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β -transformation and related topics

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SECTION 1 Introduction

For the Bernoulli scheme B ($\frac{1}{n},..,\frac{1}{n}$) there have been studied various kind of properties : the transversal flow, the central limit theorem, the construction of normal sequence etc. It is the case for the schemes B ($p_1,..,p_n$) , $p_i \geq 0$, $\sum p_i=1$ except that the transversal flow must be replaced by the transversal field and that the construction of normal sequence is slightly complicated. The difference occurs in that the metrical entropy does or does not coincide with the topological entropy. (For such difference a synthetic treatment will be given in section 6 by introducing a notion of statistical physics, namely, free energy or pressure.)

The situation is similar for the Markov schemes, and so for the dynamical systems of finite type with invariant measures of maximal entropy (or free energy). Let us restrict ourselves to study the dynamical systems which can be realized in a certain natural way by subshifts not of finite type, but close to finite type, and, finding natural invariant measures, we shall investigate the properties mentioned above for Bernoulli schemes.

The main objects which will be treated are β -transformations ($\beta > 1$) and $\underline{\underline{\beta}}$ -transformations ($\underline{\underline{\beta}} = (\beta_1, .., \beta_n)$, $\beta_i > 1$) since they have the following properties which fit for our purpose:

(1) Except for a countable number of β's or $\underline{\underline{\beta}}$'s (which are algebraic and correspond to the case of finite type) they ate not of finite type, but close to finite type so that the sequence space obtained by realization can be well approximated by Markov subshifts.

(2) The β - and the $\underline{\underline{\beta}}$-transformatioms are realized by the sequence spaces of the same type and the natural invariant measures, and correspond to the measures on the sequence space with and without the maximal entropy, respectively.

(3) In general every measure preserving transformations with finite entropy can be represented as f-transformations for some increasing f on [0,1] , and the β - and the $\underline{\underline{\beta}}$ -transformations are elementary and typical examples among f-transformations.

SECTION 2 Definitions and Preliminaries

In this section we state the definitions and fundamental theorems for β -transformations and related notions for the sake of future.

Let β be a real number such that $s-1<\beta \leq s$ for some integer $s \geq 2$, the β-expansion of a real number t is the expression of the form:

(1) $$t = a_{-1}+\sum_{n\geq 0} a_n \beta^{-n-1}$$

where a_{-1} is an integer and, for $n\geq 0$, $a_n \in A = \{0,1,\ldots,s-1\}$
The expression can be uniquely determined via β-transformation which has been studied by Renyi [9] and W.Parry [8], and is defined on the unit interval $[0,1)$ by the relation:

(2) $$T_\beta t \equiv \beta t \qquad (\mathrm{mod}\ 1)$$

Let π_β be the map of the unit interval $[0,1)$ into the infinite product space $\Omega = A^N = \{0,1,\ldots,s-1\}^N$ defined as follows:

(3) $$\pi_\beta(t)(n) = k, \qquad \text{if} \quad k\beta^{-1} \leq T^n t < (k+1)\beta^{-1}$$

where $T^0_\beta t = t$, $T^{n+1}_\beta t = T_\beta(T^n_\beta t)$ $(n\geq 0))$, and $\pi_\beta(t)(n)$ is a n-th coordinate of sequence $\pi_\beta(t)$.

It is proved in [8] that the expression (1) holds for $a_n = \pi_\beta(t)(n)$, $n\geq 0$, $a_{-1}=0$ in the case of $t\in[0,1)$;
in other words

(1') $$t = \rho_\beta(\pi_\beta(t))$$

where

(4) $$\rho_\beta(\omega) = \sum_{n\geq 0} \omega(n)\beta^{-n-1}$$

for $\omega \in A^N$.

Let Y_β be the image $\pi_\beta([0,1))$ and X_β its closure in the product space with the product topology.

Definition 2.1. The subshift (X_β, σ) will be called β-subshift.

In general the transformation of the type

$$T_f(t) \equiv f(t) \qquad \pmod 1$$

where f is a map of $[0,1)$ to $[0,+\infty)$ is called f-transformation.

If f is piece-wise continuously differentiable, $f(0) = 0$, and f' is positive and bounded, then the symbolic structure is the same as X_β for some β.

The space Ω is endowed with the lexicographical order $\omega > \omega'$ if and only if there exists an integer n such that $\omega(k) = \omega'(k)$ for $k < n$ and $\omega(n) > \omega'(n)$. The shift transformation on the space $\Omega = A^N$ will be denoted by σ: $\sigma\omega(n) = \omega(n+1)$. We set

(5) $$T_\beta^n 1 = \lim_{t \uparrow 1} T_\beta^n t$$

and $$\pi_\beta(1) = \max X_\beta = \omega_\beta$$

Proposition 2.2. (Representation theorem)

1). $\sigma \circ \pi_\beta = \pi_\beta \circ T_\beta$ on $[0,1)$.

2). $\pi_\beta : [0,1] \to X_\beta$ is an injection and is strictly order-preserving, i.e. $t < s$ implies that $\pi_\beta(t) < \pi_\beta(s)$.

3). $\rho_\beta \circ \pi_\beta$ is identity on $[0,1]$.

4). $\rho_\beta \circ \pi_\beta = T_\beta \circ \rho_\beta$ on Y_β.

5). $\rho_\beta : X_\beta \to [0,1]$ is a continuous surjection and is order-preserving, i.e. $\omega < \omega'$ implies that $\rho_\beta(\omega) \leq \rho_\beta(\omega')$.

6). The inverse image $\rho_\beta^{-1}(t)$ of $t \in [0,1]$ consists either of a one point $\pi_\beta(t)$ or of two points $\pi_\beta(t)$ and $\sup_{s<t} \pi_\beta(s)$.

The latter case occurs only when $T_\beta^n t = 0$ for some $n \geq 0$.

7). In particular, $\rho_\beta(\omega)$ is one-to-one except for a countable number of point $\omega \in X_\beta$.

The sequence ω_β is called β-expansion of one, and plays an elementary role for the analysis of sequence space X_β as we shall see later. For example,

Proposition 2.3.

(6) $$X_\beta = \left\{ \omega \in A^N \middle| \sigma^n \omega \leq \omega_\beta \text{ for all } n \geq 0 \right\}$$

For the sake of future we introduce an important class of symbolic dynamics. A subshift is a pair (X,σ) where X is a σ-invariant closed subset of the product space A^Z or A^N and the letter σ stands for the restriction $\sigma | X$ to the subset X of the shift transformation.

Thus subshift are topological dynamics with canonical generator $\left\{ [a] \middle| a \in A \right\}$. Here for a word u over the alphabet set A, in other words for $u \in \bigcup_{n \geq 1} A^n$, [u] denotes the corresponding cylinder set :

$$[u] = \left\{ \omega \middle| \omega(k) = a_k, \quad 0 \leq k \leq n \right\}$$

if $u = (a_0, \ldots, a_n)(a_k \in A)$.

Definition 2.4. A subshift (X,σ) will be called Markov subshift (or subshift of finite type) of order p if there is a subset W of A^{p+1} ($p > 0$) such that

(7) $$X = M(W) = \left\{ \omega \middle| (\omega(n), \ldots, \omega(n+p)) \in W \quad \text{for any } n \right\}.$$

The set W will be called the structure set of Markov subshift (X,σ). The structure matrix $M = (M_{uv})_{u,v \in A^p}$ is defined as follows:

(8) $$M_{uv} = \begin{cases} 1 & \text{if } u = (a_0, \ldots, a_{p-1}), \quad v = (a_1, \ldots, a_p) \\ & \quad \text{for some } (a_0, \ldots, a_p) \in W \\ 0 & \text{otherwise.} \end{cases}$$

Using above definition, we can see the following result.

