数理解析研究所講究録
第 204 巻 1974 年 115-121

The Boltzmann Equation of Gas Mixture of Hard Balls

Shigeru Tanaka (Tsuda College)

The Boltzmann equation of gas of hard balls (spacially homogeneous case) is

$$(0)\quad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{1}{2}\delta^2 N \displaystyle\int_{S^2\times R^3} |(y-x,\ell)| \{ u(t,x^*)u(t,y^*) \\ \qquad\qquad - u(t,x)u(t,y)\} \, d\ell \, dy , \\ u(0,x) = f(x) , \end{cases}$$

where $u(t,x)$ is the density of distribution of molecules with speed $x \in R^3$ at time t, $d\ell$ is the uniform distribution on S^2, and $(x,y) \longrightarrow (x^*, y^*)$ gives the change of speed of molecules by elastic collision :

$$(2)\quad \begin{cases} x^* = x + (y-x,\ell)\ell \\ y^* = y - (y-x,\ell)\ell \end{cases} \qquad \ell \in S^2$$

and δ, N is the radius and number of molecules, respectively.

When we set $u(t,\Gamma) = \int_\Gamma u(t,x)\,dx$, we can easily extend (0) to the equation of measures,

$$(1)\quad \begin{cases} \dfrac{\partial u(t,\Gamma)}{\partial t} = \dfrac{1}{2}\delta^2 N \displaystyle\int_{S^2\times R^3\times R^3} |(y-x,l)|\{\delta(x^*,\Gamma)-\delta(x,\Gamma)\}\, dl\, u(t,dx)\, u(t,dy) \\ \\ u(0,\Gamma) = f(\Gamma) \end{cases}$$

We assume that $f(\Gamma)$, initial distribution, is the probability measure on R^3.

We say that $u(t,\cdot)$ preserves mass, momentum or energy if the following condition,

$$\int_{R^3} u(t,dx) = \int_{R^3} f(dx),$$

$$\int_{R^3} x\, u(t,dx) = \int_{R^3} x f(dx),$$

$$\int_{R^3} |x|^2 u(t,dx) = \int_{R^3} |x|^2 f(dx),$$

is satisfied, respectively.

Povzner's results are the following three theorems. ([1], see also [2])

<u>Theorem 1.</u> If $\int_{R^3} |x|^2 f(dx)$ is finite, then

there exists a solution of (1) which preserves mass and momentum.

Theorem 2. If $\int_{R^3} |x|^\alpha f(dx)$ is finite for some $\alpha \geqq 3$, then there exists a solution of (1) which makes $\mu^{(\alpha)}(t)$ locally finite, where $\mu^{(\alpha)}(t) = \int_{R^3} |x|^\alpha u(t, dx)$, and this solution preserves mass, momentum and energy.

Theorem 3. If $\int_{R^3} |x|^4 f(dx)$ is finite, then the solution of (1) which makes $\mu^{(4)}(t)$ locally finite is unique, and morever if f has density, so has $u(t, \cdot)$.

We extend these results to the case of gas mixture. In the case of gas mixture of two species of molecules, (1), (2) and (3) are extended to

$$(4)\quad \begin{cases} \dfrac{\partial u_i(t,\Gamma)}{\partial t} = \displaystyle\sum_{j=1}^{2} a_{ij} N_j \int_{S^2\times R^3\times R^3} |(y-x, \ell)| \{ \delta(x^{ij}, \Gamma) - \delta(x, \Gamma) \} \, d\ell \, u_i(t, dx)\, u_j(t, dy), \\ u_i(0, \Gamma) = f_i(\Gamma) \qquad\qquad (i = 1, 2) \end{cases}$$

where $a_{11} = \frac{1}{2}\delta_1^2$, $a_{22} = \frac{1}{2}\delta_2^2$ and $a_{12} = a_{21} = \frac{1}{2}\left(\frac{\delta_1+\delta_2}{2}\right)^2$,

$$(5)\ \begin{cases} x^{ij} = x + \dfrac{2m_j}{m_i+m_j}(y-x,\ell)\ell , \\ y^{ij} = y - \dfrac{2m_i}{m_i+m_j}(y-x,\ell)\ell , \end{cases} \qquad (\ell \in S^2)$$

