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On ergodic automorphisms of compact abelian groups

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§1. Partial results towards proving that ergodic automorphisms of the torus (a compact abelian group) are Bernoulli shifts were obtained in [2, 3, 4, 6, 9, 11, 12, 13] .

Sinai [12] proved that the entropy of an automorphism of a finite-dimensional torus is equal to $\sum_{|\lambda_j|>1} \log |\lambda_j|$ where λ_j , $j = 1, 2, \ldots , k$, are characteristic values of the matrix which are in absolute value greater than unity. This problem was extedded to the general case by Geniss [3]. Under those facts, Rohlin [9] proved ~~that an ergodic automorphism of a compact metric~~ abelian group is a Kolmogorov automorphism, and Juzvinskii [4] showed more generally the above statement. That ergodic automorphisms of the 2-dimensional torus is mixing Markov shifts was proved by Adler and Weiss [13]. The work on the isomorphism problem has been dominated by the astonishing results of Ornstein [6, 7, 8]. He developed new technique to show that Bernoulli shifts with the same entropy are isomorphic, and then to extended those isomorphism results to wider classes of transformations. By using results of Ornstein, Katznelson [5] showed the following : Ergodic automorphisms of a finite-dimensional torus are Bernoulli shifts. In weiss [13], he annaunces that ergodic automorphisms of a solenoidal group are Bernoulli shifts.

We shall show that ergodic automorphisms of an infinite-dimensional torus are Bernoulli shifts.

§2. Let $(X,\mathcal{F},m)$ be a nonatomic Lebesgue probability space and let T be an invertible measure preserving transformation of X. If T is ergodic on a σ-field $\mathcal{F}$ and $\mathcal{F}$ is a increasing union of invariant sub-σ-fields $\mathcal{F}_i$ such that each dunamical system $(X,\mathcal{F}_i,m,T)$ is Berhoulli, then T itself is Bernoulli.

If the entropy of T is finite on each $\mathcal{F}_i$, then this lemma was proved by Ornstein. Hence we shall show the case of T with the infinite entropy on each $\mathcal{F}_i$.

Let $\hat{G}$ be the character group of a compact metric abelian group G. Since G is metrizable, $\hat{G}$ is countable, $\hat{G} = \{f_1, \dots, f_n, \dots\}$. We denote by $gp\{Q\}$ the subgroup generated by the subset Q of $\hat{G}$.

<u>Proposition 1.</u> Let $\hat{G}$ be the character group of a compact metric abelian group G such that $\hat{G} = gp\{U_\sigma^m f_i : m = 0, 1, 2, \dots ; i = 1, 2, \dots, n\}$ and $U_\sigma(\hat{G}) = \hat{G}$. If $\sigma : G \longrightarrow G$ is ergodic, then is the Bernoulli shift.

Let $\hat{G}$ be a compact metric abelian group, let σ be an automorphism of G and let $\hat{G}$ be the character group of G. Then each element g of $\hat{G}$ satisfies always one of the following conditions :

$$(*) \quad g^{n_o} = U_\sigma g^{n_1} \cdots U_\sigma^k g^{n_k}$$

for some integers $n_o, n_1, \dots, n_k$, and

$$(**) \quad 1 = g^{n_o} U_\sigma g^{n_1} \cdots U_\sigma^k g^{n_k}$$

for all integers $n_o, n_1, \dots n_k$.

We denote by $\langle g \rangle$ the free cyclic group of an element g. Suppose that there exists a character g of G such that g satisfies the condition (**). Then a cyclic group of g, $\langle g \rangle$, is free and an infinite product group $\otimes_{-\infty}^{\infty} U_\sigma^j \langle g \rangle$ is a subgroup of $\hat{G}$. Let

$\hat{K} = \otimes_{-\infty}^{\infty} U_{\sigma}^{j} \langle g \rangle$ and let K' be the character group of $\langle g \rangle$. Clearly, the character group of $K = \otimes_{-\infty}^{\infty} \sigma^{j}(K')$ is $\hat{K}$. $\beta = \mathcal{E} \otimes (\otimes_{i \neq 0} \sigma^{i}(2))$ where 2 is the trivial field of K' and $\mathcal{E}$ is the σ-field of K', is a sub σ-field of K, and β is a Bernoulli generator for (K, σ). Here σ is an automorphism of K. β is a Bernoulli generator for σ $(\sigma : K \longrightarrow K)$.