Theorem 2.5. Let $\beta > 1$. Then the following three conditions are equivalent $(p \geq 1)$:

1). The subshift (X_β, σ) is Mardov and its order is strictly equal to p.

2). There exist integers a_i, $i = 0,\dots,p$. $0 \leq a_i < s$, such that

a) $1-\beta^{-p-1} = \sum_{j=k}^{p} a_j \beta^{-j-1}$

b) $1-\beta^{-p-1} > \sum_{j=0}^{k} a_{j+k}\beta^{-j-1}$ $(k=1,\dots,p)$

where we set $a_{n+p+1} = a_n$ for $n \geqslant 0$.

3) The sequence ω_β is periodic with period $p+1$,

i.e.

a') $\sigma^{p+1}\omega_\beta = \omega_\beta$

and

b') $\sigma^{q}\,\omega_\beta < \omega_\beta$ for any $q = 1,\dots,p$.

Thus the β-shifts are Markovian for countably and densely many $\beta \in (1,+\infty)$. The following proposition shows that the set X_β is well-approximated by Markovian X_{β_n}'s.

Proposition 2.6.

1). If $1<\beta \leqslant \alpha$, then $Y_\beta \subset Y_\alpha$

2). For any $\beta > 1$,

a) $X_\beta = \overline{\bigcup_{\alpha<\beta} X_\alpha}$. b) $X_\beta = \bigcap_{\alpha>\beta} X_\alpha$

Let $W_n(X_\beta)$ is the words of length n which appears in β-subshift (X_β, σ), $(n \geqslant 1)$. These word sets are also endowed with lexicographical order. Let

$$W_n^0 = \left\{ (a_1,\dots,a_n) \in W_n(X_\beta) \;\middle|\; (a_1,\dots,a_{n-1},a_n+1) \in W_n(X_\beta) \right\}.$$

(9)

$$W_n^0(u) = \left\{ u\cdot v \in W_{n+k}(X_\beta) \;\middle|\; u\cdot v \in W_{n+k} \right\}$$

where $n \geqslant 1$. $u \in W_k(X_\beta)$, $k \geqslant 0$, and the symbol "$\cdot$" denotes the concatenation, i. e.

(10) $$u\cdot v = (a_1,\dots,a_n,b_1,\dots,b_m)$$

if $u = (a_1,\dots,a_n)$ and $v = (b_1,\dots,b_m)$ (the number m may be infinite).

The empty word ε is a symbol such that $\varepsilon \cdot u = u \cdot \varepsilon = u$ for any word u.

Finally we set $\quad W_0^o(u) = \{ u \}$.

Proposition 2.7. For any $k \geqslant 0$ and a word $u \in W_k(X_\beta)$

$$W_n(u) = \bigcup_{k=1}^{n} W_k^o(u) \cdot \omega_\beta[0,n-k) \cup \{\max W_n(u)\}$$

In particular $\quad W_n(\varepsilon) = W_n = \bigcup_{k=0}^{n} W_k^o \cdot \omega_\beta[0,n-k)$

where

$$\text{(11)} \qquad \omega_\beta[0,j) = \begin{cases} (\omega_\beta(0),\ldots,\omega_\beta(j-1)) & (j > 1) \\ \\ \varepsilon \text{ (empty word)} & (j=0) \end{cases}$$

We note that the sets $[w] = \{ \omega \in X_\beta \mid (\omega(0),\ldots,\omega(n) = w \}$, $w \in W_{n+1}(X_\beta)$ form a partition of the set X and that $\rho_\beta([w]) = \{ \rho_\beta(\omega) \mid \omega \in [w] \}$, $w \in W_{n+1}(X)$ form a covering of the unit interval by intervals, any two of which have at most one common point. Let $R_\beta(w)$ be the length of interval $\rho_\beta([w])$.

Now we can obtain a convergence theorem on the number of words.

Theorem 2.8. Let $u \in W_k(X_\beta)$ and $M_\beta = \sum_{n \geqslant 0} (n+1)\omega_\beta(n)\beta^{-n-1}$.

a) $$\lim_{n\to\infty} \beta^{-n} \mathrm{Card}\,(W_n^o(u)) = \frac{\beta^k R_\beta(u)}{M_\beta}$$

b) $$\lim_{n\to\infty} \beta^{-n} \mathrm{Card}\,(W_n(u)) = \frac{\beta^k R_\beta(u)}{M(1-\beta^{-1})}$$

Renyi [9] showed that there exists an invariant probability measure for β-transformation on the unit interval $[0,1)$ which is absolutely continuous with respect to Lebesgue measure and has entropy $\log \beta$.

That measure induces a shift-invariant probability measure

$d\mu_\beta(\omega) = f_\beta(\omega)\, d\rho_\beta(\omega)$ on X_β, where

$$\text{(12)} \qquad \rho_\beta(\omega) = \sum_{n \geqslant 0} \omega(n)\beta^{-n-1}$$

$d\rho_\beta$ is Stieltjes integral on the ordered space X_β .

(13) $$f_\beta(\omega) = M_\beta^{-1} \sum_{n\geq 0} \beta^{-n-1} I(\omega \leq \sigma^n \omega_\beta)$$

(14) $$M_\beta = \sum_{n\geq 0} (n+1)\, \omega_\beta(n)\, \beta^{-n-1}$$

and $I(\omega \leq n)$ denotes the indicator function of the set $\{\omega \mid \omega \leq n\}$.

The proof of invariance of μ_β is immediate in our symbolical form.

We note that the density function $f_\beta(\omega)$ is the unique solution (up to scalar multiplication) of Mf = f where

$$Mf(\omega) = \beta^{-1} \sum_{a.\omega \in X_\beta} f(a\cdot\omega)$$

(See Section 4 and Section 6)

Definition 2.9. The endomorphism $(X_\beta, \mu_\beta, \sigma)$ will be called β-endomorphism and its natural extension $(\overline{X}_\beta, \overline{\mu}_\beta, \overline{\sigma})$ β-automorphism

The $\underline{\beta}$ -transformation ($\underline{\beta} = (\beta_0, \ldots, \beta_{s-1})$, $\beta_i > 1$ and $\sum_{i<s-1} \beta_i^{-1} < 1 < \sum_{i\leq s-1} \beta_i^{-1}$) is an f-transformation with

$$f(t) = (\beta_k \cdot t) + k \qquad \text{if} \quad \sum_{i<k} \beta_i^{-1} \leq t < \sum_{i\leq k} \beta_i^{-1} \qquad k=0,\ldots,s-1,$$

It has also a natural invariant measure μ_β in the sense that it is absolutely invariant with respect to the Lebesgue measure. (See Section 4)

The realized subshift is (X_β, σ) for some $\beta > 1$. The difference in the measure-theoretical study of them reflects the fact that the entropy $h(\mu_{\underline{\beta}})$ less than the topological entropy $\log\beta = h(\mu_\beta)$. From another point of view, there are various shift invariant measures μ_f on the same space X_β and corresponding endomorphisms $(X_\beta, \sigma, \mu_\beta)$ (or automorphisms $(\overline{X}_\beta, \overline{\sigma}, \overline{\mu}_f)$) among which $(X_\beta, \sigma, \mu_\beta)$ is characterized as the system with maximal entropy.

