q(\gamma+it,\chi_0)|^2}{|\gamma+it|^2} dt \right\} \qquad (38)$$

*We now apply Gallagher's mean value theorem to the above integrals .*

*Then we have*

$$\sum_{\chi \bmod \tilde{f}} \int_{-T}^{T} |Q(\gamma+it,\chi)|^2 dt \ll \sum_{N(\mathfrak{r}) < N(f)T^n} \left(h(\tilde{f})T + N(f)^{\frac{1}{n}} T N(\mathfrak{r})^{1-\frac{1}{n}} + N(\mathfrak{r})\right) \frac{1}{N(\mathfrak{r})^{2\gamma}}$$

$$\ll N(f)T \qquad (39)$$

*and*

$$\sum_{\chi \bmod \tilde{f}} \int_{-T}^{T} \frac{\left| \sum_{N(f)T^n \le N(\mathfrak{r}) \le (N(f)T^n)^2} b(\mathfrak{r})\chi(\mathfrak{r})N(\mathfrak{r})^{-\gamma-it} \right|^2}{|\gamma+it|^2} dt$$

$$\ll \sum_{N(f)T^n \le N(\mathfrak{r}) \le (N(f)T^n)^2} \left(h(\tilde{f})T + N(f)^{\frac{1}{n}} T N(\mathfrak{r})^{1-\frac{1}{n}} + N(\mathfrak{r})\right) \frac{|b(\mathfrak{r})|^2}{N(\mathfrak{r})^{2\gamma}}$$

$$\ll \log^4 x \qquad (40)$$

(*We use* $\sum_{N(\mathfrak{r}) < x} |b(\mathfrak{r})|^2 \ll \sum_{N(\mathfrak{r}) \le x} |d(\mathfrak{r})|^2 \ll x \log^3 x$ )

*The last integral is trivially estimated and we obtain*

$$\sum_{\chi \bmod \tilde{f}} I_\gamma(x) \ll x^\gamma \left(\log^4 x + \frac{N(f)}{T}\right) \qquad (41)$$

## 6 . 7 *Estimation of* $\psi(x, \mathfrak{l})$

*Now we collect the results of preceding paragraphs , then we have*

$$\psi_1(x,\mathfrak{l}) = \frac{x}{h(f)} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f})\beta_1^2} + O\left(\frac{x}{T}\log^2 x\right)$$

$$+\,O\left(x^{1-\frac{1}{2n}} N(f)^6\, T^{\frac{5}{3}} \log^6 x\right) + O\left(\frac{x^6}{h(\tilde{f})} \log^4 x\right) \quad (42)$$

*Taking* $T = N(f)^4$, *we have*

$$\frac{x}{T}\log^2 x \ll \frac{x}{h(\tilde{f})}\, e^{-c_3 \frac{\log x}{\log N(f)}} \quad (43)$$

$$x^{1-\frac{1}{2n}} N(f)^6\, T^{\frac{5}{3}} \log^5 x \ll \frac{x}{h(\tilde{f})} N(f)^{-c_4 \log\log N(f)}$$
$$\ll \frac{x}{h(\tilde{f})}\, e^{-c_5 \frac{\log x}{\log N(f)}} \quad (44)$$

*and*

$$\frac{x^{\gamma}}{h(\tilde{f})} \log^4 x \ll \frac{x}{h(\tilde{f})}\, e^{-\frac{A}{2} \frac{\log x}{\log N(f)}} \quad (45)$$

*Therefore we get*

$$\psi_1(x, \ell) = \frac{x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f})\beta_1^2} + O\left(\frac{x}{h(\tilde{f})} e^{-c_6 \frac{\log x}{\log N(f)}}\right) \quad (46)$$

7 . *Estimation of* $\psi(x, \ell)$

*Proposition 1* . *For all* $c^* > 0$ , *there exists a constant* $c_1^*$ $0$ ,

*such that , under the condition*

$$c_1^* \log N(f) \log\log N(f) \leqq \log x \leqq c^* (\log N(f))^2$$

$$\psi(x, \ell) = \frac{x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f})\beta_1} + O\left(\frac{x}{h(\tilde{f})} e^{-c_2^* \frac{\log x}{\log N(f)}}\right) \quad (47)$$

*proof )* *Let* $c_2 = 2c^*$ *and take* $\theta$ *, such that* $0 \leqq \theta \leqq 1$ ,

*we have by ( 46 ) ,*

$$\psi_1(x+\theta x, \ell) - \psi_1(x, \ell) = \frac{\theta x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f})\beta_1}\left(\theta + O(\theta^2)\right)$$
$$+\,O\left(\frac{x}{h(\tilde{f})}\, e^{-c_6 \frac{\log x}{\log N(f)}}\right) \quad (48)$$

*And by the definition of* $\psi_1(x, \ell)$ , *we get*

$$\psi_1(x+\theta x, l) - \psi_1(x, l) = \sum_{N(\mathfrak{a})<x,\ \mathfrak{a}\in l} \Lambda(\mathfrak{a}) \log(1+\theta) + \sum_{\substack{x\le N(\mathfrak{a})<x+\theta x \\ \mathfrak{a}\in l}} \Lambda(\mathfrak{a}) \log \frac{x}{N(\mathfrak{a})} \quad (49)$$

*Using theorem 3 , we obtain*

$$\sum_{\substack{x\le N(\mathfrak{a})<x+\theta x \\ \mathfrak{a}\in l}} \Lambda(\mathfrak{a}) \log \frac{x}{N(\mathfrak{a})} = O\left( \theta \log x \left( \frac{\theta x}{h(\mathfrak{f})} + \frac{N(\mathfrak{f})^{\frac{1}{n}}}{h(\mathfrak{f})} x^{1-\frac{1}{n}} + 1\right)\right) \quad (50)$$

*and*

$$\sum_{N(\mathfrak{a})\le x,\ \mathfrak{a}\in l} \Lambda(\mathfrak{a}) \log(1+\theta) = \theta\psi(x, l) + O\left(\frac{\theta^2 x \log x}{h(\mathfrak{f})}\right) \quad (51)$$

*The proposition is immediately deduced , taking*

$$\theta = e^{-\frac{C_6}{2} \frac{\log x}{\log N(\mathfrak{f})}} \quad (52)$$

*The theorem is easily proved from this proposition .*

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