The above conditions (*) and (**) will use essentially to study the existence of weak Bernoulli partitions for (G, σ). For characters $f_1, \ldots, f_n$ of $\hat{G}$, let

$$\hat{G} = gp\{U_{\sigma}^{m} f_i : m = 0, \pm 1, \pm 2, \ldots ; i = 1, 2, \ldots, n\}.$$

If the automorphism $\sigma : G \longrightarrow G$ has finite entropy, then each of f_i satisfies the condition (*).

Let G be an infinite-dimensional torus and let

$$\hat{G}_1 = \{ f \in \hat{G}: f^{n_0} U_{\sigma} f^{n_1} \ldots U_{\sigma}^{k} f^{n_k} = 1 \text{ for some } n_0, \ldots, n_k \}.$$

If $G_1 = \mathrm{ann}(\hat{G}_1)$, then the character group of the factor group G/G_1 is $\hat{G}_1$, and the character group of G_1 is $\hat{G}/\hat{G}_1$.

For an arbitrary $g\hat{G}_1 \in \hat{G}/\hat{G}_1$, the direct product group $\hat{K} = \otimes_{-\infty}^{\infty} U_{\sigma}^{j} \langle g \rangle$ is the subgroup of $\hat{G}$. Here we have the following

<u>Proposition 2.</u> If G is an infinite-dimensional torus with the character group of the form $\hat{G} = \hat{G}_1 \circledS \hat{K}$ where $\hat{G}_1$ and $\hat{K}$ are as in the preceding notations and if σ is an ergodic, then σ is the Bernoulli shift.

<u>Proposition 3.</u> If G is an infinite-dimensional torus and if its character group $\hat{G}$ is $\hat{G}_1 \circledS \hat{K}_1 \circledS \ldots \circledS \hat{K}_n$ where $\hat{G}_1$ and $\hat{K}_i$, $i = 1, 2, \ldots, n$, are as in the preceding notations, then σ is the Bernoulli shift.

Let G be an infinite-dimensional torus with the caracter group $\hat{G} = \hat{G}_1 \circledS \hat{K}_1 \circledS \dots \circledS \hat{K}_n$. If there exists the least integer $k > 0$ such that

$$(\hat{G}_1 \circledS \hat{K}_1 \circledS \dots \circledS \hat{K}_k) \cap (\hat{K}_{k+1} \circledS \dots \circledS \hat{K}_n) = 1$$

where $\hat{G}_1$ and $\hat{K}_i$, $i = 1, 2, \dots, n$, are defined as in the preceding notations, then $\hat{G} = (\hat{G}_1 \circledS \hat{K}_1 \circledS \dots \circledS \hat{K}_k) \otimes (\hat{K}_{k+1} \circledS \dots \circledS \hat{K}_n)$. We write

$$\tilde{K} = \mathrm{ann}(\hat{G}_1 \circledS \hat{K}_1 \circledS \dots \circledS \hat{K}_k) ,$$

$$\tilde{G} = \mathrm{ann}(\hat{K}_{k+1} \circledS \dots \circledS \hat{K}_n) .$$

Then it is clear that G is the direct product group of $\tilde{K}$ and $\tilde{G}$, $\sigma(\tilde{K}) = \tilde{K}$ and $\sigma(\tilde{G}) = \tilde{G}$. There exists the largest integer $q < k$ such that

$$\hat{G}_1 \circledS \hat{K}_1 \circledS \dots \circledS \hat{K}_k = \hat{G}_1 \otimes \hat{K}_1 \otimes \dots \otimes \hat{K}_q \circledS \dots \circledS \hat{K}_k ,$$

$$(\hat{G}_1 \otimes \hat{K}_1 \otimes \dots \otimes \hat{K}_q) \cap \hat{K}_i = 1 , \quad i = q+1, \dots , k .$$

<u>Proposition 4.</u> Let $\tilde{G}$ be as in the above notations. If $\sigma : \tilde{G} \longrightarrow \tilde{G}$ is an ergodic automorphism, then σ is the Bernoulli shift.

<u>Proposition 5.</u> Let G be an infinite-dimensional torus and let σ be an automorphism of G. If its character group $\hat{G}$ is of the form $\hat{G}_1 \circledS \hat{K}_1 \circledS \dots \circledS \hat{K}_n$, and if σ is ergodic, then σ is the Bernoulli shift.

<u>Proposition 6.</u> Let G be an infinite-dimensional torus. If $\hat{G}/\hat{G}_1$ is infinite countable and if $\sigma : G \longrightarrow G$ is ergodic, then σ is the Bernoulli shift.

Here the main result of this paper is summarized as follows

<u>Theorem.</u> An ergodic automorphism of an infinite-dimensional torus is the Bernoulli shift.

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