SECTION 3 Periodic Points and Normal Sequence

Let us show two results obtained from the symbolic structrre of β-

shifts. The location (distribution) of periodic points is important for the study of dynamical systems. For example, from the denseness it follows that there exists a regular invarant measure and the increasing order of periodic plints is reflected to the ergodic property of invariant measures.

These properties of periodic points are, however, known only for the dynamical systems of finite type, i. e. the systems which can be realized by Markow subshifts. The subshift (X_β, σ) offers an example of non-Markov subshift with these properties.

On the other hand normal sequences are constructed for Bernoulli schemes (Champernowne) and Markov schemes (Postnikov). The construction depends on the fact that the invariant measure is the most unifoumly distributed on the given sequence space. This is also possible for our systems.

Now we sketch the two facts mentioned above. Let

$$P_n = \{\omega \mid \sigma^n \omega = \omega,\ \omega \in X_\beta\}$$

then

$$P_n = W_n^o \cup \bigcup_{k=1}^{n-1} \{u \in W_{n-k}^0,\ \omega_\beta[0,k)\cdot u \in W_n\}$$

and

$$\mathrm{Card}(P_n) = \mathrm{Card}\,(W_n^o) + \sum_{k=1}^{n-1} \mathrm{Card}\ W_{n-k}^o(\omega_\beta[0,k))$$

Applying the property 2.8. of Section 2 we obtain

Theorem 3.1. The periodic points of (X_β, σ) is dense in X_β, and they increase exponentially:

$$\lim_{n\to\infty} \beta^{-n} \mathrm{Card}\ (\mathrm{Per}_n(X_\beta, \sigma)) = 1$$

In particular

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Card}\ (\mathrm{Per}_n(X_\beta, \sigma)) = \log \beta = \mathrm{ent}(X_\beta, \sigma)$$

We note that the conclusion of Theorem 3.1. holds also for Markov subshift.

Generally we can show the following property on the periodic points.

(1) For a subshift (X,σ) the periodic points are dense in X if and only if it can be accesible from below by Markov subshifts in the sense that

$$X = \text{closure} \bigcup [\, Y : (Y,\sigma) \text{ Markov}]$$

(2) If the periodic points are dense in X and

$$\sup_{Y:\text{Markov}} \text{ent}\,(Y,\sigma) = \text{ent}\,(X,\sigma)$$

then the periodic points increase exponentially;

$$\lim_{n\to\infty} \frac{1}{n} \log \text{Card}\, [\,\text{Per}_n\,(X,\sigma)] = e(X,\sigma).$$

For the Bernoulli scheme $B(\frac{1}{p},\frac{1}{p},\ldots,\frac{1}{p})$, Champernowne constructed the following sequence : Let V_n be the set of all words of length n, and let $v_1^n \leqslant v_2^n \leqslant \cdots\cdots \leqslant v_{p^n}^n$ be its elements in lexicographical order. Then

$$\omega = v_1^1 \circ \cdots\cdots \circ v_p^1 \circ v_1^2 \cdots v_{p^2}^2 \cdots\cdots \circ v_1^n \circ \cdots\cdots \circ v_{p^n}^n \circ v_1^{n+1} \cdots$$

The normality follows from the "uniform denseness" of the orbit of ω in the sequence space.

Analogically, setting $V_n = \left\{ v_1^n \leqslant \cdots\cdots \leqslant v_{N_n}^n \right\}$ to be the set of all words of length n in X_β, we can prove:

Theorem 3.2. The sequecce $\omega = v_1^1 \circ \cdots\cdots \circ v_{N_1}^1 \circ v_1^2 \circ \cdots \circ v_{N_2}^2 \circ \cdots$ is the β-normal sequence, i. e.

$$\lim_{n\to\infty} \frac{1}{n} \text{Card} \left\{ \ell \mid 0 \leqslant \ell \leqslant n-k \;,\; \omega[\ell,\ell+k) = u \right\} = \mu_\beta[u]$$

SECTION 4 Ergodic Properties and Isomorphism Problem

In this section we study the ergodic properties, including central limit theorem. We shall sketch the construction of an isomorphism of β-transformation $(X_\beta, \sigma, \mu_\beta)$ to a mixing Markov automorphism using the uniqueness of invariant measure with maximal entropy. Although the result can be generalized to $\boldsymbol{\beta} = (\beta_0, \dots, \beta_{s-1})$-automorphism we must introduce a new notion, free energy. A direct verification of Ornstein's weak Bernoulli condition is given in 4.3. In the last 4.4., the central limit theorem is studied for our cases.

Theorem 4.1. A β-automorphism is isomorphic to a mixing Markov automorphism.

Theorem 4.2. A $\boldsymbol{\beta}$-automorphism is isomorphic to a mixing Markov automorphism.

Let $a_0, a_1, \dots,$ be the β- or $\boldsymbol{\beta}$-expansion of one,

and

$$\tau(\omega) = \sup \{ i : i \geq 1, \ (\omega(-i), \dots, \omega(-1)) = (a_0, \dots, a_{i-1}) \}$$
$$\in \{ 0, 1, \dots, \infty \}$$

The isomorphism φ is given by

$$\varphi(\omega)(n) = \begin{cases} i & \text{if} \quad \omega \in \sigma^n(\tau = i) \quad i \geq 1 \\ -\omega(-1) & \text{if} \quad \omega \in \sigma^n(\tau = 0) \end{cases}$$

Then the image of φ is contained in a Markov subshift $(\mathcal{M}(M), \sigma)$ over a symbol set $I = \{ -(s+1), \dots, -1, 0, 1, \dots, \infty \}$ where

(1) $$M_{ij} = \begin{cases} 1 & \text{if} \quad i = j+1 < \infty, \quad \text{if} \quad i \leq 1, \quad j \leq 0, \quad \text{or if} \quad i = j = \infty \\ 0 & \text{if} \quad 1 < i \leq \infty \text{ and } i = j+1, \quad \text{if} \quad i = 1 \quad \text{and } 1 \leq j \leq \infty, \\ & \text{or if} \quad i = \infty \text{ and } j < \infty \quad , \end{cases}$$

$$M_{i\infty} = \limsup_{j \to \infty} M_{ij}$$

and

(2) $$M_{ij} = \begin{cases} 1 & \text{if} \quad a_j > |i| = -i \\ 0 & \text{otherwise} \end{cases}$$

Lemma 1 If M verifies (1), then the invariant measure λ_M of $(\mathfrak{M}(M).\sigma)$ with maximal entropy is unique and Markovian : the stationary measure $(\pi_i, i \in I)$ and transition propability $(p_{ij}, \ i,j \in I)$ is

$$\pi_i = x_i y_i \Big/ \sum_j x_j y_j \ , \qquad p_{ij} = M_{ij} x_j \Big/ \rho\, x_i$$

where

$$x_i = \begin{cases} \rho^{-i} & (i \geqslant 1) \\ \sum_{k \geqslant 0} M_k \rho^{-k-1} & (i \leqslant 0) \end{cases}$$

$$y_j = \begin{cases} \rho^j \sum_{k \geqslant j} M_k \rho^{-k-1} & (j \geqslant 1) \\ 1 & (j \leqslant 0) \end{cases}$$

and the value ρ is the unique positive solution of the equation

$$1 = \sum_{k \geqslant 0} M_k \rho^{-k-1} \ ; \qquad M_k = \sum_{i \leqslant 0} M_{ik} \ .$$

The entropy $h(\mu) = \log \rho = \mathrm{ent}\ (\mathfrak{M}(M), \sigma)$.

