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An ergodic theorem related to spectra of a discrete random system

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§ 1. Introduction

Let Z^ν be the ν-dimensional lattice space. Each point of Z^ν is denoted by $a = (a_1, a_2, \cdots, a_\nu)$. Consider the operator

$$(H^0u)(a) = \sum_{i=1}^{\nu} \kappa_i \{u(a_1, \cdots, a_i - 1, \cdots, a_\nu) - 2u(a) + u(a_1, \cdots, a_i + 1, \cdots, a_\nu)\}, \quad a \in Z^\nu, \quad u \in C_0(Z^\nu),$$

where κ_i, $i = 1, 2, \cdots, \nu$, are positive constants fixed throughout this paper and $C_0(Z^\nu)$ is the space of functions on Z^ν with finite supports.

We then define a transformed operator

$$(1) \quad (Hu)(a) = \frac{1}{m(a)}\{(H^0u)(a) - q(a)u(a)\}$$

with a positive function $m = m(a)$ and a real valued function $q = q(a)$ on Z^ν. As a model of crystals, the vector $X(a) = (m(a), q(a))$ appeared in (1) is assumed to be translationally periodic, in which case we say that the operator H represents a (discrete) <u>regular system</u>. However there are many physical phenomena where the translational symmetry is violated such as the cases of glasses, alloys and tight binding electrons. These systems have been called (discrete) <u>disordered systems</u> or (discrete) <u>random systems</u>. We refer the reader to [1],[2] and [3] for more detailed information on random systems.

Mathematically the transfer from a regular system to a disordered one amounts to a randomization of $X(a)$ by taking up a stationary stochastic process $X(a, \omega) = (m(a, \omega), q(a, \omega))$ with multidimensional parameter $a \in Z^{\nu}$. We thus start with an ensemble of operators $\{H_{\omega}\}$. One of the most important notions in the theory of the disordered system is the <u>spectral distribution function</u> (or the <u>distribution of eigenstates</u>), which is defined as the almost sure limit of the normalized distribution of eigenvalues of the operator H_{ω} restricted to each of expanding bounded domains under some admissible boundary conditions.

In this paper, we first show that the above limit actually exists and second we give an <u>identification</u> of the limit function, namely, a description of the limit in terms of the spectral family $\{E_x^{\omega} ; -\infty < x < \infty\}$ associated with the self-adjoint operator $-\bar{H}_{\omega}$ on $L^2(Z^{\nu} ; m)$. We essentially follow the lines laid by L.Pastur [3], who has treated the Schrödinger operator $\Delta - q$, the potential q being a stationary stochastic process with multidimensional parameter space R^{ν}. In our discrete case however, no restriction on the sample function of the stationary process $X = (m, q)$ will be imposed except that, in Theorem 2, $\inf_{a \in Z^{\nu}} m(a, \omega)$ is assumed to be strictly positive.

The identification mentioned above is important partly because it has provided us with a basis to derive exponential characters of the tails of the spectral distribution functions (L. Pastur [3] and the auther [4]).

I would like to express my hearty thanks to Professor H. Matsuda who has called my attention to the present identification problem of spectra.

§ 2. Statement of Theorems

Let $X(a, \omega) = (m(a, \omega), q(a, \omega))$, $a \in Z^\nu$, be a stationary process taking values in $(0, \infty) \times (-\infty, \infty)$. The basic probability space $(\Omega, \mathcal{B}, P)$ can always be realized as follows [5] : $X(a) = X(a, \omega)$ is the a-th coodinate map on the space $\Omega = \{(0, \infty) \times (-\infty, \infty)\}^{Z^\nu}$, $\mathcal{B}$ is the smallest σ-field making all $X(a)$'s measurable and the probability measure P is preserved under the shifts T_i, $i = 1,2,\cdots, \nu$, on Ω defined by $X(a, T_i\omega) = X((a_1,\cdots,a_i+1,\cdots a_\nu), \omega)$. Denote by $\mathcal{I}$ the σ-field of invariant sets: $\mathcal{I} = \{B \in \mathcal{B}; P(T_iB \ominus B) = 0, i = 1,2,\cdots, \nu\}$. If $\mathcal{I}$ is trivial, the stationary process $X(a)$ is called metrically transitive. This is the case when $X(a)$, $a \in Z^\nu$, are mutually independent and identically distributed.

