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A CHARACTERIZATION OF UNITARY OPERATORS INDUCED BY NONSINGULAR TRANSFORMATIONS AND ITS APPLICATIONS.

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In this talk we give a necessary and sufficient condition for a unitary operator on an L^2-space to be induced by a non-singular transformation and its applications.

Let (X,m) be a σ-finite measure space, $L^2(m)$ be the complex Hilbert space of all square summable functions and $L^\infty(m)$ be the algebra of all bounded measurable functions on (X,m). Then for every α in $L^\infty(m)$ we associate a bounded linear operator $T[\alpha]$ on $L^2(m)$ by

$$T[\alpha] : \xi(x) \to \alpha(x)\xi(x), \qquad \xi \in L^2(m).$$

As is well-known, the correspondence between $L^\infty(m)$ and

$$\Lambda(m) = \{T[\alpha] ; \quad \alpha \in L^\infty(m)\}$$

is isomorphic and $\Lambda(m)$ is a commutative von Neumann algebra.

A transformation f of (X,m) is a nonsingular transformation if it satisfies the following conditions :

(N.1) There exists a null set N such that f is a bijective transformation of $X-N$ onto itself.

(N.2) f is bimeasurable.

(N.3) $m(E) = 0$ if and only if $m\circ f(E) = m(f(E)) = 0$.

We say $[f;q]$ is a nonsingular pair if f is a nonsingular transformation and $q = q(x)$ is a complex measurable function such that

$$|q(x)|^2 = \frac{dm\circ f}{dm}(x), \qquad \text{a.e.}(m).$$

For every nonsingular pair $[f;q]$ we define a unitary operator $U[f;q]$ on $L^2(m)$ by

$$U[f;q] : \xi(x) \to q(x)\xi(f(x)), \qquad \xi \in L^2(m).$$

We first prove the following theorem :

THEOREM 1. Let (X,m) be a σ-finite abstract Lebesque space and U be a unitary operator on $L^2(m)$. Then there exists a nonsingular pair $[f;q]$ such that

$$U = U[f;q]$$

if and only if

$$U^{-1}\Lambda(m)U \subset \Lambda(m).$$

As applications of Theorem 1, we prove the following theorems.

THEOREM 2. Let (X,m) be a non-atomic σ-finite abstract Lebesque space and U be a unitary operator on $L^2(m)$ such that $U-I$ is a compact operator. Then U is induced by a nonsingular pair if and only if U is the identity operator.

This theorem implies that any non-trivial finite dimensional unitary operator on such an L^2-space is never induced by a nonsingular pair. Furthermore, let U be a unitary operator on such a space which is represented by a unitary matrix (α_{ij}) such that

$$\sum_{i,j=1}^{\infty} |\alpha_{ij} - \delta_{ij}|^2 < +\infty .$$

Then U is not induced by a nonsingular pair unless U is the identity.

THEOREM 3. Let $[f;q]$ be a nonsingular pair of (R^1, dx) and F be the Fourier transform. Furthermore, assume that $U[f;q]$ is a rotation of S, the nuclear space of all rapidly decreasing functions on the real line. Then $F^{-1}U[f;q]F$ is again induced by a nonsingular pair if and only if $[f;q]$ is given by

$$f(x) = \alpha x + \beta,$$

$$q(x) = \sqrt{|\alpha|}\, e^{i(\theta x+\tau)},$$

where $\alpha(\neq 0)$, β, θ and τ are real constants.

This theorem enables us to construct two one-parameter unitary groups, one of which is induced by nonsingular pairs and the another is not, and still both of them have the same spectral type.

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