Then the image $\varphi(\mu_\beta)$ must coincide with λ_M since $h(\varphi(\mu_\beta)) = \log \beta$ and $\rho = \beta$ in our case (2).

In the case of β-transformation we use the notion of free energy instead of topological entropy, i. e. we seek a measure λ for which the function

$$f_v(\lambda) = h(\lambda) - \int U d\lambda$$

attains the maximam for a suitable function U on $\mathfrak{M}(M)$. (See Section 6 for the general definition.)

Let $U_\beta(\omega) = U_\beta(\omega(0), \omega(1))$ be given by

$$U_{\beta}(i,j) = \begin{cases} \log \beta_{-i} & i \leqslant 0 \quad j \leqslant 0\ , \quad \text{or} \quad j > 0 \quad \text{and} \quad a_j > |i| \\ \log \beta_N & i = 1\ , \quad j \leqslant 0 \\ \log \beta_{a_{j-1}} & i = j-1 \geqslant 1 \\ +\infty & \text{otherwise} \end{cases}$$

Lemma 2. The invariant measure $\mathcal{V}_U$ with maximal free energy for U is unique and Markov ;

$$y_i = \begin{cases} 1 & i \leqslant 0 \\ \\ T^i_{\beta} 1 & i \geqslant 1 \end{cases} \qquad x_i = \begin{cases} \beta^{-1}_{-i}\beta_N + \sum_{j\geqslant 1} M_{ij}\, \beta^{-1}_{a_0} \cdots \beta^{-1}_{a_{j-1}} & i \leqslant 0 \\ 1 & i = 1 \\ \beta^{-1}_{a_0} \cdots \beta^{-1}_{a_{i-1}} & i \geqslant 2 \end{cases}$$

In [3], there is given a general condition for the uniqueness on U, called Perron-Frobenius type.

Lemma 3 ([3]) Let U be of Perron-Frobenius type on I^Z. An automorphism (X, T, μ) is isomorphic to $(I^Z, \sigma, \mathcal{V}_U)$ if there is bi-Borel injection φ of X into I^Z, such that for some μ-integrable functions V(x) and F(x)

(1) $\sigma \circ \varphi = \varphi \circ T$ $\qquad \mu$-a.e. ,

(2) $U(\varphi(x)) = V(x) + F(x)$ $\qquad \mu$-a.e. and $\int F\, d\mu = 0$,

(3) $f_U(\mathcal{V}_U) = h(\mu) - \int V\, d\mu$.

Now Theorem 4.2. follows easily from these lemmas setting $\varphi = \varphi_{\beta}$ and $U = U_{\beta}$. It is now obvious that $(\bar{X}_{\beta}, \bar{\sigma}, \bar{\mu}_{\beta})$ is Bernoullian. But we can also show the following.

Theorem 4.3. The β-automorphism $(\bar{X}, \bar{\sigma}, \bar{\mu}_{\beta})$ satisfies the Ornstein's weak Bernoulli condition :

$$\limsup_{k\to\infty}\ \sup_{p} \sum_{u,v\in A^p} \left| \mu([U] \cap \sigma^{-k-p}[v]) - \mu([v])\mu([v]) \right| = 0$$

For the proof, let us introduce an operator $S = S_\beta$,

$$S\varphi(\omega) = \beta^{-1}\sum \varphi(a\cdot\omega)$$

where the sum is taken over $a \in A$ such that

$$a\cdot\omega = a\cdot\omega(0)\cdot\omega(1)\ldots \quad \in X_\beta$$

Lemma 4. S_β is a nonnegative operator on the space of Borel functions on X_β and satisfies the following properties :

a) $$\int_{X_\beta} S_\beta \phi(\omega).\psi(\omega)\, d\mathfrak{f}_\beta(\omega) = \int_{X_\beta} \phi(\omega)\psi(\omega)\, d\mathfrak{f}_\beta(\omega)$$

wherever $\phi \in L^1(X_\beta, d\mathfrak{f}_\beta)$ and $\psi \in L^\infty(X_\beta, d\mathfrak{f}_\beta)$.

In particular, S_β is a nonnegative contraction operator on $L^1(X_\beta, d\mathfrak{f}_\beta)$ such that

$$\int_{X_\beta} S_\beta \phi(\omega)\, d\mathfrak{f}_\beta(\omega) = \int_{X_\beta} \phi(\omega)\, d\mathfrak{f}_\beta(\omega)$$

b) S_β is a bounded operator on $L^\infty(X_\beta, d\mathfrak{f}_\beta)$ and

$$\limsup_{n\to\infty \, \|\phi\|_\infty \leq 1} \left| S_\beta^n \phi(\omega) \right| = \frac{1}{S_\beta(1-\beta^{-1})}$$

c) Let ϕ be a continuous function on X_β.

$$\lim_{k\to\infty} \left\| S_\beta^k \phi - C(\phi) f_\beta \right\| = 0 \qquad \text{where } C(\phi) = \int \phi \, d\mathfrak{f}_\beta ,$$

and the convergence is uniform on the set $\Phi = \bigcup_{n\geq 1} S^n \Phi_n$

where

$$\Phi_n = \left\{ \Phi \,\middle|\, \|\phi\|_\infty < 1, \quad \phi(\omega) \text{ depends only on } \omega(k), \quad 0 \leq k \leq n \right\}$$

Sketch of the proof.

a) is trivial. b) and c) follow from Theorem 2.8. on the classification of words in X_β and the convergence of β^{-n} Card $(W_n(u))$. We only note that the proof is completed by approximating S^n by the operators $S^n(m)$.

$$S^n(m)\varphi(\omega) = \sum_{j=0}^{m} \beta^{-k} \sum_{v \in W^0_{k-j}} \varphi(v\cdot\omega_\beta[0,j)\cdot\omega) .$$

Proof of Theorem. The Ornstein's is equivalent to the following condition.

(*) $$\lim_{k\to\infty} \sup_{n} \sup_{\phi\in\Phi_n} \sum_{v\in A^n} \left| \int_{\sigma^{-k-n}[v]} \phi \, d\mu - \int \phi \, d\mu \, \mu([v]) \right| = 0$$

Now we prove that the measure $\bar{\mu}_\beta$ satisfies this condition. It suffices to show (*) for μ_β. It follows from Lemma 4.a).

$$\int_{\sigma^{-k-n}[v]} \phi \, d\mu_\beta = \int \phi(\omega) f_\beta(\omega) 1_{[v]}(\sigma^{k+n}\omega) \, d\rho_\beta(\omega)$$
$$= \int_{[v]} S^{k+n} (\phi f_\beta) \, d\rho_\beta .$$

Therefore

$$\Delta_n^k(\phi) = \sum_{v\in A^n} \left| \int_{\sigma^{-k-n}[v]} \phi \, d\mu_\beta - \int \phi \, d\mu_\beta \, \mu_\beta([v]) \right|$$
$$= \sum_{v\in A^n} \left| \int_{[v]} \left\{ S^{k+n}(\phi f_\beta) - \left(\int \phi \, d\mu_\beta\right) f_\beta \right\} d\rho_\beta \right|$$
$$\leq \int \left| S^{k+n}(\phi f_\beta) - \left(\int \phi \, d\mu_\beta\right) f_\beta \right| d\rho_\beta$$

Applying c) of Lemma 4, we can show

$$\Delta_n^k(\varphi) \longrightarrow 0$$

as $k \to \infty$ uniformly in $n \geq 1$.