$$(6)\ \begin{cases} \displaystyle\int_{R^3} u_1(t,dx) = \int_{R^3} f_1(dx) \text{ and } \int_{R^3} u_2(t,dx) = \int_{R^3} f_2(dx), \\ \displaystyle\sum_{i=1}^{2} N_i m_i \int_{R^3} x u_i(t,dx) = \sum_{i=1}^{2} N_i m_i \int_{R^3} x f_i(dx), \\ \displaystyle\sum_{i=1}^{2} N_i m_i \int_{R^3} |x|^2 u_i(t,dx) = \sum_{i=1}^{2} N_i m_i \int_{R^3} |x|^2 f_i(dx), \end{cases}$$

where m_1, m_2 is the mass of two species of molecules, respectively. ([3])

And we can extend Th.1 ~ Th.3 in the following,

<u>Theorem 4.</u> If $\int_{R^3} |x|^2 f_i(dx)$ is finite $(i=1,2)$, then there exists a solution of (4) which preserves mass and momentum.

<u>Theorem 5.</u> If $\int_{R^3} |x|^\alpha f_i(dx)$ is finite $(i=1,2)$ for some $\alpha \geq 3$, then there exists a solution of (4) which makes $\mu_1^{(\alpha)}(t)$ and $\mu_2^{(\alpha)}(t)$ locally finite, where $\mu_i^{(\alpha)}(t) = \int_{R^3} |x|^\alpha u_i(t,dx)$ $(i=1,2)$, and this

solution preserves mass, momentum and energy.

Theorem 6. If $\int_{R^3} |x|^4 f_i(dx)$ is finite $(i=1,2)$, then the solution of (4) which makes $\mu_1^{(4)}(t)$ and $\mu_2^{(4)}(t)$ locally finite is unique, and morever if f_1 and f_2 have density, so have $u_1(t,\cdot)$ and $u_2(t,\cdot)$.

The proof of these theorems is essentially the same as that of H. Tanaka [2], if we transform (4), (5) and (6) in the following manner.

Let R_1 and R_2 be two copies of R^3 and let Q be the disjoint union of R_1 and R_2. And let us considere the following measures on Q,

$$\begin{cases} u(t,\Gamma) = \dfrac{1}{N_1+N_2}\{N_1 u_1(t,\Gamma\cap R_1) + N_2 u_2(t,\Gamma\cap R_2)\} \\ f(\Gamma) = \dfrac{1}{N_1+N_2}\{N_1 f_1(\Gamma\cap R_1) + N_2 f_2(\Gamma\cap R_2)\} \end{cases}$$

then the equation (4), (5) and (6) can easily be transformed to

$$(7)\quad \begin{cases} \dfrac{\partial u(t,\Gamma)}{\partial t} = \displaystyle\int_{S^2\times Q\times Q} g(x,y,l)\{\delta(x^*,\Gamma)-\delta(x,\Gamma)\}\,dl\, u(t,dx)\,u(t,dy), \\ u(0,\Gamma) = f(\Gamma) \end{cases}$$

where $g(x,y,l) = a_{ij}|(y-x,l)|$, if $x \in R_i$ and $y \in R_j$,

$$(8)\quad \begin{cases} x^* = x^{ij} & (\text{as element of } R_i), \\ y^* = y^{ij} & (\text{as element of } R_j), \end{cases}$$
if $x \in R_i$ and $y \in R_j$,

$$(9)\quad \begin{cases} \int_{R_i} u(t,dx) = \int_{R_i} f(dx), \quad (i=1,2) \\ \int_Q m_x x\, u(t,dx) = \int_Q m_x x f(dx), \\ \int_Q m_x |x|^2 u(t,dx) = \int_Q m_x |x|^2 f(dx), \end{cases}$$

where $m_x = m_i$ if $x \in R_i$.

It is easily seen that, we can get the same results for gas mixture of more than two species of molecules in the same manner.

References

[1] A. Ya. Povzner: On Boltzmann's Equation in the Kinetic Theory of Gases. Mat. Sb., 58 (1962), 63-86.

[2] H. Tanaka : The Boltzmann Equation of Gas of Hard Balls. R.I.M.S. Kôkyûroku 174 (1973), 1-20. (in Japanese)

[3] L. Boltzmann : Vorlesungen über Gastheorie. Leipzig (1896).