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Subsequences of normal sequences

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Let Σ be any compact metric space. Let $N=\{0,1,2,\cdots\}$ be the set of non-negative integers. By Σ^N, we mean the product space of Σ with the product topology. The i-th coordinate ($i\varepsilon N$) of $\alpha\varepsilon\Sigma^N$ is denoted by $\alpha(i)$. An element of Σ^N is called a <u>sequence</u>. Let T be the <u>shift</u> on Σ^N; $(T\alpha)(i)=\alpha(i+1)$ for any $\alpha\varepsilon\Sigma^N$ and $i\varepsilon N$.

By a <u>measure</u> on a topological space, we always mean a probability Borel measure. Let W be an arbitrary compact space and let ν_n ($n=0, 1,\cdots$) and ν be measures on W. We say that ν_n <u>converges weakly</u> [11] to ν as $n\to\infty$, and denote $\text{w-}\lim_{n\to\infty}\nu_n=\nu$, if for any real-valued continuous function f on W, $\int fd\nu_n$ converges to $\int fd\nu$ as $n\to\infty$. For $x\varepsilon W$, δ_x is the unit measure at x. By a <u>non-degenerate</u> measure on W, we mean a measure which is not a unit measure.

Let $\alpha\varepsilon\Sigma^N$. Let Ξ_α denote the family of all infinite subsets S of N such that

$$\mu_\alpha^S=\text{w-}\lim_{\substack{n\varepsilon S\\ n\to\infty}}\frac{1}{n}\sum_{i=0}^{n-1}\delta_{T^i\alpha} \tag{1.1}$$

exists. Note that $\Xi_\alpha\neq\phi$ for any $\alpha\varepsilon\Sigma^N$ since the space of measures on a compact metric space is compact in the topology of weak convergence (see [11]). Also, note that μ_α^S is a T-invariant measure for any $\alpha\varepsilon\Sigma^N$ and $S\varepsilon\Xi_\alpha$. We call $\alpha\varepsilon\Sigma^N$ a <u>stochastic sequence</u> [3] (or sometimes a <u>quasi-regular point</u> in Σ^N with respect to T [9]) if $N\varepsilon\Xi_\alpha$. In this case, we denote $\mu_\alpha = \mu_\alpha^N$. Let P be a non-degenerate measure on Σ. A stochastic sequence $\alpha \in \Sigma^N$ is called a P-normal sequence if μ_α equals P^N, the product measure of P on Σ^N. Note that if $\Sigma = \{0,1,\cdots,r-1\}$ and $P(\{i\}) = 1/r$ for $i=0,1,\cdots,r-1$, then the notion of P-normal sequence coincides with the usual notion of r-adic normal sequence. The set of all P-normal sequence is denoted by Nor_P. A strictly increasing function from N to N is called a selection function. Let τ be a selection function. For $\alpha \in \Sigma^N$, the subsequence of α selected by τ is defined by $(\alpha\circ\tau)(i) = \alpha(\tau(i))$ for any $i \in N$ and denoted by $\alpha\circ\tau$. Following von Mises, $\alpha \in \Sigma^N$ is called a τ-collective if

$$\text{(1.2)} \qquad \operatorname*{w-lim}_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\alpha(i)} = \operatorname*{w-lim}_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\alpha(\tau(i))}.$$

Our problem is to characterize a selection function τ which satisfies the following conditions. Denote $\mathrm{Nor}_P\circ\tau = \{\alpha\circ\tau;\ \alpha \in \mathrm{Nor}_P\}$.

Condition 1. Any $\alpha \in \mathrm{Nor}_P$ is a τ-collective.

Condition 2. $\mathrm{Nor}_P\circ\tau \subset \mathrm{Nor}_P$

Condition 3. $\mathrm{Nor}_P\circ\tau = \mathrm{Nor}_P$

Clearly, condition 3 implies condition 2. It is also easy to verify that condition 2 implies condition 1.

In this paper, we prove that each of the above three conditions is equivalent to condition 4 stated below under a reasonable restriction that $\overline{\lim_{n\to\infty}} \frac{\tau(n)}{n}<\infty$ (Theorem 4). It should be remarked that the fact that condition 4 implies condition 2 under the restriction stated above was already obtained by Benjamin Weiss [14] in 1971.

To state condition 4, some more notions are necessary. For a selection function τ, denote by $\theta_\tau \varepsilon \{0,1\}^N$ the 0-1-sequence defined by

$$\theta_\tau(i)=\begin{cases} 1 & \text{if } i\varepsilon\{\tau(j);\ j\varepsilon N\} \\ 0 & \text{else} \end{cases} \qquad (i\varepsilon N). \tag{1.3}$$

That is, $\theta_\tau(i)=1$ if and only if the i-th coordinate is selected as a subsequence by the selection function τ. For a T-invariant measure μ on $\{0,1\}^N$, where T is the shift on $\{0,1\}^N$, the entropy of the measure-preserving transformation T on the measure space $(\{0,1\}^N,\mu)$ is denoted by $h_\mu(T)$. That is,

$$h_\mu(T)= \lim_{n\to\infty} \frac{1}{n} \sum_{\xi\varepsilon\{0,1\}^n} -\mu(\Gamma_\xi)\cdot\log \mu(\Gamma_\xi), \tag{1.4}$$

where for $\xi=(\xi_0,\xi_1,\cdots,\xi_{n-1})\varepsilon\{0,1\}^n$,

$$\Gamma_\xi=\{\beta\varepsilon\{0,1\}^N;\ \beta(i)=\xi_i \text{ for } i=0,1,\cdots,n-1\}. \tag{1.5}$$

The above limit is known to exist [2]. Following [14], $\beta \in \{0,1\}^N$ is said to be <u>completely deterministic</u> if $h_\mu(T)=0$ for any $\mu \in \{\mu_\beta^S;\ S \in \Xi_\beta\}$. Now, we state condition 4.

Condition 4. θ_τ <u>is completely deterministic</u>.

Note that condition 4 is indifferent to what Σ and P are. This condition is not only simple but also easy to check. Various types of sequences are known to be completely deterministic (Example 1).

ON KOLMOGOROV'S COMPLEXITY AND INFORMATION

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About Kolmogorov's complexity measure *K*, we prove the following theorem, which seems rather eccentric.

Theorem. *For any C, there exists a sentence y such that*

$$K(y)-K(y|x)>C$$

holds except for finitely many sentences x.