Finally we show the central limit theorem for β-transformations. The proof depends only on the following estimate of the operator $S = S_\beta$ or $\bar{S}_\beta$.

~~Lemma 4'~~ There exists a sequence $\psi(k)$, $k \geq 0$ such that

(1) $$\sum_{k=1}^{\infty} \psi(k)^{\frac{\delta}{2+\delta}} < +\infty \qquad \text{for some } \delta > 0$$

(2) $$\left\| S^{k+n}\varphi - c(\varphi) f_\beta \right\|_\infty \leq \psi(k) \|\varphi\|_\infty ,$$

for any $n \geq 0$ and any tame function φ on X_β of length n, where $c(\varphi) = \int \varphi \, d\rho_\beta$.

From Lemma 4' we can easily show the followings :

Lemma 5. For any Borel subset B of [0,1)

$$|\mu_\beta(B) - \lambda(T_\beta^{-k}B)| \leqq \psi(k)\lambda(B) \qquad (k \quad 0)$$

where λ is Lebesgue measure.

Lemma 6. The canonical generator R on [0,1) satisfies the strong mixing condition of Rosenblatt ; there exists a sequence $\varphi(k)$, $k \geq 1$ such that for any $h, k \geq 1$, $A \in \mathcal{F}(\bigvee_{i=0}^{h-1} T^{-i}R)$ any measurable B,

$$|\mu_\beta(A \cap T^{-(k+h)}B) - \mu_\beta(A)\mu_\beta(B)| \leqq \varphi(k)$$

and

$$\sum_{k=1}^{\infty} \varphi(k)^{\frac{\delta}{2+\delta}} < +\infty \qquad \text{for some} \quad \delta > 0 \ .$$

Then we can apply the following theorem (Ibragimov Linnik [15]) and prove the following.

Theorem 4.4. Let $\delta > 0$ be as in Lemma 6.

If $f \in L^{2+\delta}([0,1),\mu_\beta)$, and

$$\sum_{k=1}^{\infty} \| f - E_{\mu_\beta}[f \mid \bigvee_{0}^{k-1} T^i f] \|_{L^{\frac{2+\delta}{1+\delta}}([0,1),\mu_\beta)} < +\infty$$

then

$$\sigma^2 = E(f - E_{\mu_\beta}(f))^2 + 2\sum_{j=1}^{\infty} E_{\mu_\beta}[(f - E_{\mu_\beta} f)(f \circ T_\beta^j - E_{\mu_\beta} f)] < +\infty$$

and

$$\lim_{n\to\infty} \mu_\beta\left\{\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (f \circ T_\beta^j - E_{\mu_\beta} f) < z\right\} = \Phi_\sigma(z)$$

$$\lim_{n\to\infty} \lambda\left\{\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (f \circ T_\beta^j - E_{\mu_\beta} f) < z\right\} = \Phi_\sigma(z)$$

where $\Phi_\sigma(z) = \dfrac{1}{\sqrt{2\pi}\,\sigma} \displaystyle\int_{-\infty}^{z} e^{-\frac{t^2}{2\sigma^2}}\, dt$, $\Phi_0(z) = 1 \ (z > 0)$.

Using the property of $\bigvee_{i=0}^{n-1} T_\beta^{-i}R$, we can show that Hölder cintinuous functions and functions of bounded total variation satisfy the condition of Theorem 4.4.

If the expansion of one is periodic, then the natural generator R satisfies the uniform mixing condition and the central limit theorem holds for wider class of functions.

SECTION 5 Construction of a Transversal Flow

For automorphisms (X,T,μ) with intensive mixing property it is often the case that there exists a flow $\{Z_t\}_{-\infty<t<+\infty}$ on (X,T,μ) such that

$$Z_t T = T Z_{\lambda t} \qquad -\infty<t<+\infty$$

Y.G.Sinai [14] proved that an automorphism is K-system if there exists such a flow, which is called transversal flow of T. The typical examples are the quasi-periodic motions for toral automorphisms ,the holocycle flow for the geodesic flow on the compact Riemannian manifold with constant curvature, stationary Markov chains with maximal entropy,(over Markov subshifts) etc. All these examples are automorphisms of finite type. But for non-Markovian automorphisms can a transversal flow be constructed ?

We shall show the existence of natural transversal flow for β-automorphisms which are not of finite type in general. In order to construct a β-transversal flow for β-automorphism we use S-representation.

Let us define the basic automorphism B on X_β^- :

$$B\omega_- = \omega'_- \qquad \text{if} \quad \omega \in C_k$$

and $\omega'_-=(\ldots,\omega(-k-2),\omega(-k-1)+1,0,\ldots,0)$

(See section 4 for the definition of C_k)

The transformation B is a generalization of the adding machine. Then a probability measure ν_β on (X_β^-,B) is defined, but the details are omitted here. The ceiling function g_β on (X_β^-,B,ν_β) is given by

$$g_\beta(\omega) = \xi_\beta(T^k \omega_\beta) \qquad \text{if} \quad \omega\in C_k$$

The S-flow constructed by (X_β^-,B,ν_β) and g_β gives a transversal flow since the β-automorphism is represented as the transformation

$$(\omega^-,y) \qquad (\omega^-,a,T_\beta y)$$

$a = \Pi_\beta(y)(0)$ on the space $\{(\omega^-, y) \mid \omega^- \in X_\beta^-,\ 0 \leq y < \rho_\beta(\omega^-)\}$

(Note that the ceiling function g_β is generally not bounded away from zero.)

Thus we obtain the following

Theorem 5.1 For β-automorphisms $(X_\beta, \sigma, \mu_\beta)$

(1) We can construct a transversal flow $\{Z_\beta^t; -\infty < t < +\infty\}$

(2) The transversal flow $\{Z_\beta^t; -\infty < t < +\infty\}$ is ergodic and of entropy 0.

(3) If $\beta_n \to \beta$, then $Z^t_{\beta_n}\omega \to Z^t_\beta \omega$ in the natural topology of A^Z for all t and a.e. $\omega \in X$.

Remark The spectrum type of the flows $\{Z_\beta^t; -\infty < t < +\infty\}$ or of the basic automorphisms B is unknown except for the case where $\beta = \frac{\sqrt{5}+1}{2}$. In this case it has discrete spectrum. ([7])

SECTION 6 Equilibrium measure and Transfer Matrix Method

In the proceeding sections we have studied linear mod. 1 transformations endowed with natural invariant measures in the sense of absolute continuity with respect to Lebesgue measure. These transformations have been considered as pairs of subshift and function called potential, and the natural invariant measures are characterized by the maximality of free energy. This approach to the measure-theoretical study of dynamical systems is not original and can be found in the theory of statistical mechanics, where the natural invariant measures are limiting Gibbsian distributions so that it is meaningless to speak of absolute continuity.

Let us introduce several terminology borrowed from statistical mechanics and give a brief sketch of the relation to our results, and finally the f-expansions are discussed.

6.1. Equilibrium measures

Let U be a continuous function on a sequence space X which will be assumed a subshift of finite type, i.e., a Markov subshift over finite symbols.

DEFINITION The free energy of shift invariant probability measure μ on X for potential U is

$$f_U(\mu)=h(\mu,\sigma)-\int U d\mu \qquad (\sigma \text{ being shift})$$

If $f_U(\mu)=p(U)=\sup f_U(\nu)$, then μ is called equilibrium measure.

DEFINITION [Sinai's form] An invariant measure μ is called (limiting)Gibbsian measure with boundary condition ν if it is a vague limit point of measures $\mu_{n,m}$, $n,m \geq 0$.

$$\int g d\mu_{n,m}=\int g \exp\left\{-\sum_{-n}^{m} U\circ\sigma^k\right\}d\nu \Big/ \int \exp\left\{-\sum_{-n}^{m} U\circ\sigma^k\right\} d\nu$$

DEFINITION An invariant measure μ is called (limiting) Gibbsian measure with periodic boundary condition if it is a vague limit point of the measures $\mu^{(n)}$, $n \geq 0$.

$$\int g d\mu^{(n)}=\sum_{per_n(x)} \exp\left\{-\sum_{k=0}^{n-1} U\circ\sigma^k \circ g\right\} \Big/ \sum_{per_n(x)} \exp\left\{-\sum_{k=0}^{n-1} U\circ\sigma^k\right\} .$$

It is easy to see that p(U) is convex,monotone increasing continuous functon of U with respect to the sup-norm and is invariant under the additon of functions of the form $F-F\circ\sigma$.

Let us introduce a class of potentials D: $U \in D$ iff $\sum_{n\geq 0}[U]_n < \infty$,

$$[U]_n=\sup\left\{|U(\omega)-U(\omega')| : \omega(k)=\omega'(k) \quad |k|<n\right\}$$

Then we can find , for any $V\in D$,a potential $U\in D$ such that $U(\omega)$ depends only on the semi-infinite sequence $\omega(n)$, n=0,1,...

We obtain the following theorem : Let $U(\omega)=U(\omega(n),\ n\geq 0)$.

There is an equilibrium measure for U if and only if the Jacobian

$$j_\mu(\omega)=\mu(\omega(0)/\sigma\omega)$$

is $L(\mu)$-limit of the functions $F_n \circ \exp[-U-p(U)]/F_n \circ \sigma$, $n=0,1,\ldots$ In particular if there exists a function h such that

$$\sum_a h(a\omega)\exp[-U(a\omega)-p(U)]/h(\omega)=1$$

then $j=h\exp[-U-p(U)]/h\circ\sigma$ for any equilibrium measure.

6.2 Transfer Matrix Method

The existence of equilibrium measure follows from the compactness of the sequence space X and the lower semicontinuity of potential U.It is difficult to obtain the uniqueness,which may actually be broken as is well known in statistical mechanics. (See also Appendix)The following theorem,which can be proved by so-called transfer matrix method,seems the most general except the Lee-Young theorem:

Theorem 6.1. [ ] Let X be a Markov subshift over finite symbols which is "uniformly transitive", i.e., for any cylinder sets[U] and [V] of length n,$[U]\cap\sigma^{-n-t}[V]\neq\phi$ if $X\cap[U]\neq\phi$ and $X\cap[V]\neq\phi$,where the number t depends only on X and does not on particular cylinder sets. If a potential U belongs to the class D,then there exists a unique equilibrium measure μ_U and the system (X,σ,μ_U) is Bernoullian.

Furthermore the Gibbsian measure is unique and coincides with μ_U for any boundary condition ν and also for periodic boundary condition.

Remark (i) The Theorem generalizes the result of Sinai[11]. (ii) In case of $X=A^Z$ the uniqueness and K-property is already shown in Ruelle[10] and the Bernoulli property is proved by G.Gallavotti,Ising Model in one dimension, Comm. math,Phys.32 183-190(1973).

The poof is given by sharpening the estimate in Ruelle's method called transfer matrix method. Let us introduce an operator S_U on the space of functions defined on the semi-infinite sequence space X^+:

$$S_U f(\omega) = \sum_{a\omega \in X^+} f(a\omega)\exp[-U(a\omega)], \quad \omega \in X^+.$$

If the potential U is zero and $X^+ = X_\beta$,then the operator S_U is nothing but the operator S_β introduced in the previous section.

The eigenvalue problem for the "transfer matrix" S_U brings almost full information on the structure of equilibrium measures.For example,if there exists an eigenfunction $h > 0$, then,as is seen above,the Jacobian of equilibrium measure is uniquely determined,and so is the equilibrium measure itself. Furthermore the iteration S_U^n converges,under suitable normalization ,storongly and uniformly on a certain subset,and the Ornstein's weak Bernoulli condition is verified. Theorem 6.1. is easily deduced if we can show

Proposition, Under the assumption of Theorem 6.1.,there uniquely exists a positve number α_U,a probability measure ρ_U,and a positive continuous function h_U on X such that

$$S_U h_U = \alpha_U h_U, \quad \rho_U S_U = \alpha_U \rho_U, \quad \int h_U d\rho_U = 1.$$

Furthermore

$$\text{s-lim}\ \alpha_U^{-n} S_U^n f = \int f d\rho_U h_U$$

where the convergence is uniform and exponential on the set

$$[\alpha_U^{-n} S_U^N f : \|f\| \leq M, [\log f^{\pm}]_q \leq \varepsilon\ ]$$

for any $q \geq 1$, $\varepsilon > 0$ and $M < \infty$.

In order to prove Proposition the following are essential:

Lemma 1. If f is a non-trivial nonnegative function, then

$$[\log S_U^n f]_q \leqslant \sum_{k=q+1}^{q+n} [U]_k + [\log f]_{q+n}$$

for $t > n$ and $q \geqslant p$ (p being the order of Markov subshift X).

Lemma 2. There exists a constant $C=C(X,U)$ such that for any nontrivial nonnegative function f and for any $n > p+t$,

$$S_U^n f(\omega),\ S_U^n f(\omega') \leqslant C, \qquad \omega, \omega' \in X^+.$$

6.3. The measure μ_β as equilibrium measure

Although the β-shift is beyond the situation of Theorem 6.1., we already showed that Proposition of the previous subsection holds, where the potential $U=0$. Thus we can show that μ_β is equilibrium measure as well as Gibbsian measure with any boundary condition ν. We can also prove

Theorem 6.2. The measure μ_β is the unique Gibbsian measure with periodic boundary condition:

$$\int f d\mu_\beta = \lim_{n\to\infty} (\mathrm{Card}[\mathrm{Per}_n(X_\beta)])^{-1} \sum_{\mathrm{Per}_n(X_\beta)} f(\omega).$$

This Theorem 6.2. together with the results in § 3 shows that the periodic points are dence, increases exponentially and are distributed in a uniform manner, just as the periodic points of uniformly transitive Markov subshift behave themselves.

The results stated in Theorem 6.1. also holds for β-transformations corresponding to the potentials of finite range (i.e., the functions depending only on finite number of coordinates).

In the next subsection we shall see that similar results remain valid for f-transformations which correspond to the potentials of class D.

6.4. f-transformations

Let F be the family of increasing continuous functions f on the unit interval $[0,1]$ with $f(0)= 0$ such that the right-derivative f' exists and is greater than a constant > 1.

Theorem 6.3. Let $f \in F$. Then the f-transformation

$$T_f t = f(t) \qquad (\mathrm{mod}\ 1)$$

is realized by some β-shift (X_β, σ); there exist a continuous function $\rho_f(\omega)$ on X, a Borel map $\pi_f(t)$ of $[0,1)$ into X such that

$$\pi_f T_f = \sigma \circ \pi_f, \qquad \rho_f \circ \pi_f = id_{[0,1]}, \quad T_f \circ \rho_f(\omega) = \rho_f(\sigma\omega).$$

Furthermore $\psi = \rho_\beta \circ \pi_f$ is a homomorphism of [0,1] such that

$$T_\beta \circ \psi = \psi \circ T_f \qquad \text{(topological conjugacy)}$$

Now let us find an invariant measure of T_f which is absolutely continuous with respect to Lebesgue measure. Let g(t)dt be a probability measure on $[0,1)$. Since T_f is a non-singular transformation, thus its inverse image under T_f can be expressed in the form (Lg)dt:

$$(Lg)(t) = \sum_{s \in T_f^{-1} t} g(s)/f'(s)$$

The "Jacobian operator" L corresponds, by the realization ρ_f, to the transfer matrix

$$(S_U g)(\omega) = \sum_{a \cdot \omega \in X} \exp[-U(a\omega)] g(a \cdot \omega)$$

where

$$U(\omega) = \log f'(\rho_f(\omega)).$$

Thus the situation is close to the situation of Theorem 6.1., although the β-shifts are not Markov in general.

We note that $U \in D$ if the right-derivative f' is Lipschitzian on each connected component of $T_f^{-1}[0,1]$. In fact ρ_f is the solution of the functional equation

$$\rho(a\omega) = f^{-1}(a + \rho(\omega)), \qquad \omega \in X_\beta ,$$

and the class D verifies the following property:

The pointwise limit of [ ]-norm bounded sequence of functions in D also belongs to the class D.

We obtain the following

Theorem 6.4. Let $f \in F$. Assume that the right-derivative f' is Lipschitzian on each connected component of $f^{-1}[0,1)$. Then there exists an invariant measure μ_f for T_f which is absolutely continuous with respect to Lebesgue measure;the system$([0,1],T_f, \mu_f)$ is Bernoullian.

For the proof of Theorem 6.4. the only crusial step is the proof of the statement in Lemma 2 From the classification of words in § 2 it follows that

$$S_U^n 1(\omega) = e^{-\sum_{k=0}^{n-1} U(\sigma^k(\omega_\beta[0,n)\,\omega))} I(\omega \leq \sigma^n \omega_\beta) + \sum_{k=0}^{n-1} C_{n,k}(\omega) S_U^{n-k-1} 1(\omega)$$

where $C_{n,k}(\omega)$ is such that

$$e^{-[U]} C_{n,k}(\omega) \leq \sum_{a < \omega(k)} e^{-\sum_{j=0}^{k} U(\sigma^j(\omega_\beta[0,k)\, a\, \omega_\beta))}$$

$$\leq e^{[U]}(C_{n,k}(\omega))$$

The rest of the proof follows in a similar way to the proof given for β-transformations.

APPENDIX

A Caricature of Ergodicity under Phase Transition

The existence of phase transition for the classical mechanics of lattice system is well known but we have no further information on the phase transition in general, e.g., the critical temperature,or the ergodic property under the existence of phase transition.We shall give a simple example of statistical mechanics of one-dimensional lattice system for which the critical temperature and the ergodic property at all temperature and all activity is known,although the potential is not pair potential and bears an artificial character.

Let us consider the configurations on $Z=[\ldots,-1,0,1,\ldots]$ and assume that each consequent n particles has internal energy c_n.If $c_n \neq 0$ for all n, then the potential is of long range and there may actually exist a phase transition;one phase is Bernoullian with respect to the translation and the other is a point mass.Using this example and the fact that the equilibrium measure for potentials which can be expressed in the direct product spase $X=X_1 \times X_2$ $(x=(x_1,x_2),\ x_1 \in X_1, x_2 \in X_2)$ is the direct product measures $\mu = \mu_1 \otimes \nu_2$ of the equilibrium measures μ_1 for U_1 on X_1 and μ_2 for U_2 on X_2 ,we can obtain

_____ an example with phase transition for which every phase is Bernoullian(but there is a discontinuity for the mean entropy)

_____an example which has m phases,where m is an arbitrary integer $r \geq 1$.

______an example which has more than two critical temperature.

1. Thermodynamical Limit

Let $X=0.1$. $B(0)=[0]=[x:x(0)=0]$, $B(n)=[\overbrace{11\ldots1}^{n}0]$ $=[x:x(k)=1$ for $k=0,1,\ldots,n-1,\ x(n)=0]$,and $B(\infty)=[1111\ldots\ldots]=[x:x(k)=1$ for any $k]$.

Consider potentials of the form

(1) $U(x)=a(n)$ if x belongs to $B(n)$, $0\leqslant n\leqslant\infty$, $\lim\limits_{n\to\infty} a(n)=a(\infty)$

Then

$$\Phi(x(0),\ldots x(n-1))=\sum_{k=1}^{n-k} a(k)\sum_{i=0}^{n-k} x(i)x(i+1)\ldots x(i+k-1)$$

and

$$\Xi_n=\sum_{x(0),\ldots x(n-1)=0\text{ or }1}\exp[-\Phi(x(0),\ldots x(n-1))]$$

$$=\sum_{r\geqslant 0}\ \sum_{\substack{k_1+\ldots+k_r+\ell_0+\ldots+\ell_r=n\\ k_1\geqslant 1,\ldots k_r\geqslant 1\\ \ell_0\geqslant 0,\ \ell_1\geqslant 1,\ \ldots \ell_{r-1}\geqslant 1,\ \ell_r\geqslant 0}}\exp[-\Phi(\overbrace{00..0}^{\ell_0}\overbrace{1..1}^{k_1}\overbrace{0..0}^{\ell_1}\ldots\overbrace{1..1}^{k_r}\overbrace{0..0}^{\ell_r})]$$

$$=\sum_{r\geqslant 0}\ \sum_{\substack{k_1+\ldots k_r\leqslant n+r+1\\ k_1\geqslant 1,\ldots k_r\geqslant 1}}\binom{n-k_1-\ldots-k_r}{r-1}e^{-b_{k_1}-\ldots b_{k_r}}$$

where

$$b_k=\sum_{i=1}^{k} a_i.$$

Since

$$\sum_{n\geqslant 0} z^n\,\Xi_n=\frac{1-z}{z}\Big(1-\sum_{k\geqslant 0} z^{k+1}e^{-b_k}\Big)^{-1},$$

we obtain

(2) $$\limsup_{n\to\infty}\frac{1}{n}\log\Xi_n=f^*=\inf\Big\{f:\ \sum_{k\geqslant 0}e^{-b_k-(k+1)f}<1\Big\}$$

2 Free energy and equilibrium measures

It is obvious that the shift (X,ϑ) is isomorphic to the Markov subshift (Y,σ) over the symbols $0,1,\ldots,\infty$ where

$$Y=[y:\ y(n+1)=y(n)-1\quad\text{if } y(n)\geqslant 1]$$

The corresponding potential on the space Y is

$$V(y)=a(n)\qquad\text{if } y(0)=n.$$

It is not difficult to show the following

Lemma (i) If $\sum\limits_n\exp[-b(n)-(n+1)f^*]=1$

and if $\sum\limits_n e(n+1)\exp[-b(n)-(n+1)f^*]<\infty$,

then the Markov measure defined by the following transition function and stationary distribution is an equilibrium measure:

$$p(0,j)= r(j),\quad p(i,j)= \begin{cases} 1 & \text{if } j\neq i-1 \\ 0 & \text{if } j=i-1 \end{cases}$$

$$q(0) =\Big(\sum_{i\geq 0} (i+1)r(i)\Big)^{-1},\quad q(i)=q(0) \sum_{j\geq i} r(j) \quad (j\geq 1),$$

where

$$r(n) =\exp[-b(n)-(n+1)f^{*}].$$

(ii) If $f^{*}=-a(\infty)$, then the point mass $\varepsilon_{11\ldots}$ at the sequence (111...) is an equilibrium measure.

(iii) There exist no other equilibrium measures.

3. Existence condition for Phase transition

Let us introduce the inverse temparature β and activity μ, and consider a potential

$$U= \sum_{n} (\mu+\beta c(n))\chi_{B(n)}, \quad (\chi_B \text{ being the indicator of } B)$$

i.e., $a(n)=\mu+\beta c(n)$ and $b(n)=n\mu+\beta\tau_n$ where $\tau_n= \sum_{i+1}^{n} c(j)$.

By Lemma of proceeding section, if the Dirichret series (in β)

$$\text{(I)}\quad \sum_{n\geq 0} \exp-\beta[\tau_n-(n+1)c(\infty)]= +\infty$$

then the phase transition never occurs. On the other hand if (I) does not hold, then there is a β_0 ($\beta_0\geq 0$) such that

$$\text{(II)}\quad \sum_{n\geq 0} \exp-\beta[\tau_n-(n+1)c(\infty)] \begin{cases} = +\infty & \text{if } \beta<\beta_0 \\ < +\infty & \text{if } \beta>\beta_0 \end{cases}$$

Moreover there is another $\beta_1\geq\beta_0$ such that

$$\text{(III)}\quad \sum_{n\geq 0} (n+1)\exp-\beta[\tau_n-(n+1)c(\infty)] \begin{cases} = +\infty & \text{if } \beta<\beta_1, \\ < +\infty & \text{if } \beta>\beta_1 \end{cases}$$

Since β_0 and β_1 are the convergence range of Dirichret series they can be expressed in term of $\tau_n-(n+1)c(\infty)$ but we omit the expression here. If $\beta>\beta_0$, we can define the value

$$\varphi(\beta)= -\log \sum_{n\geq 0} \exp-\beta[\tau_n-(n+1)c(\infty)].$$

Let

$G=[(\beta,\nu)$: (I) holds or (II) holds and $\nu>\nu_0(\beta)$, $\beta>\beta_0$ or (IV)]

$S=[(\beta,\nu)$: (II) holds, $\beta>\beta_1$ and $\nu<\nu_0(\beta)$, or $\beta_0<\beta<\beta_1$ and $\nu\leqslant\nu_0(\beta)$]

$M=[(\beta,\nu)$: (II) holds, $\nu=\nu_0(\beta)$ and $\beta>\beta_1$]

where

(IV) $\sum_{n\geqslant 0}\exp[-n\nu-\beta\tau_n-(n+1)P(\beta,\nu)]=1$

$$\sum_{n\geqslant 0}(n+1)\exp[-n\nu-\beta\tau_n-(n+1)P(\beta,\nu)]<+\infty$$

$$\text{and } P(\beta,\nu)>-\nu-\beta c(\infty).$$

Here $P(\beta,\nu)$ is the value f^* in (2) for this case.

Theorem

(I) If $(\beta,\nu)\in G$, then the equilibrium measure is unique and isomorphic to strong mixing Markov automorphism. (Hence it is Bernouillian)

(II) If $(\beta,\nu)\in S$, then the equilibrium measure is unique and is point mass.

(III) If $(\beta,\nu)\in M$, then there exist two and only two equilibrium measures ,one of which is a point mass and the other is isomorphic to mixing Markov measure. (In particular the former has 0 entropy and the latter has positive entropy)

Examples.

$\gamma_n-(n+1)c(\infty)$		β_0	$\mu_0(\beta_0)$	β_1	$\sum(n+1)e^{-\beta(\gamma_n-(n+1)c_\infty)}$	fig.	
$\Omega(\log n)$		0		0		fig 3	$\beta=0$ may be critical
$O((\log n)^\alpha)$, $0\leqq\alpha<1$		$+\infty$		$+\infty$		fig 4	No phase transition
$\log[n(\log n)^\alpha]$	$\alpha\leqq\frac{1}{2}$	1	0	2	$+\infty$	fig 2	1 phase at C_1
	$\frac{1}{2}<\alpha<1$	1	0	2	finite	fig 2	2 phases at C_1
	$\alpha>1$	1	positive	2	finite	fig 1	2 phases at C_1

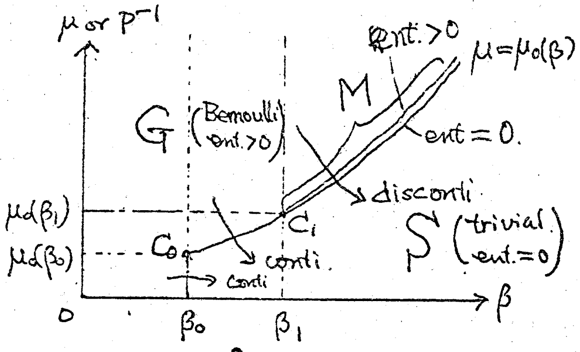

fig. 1.

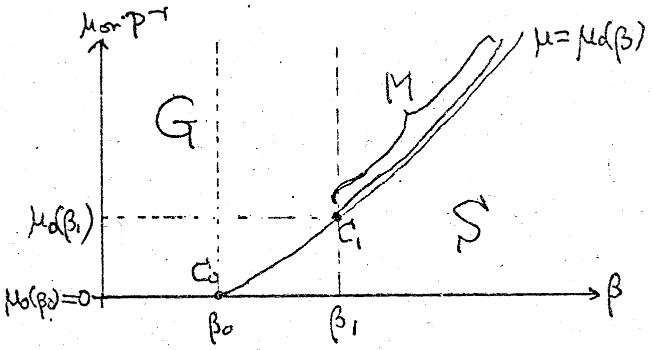

fig. 2.

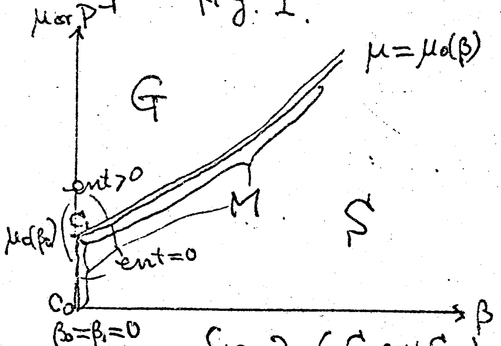

fig.3. (C_1 and C_0 may coincide)

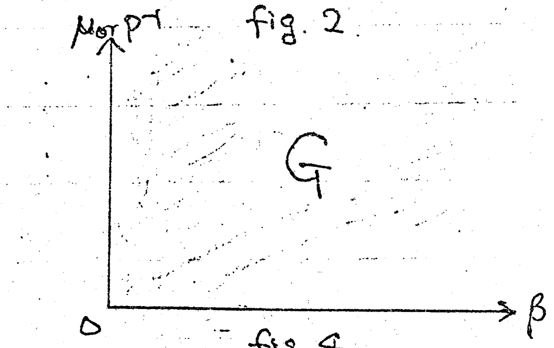

fig. 4.

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