Fixing $\omega \in \Omega$ and a bounded set $\Lambda \subset Z^\nu$, we formulate the eigenvalue problem of the operator $-H_\omega$ on Λ by making use of symmetric forms $\mathcal{E}_\Lambda^{\delta,\pi}$ with "boundary elements" δ and π. The boundary $\partial\Lambda$ of Λ is defined by $\partial\Lambda = \{a \in \Lambda ;$ there exists $a' \in Z^\nu - \Lambda$ such as $|a - a'| = 1\}$, $|a - a'|$ being the Euclidean distance. The space of real valued functions on Λ endowed with the inner product $(u, v)_{\Lambda,m} = \sum_{a \in \Lambda} u(a)v(a)m(a, \omega)$ is denoted by $L^2(\Lambda ; m)$. Let δ(resp. π) be a function on $\partial\Lambda$(resp. $\partial\Lambda \times \partial\Lambda$) satisfying

$$(2)\quad \begin{cases} 0 \leqq \delta(a) \leqq \infty, & a \in \partial\Lambda \\ 0 \leqq \pi(a,a') < \infty, & \pi(a,a') = \pi(a',a),\ a,a \in \partial\Lambda. \end{cases}$$

In the following paragraph, we introduce a symmetric form $\mathcal{E}_\Lambda^{\delta,\pi}$ which depends on δ and π but satisfies

$$(3)\quad \mathcal{E}_\Lambda^{\delta,\pi}(u,\ v) = (-H_\omega u,\ v)_{\Lambda,m}$$

whenever u and v vanishes on the boundary $\partial\Lambda$.

We put

$$(4)\quad \mathcal{E}_\Lambda^{\delta\ \pi}(u,\ v) = \frac{1}{2}\sum_{\substack{|a-a'|=1 \\ a,a'\in\Lambda}} (u(a)-u(a'))(v(a)-v(a'))\kappa(a,a')$$

$$+ \sum_{a\in\Lambda} u(a)v(a)q(a,\ \omega) + \sum_{a\in\partial\Lambda} u(a)v(a)\delta(a)$$

$$+ \sum_{a,a'\in\partial\Lambda} (u(a)-u(a'))(v(a)\ -\ v(a'))\pi(a,a'),$$

where $\kappa(a,\ a') = \kappa_i$ if $a - a' = (\overbrace{0,\cdots,\ \pm 1}^{i},\cdots,\ 0)$, $i = 1, 2,\cdots,\ \nu$. The domain $\mathcal{D}(\mathcal{E}_\Lambda^{\delta,\pi})$ of $\mathcal{E}_\Lambda^{\delta,\pi}$ is defined to be the set of those functions on Λ vanishing on the boundary point where δ is infinity. If we set $0\times\infty = 0$ by convention, $\mathcal{E}_\Lambda^{\delta,\pi}$ determines uniquely a symmetric operator $A_\Lambda^{\delta,\pi}$ acting on the subspace $\mathcal{D}[\mathcal{E}_\Lambda^{\delta\ \pi}]$ of $L^2(\Lambda\ ;\ m)$ by

$$(5)\quad \mathcal{E}_\Lambda^{\delta,\pi}(u,\ v) = (-A_\Lambda^{\delta,\pi}u,\ v)_{\Lambda,m},\quad u,v\in\mathcal{D}[\mathcal{E}_\Lambda^{\delta,\pi}].$$

In view of (3) and (5), $A_\Lambda^{\delta,\pi}$ represents a restriction of H_ω to Λ under the assignment of the "boundary condition" expressible by δ and π. We call $\delta = \delta(\partial\Lambda,\ \omega)$ and $\pi = \pi(\partial\Lambda,\ \omega)$ the <u>admissible boundary elements</u> if they satisfy (2) for fixed ω and Λ and if they are $\mathcal{B}$-measurable in $\omega\in\Omega$ for a fixed Λ. In terms of the lattice vibration, $\delta(a)$, $a\in\partial\Lambda$, indicates the rate according to which a-th atom is fixed at its equilibrium position and $\pi(a, a')$, $a, a' \in \partial\Lambda$, indicates an interaction between two atoms numbered a and a'.

We are ready to state our first theorem. γ is an eigenvalue of $A_\Lambda^{\delta,\pi}$ iff $-A_\Lambda^{\delta,\pi}u = \gamma_u$ admits a non-trivial solution u $\in \mathcal{D}[\mathcal{E}_\Lambda^{\delta,\pi}]$. Let $\gamma_1 \leqq \gamma_2 \leqq \cdots \leqq \gamma_N$ be the repeated eigenvalues of $A_\Lambda^{\delta,\pi}$ arranged in the ascending order. Obviously $|\Lambda - \partial\Lambda| \leqq N \leqq |\Lambda|$, $|\Lambda|$ being the number of points of Λ. We put, for each γ,

$$(6) \quad \mathcal{H}^{\delta,\pi}(\gamma\ ;\ \Lambda) = \sum_{\gamma_i \leqq \gamma} 1.$$

The left hand side of (6) depends not only on γ and Λ but also an $\omega \in \Omega$, $\delta(\partial\Lambda, \omega)$ and $\pi(\partial\Lambda, \omega)$. It is in fact a random variable for admissible boundary elements δ and π.

Let $\mathcal{L}$ be the family of all rectangles containing the origin with sides parallel to the axes. The length of edges of $\Lambda \in \mathcal{L}$ is denoted by $L^{(i)}(\Lambda)$, $i = 1,2,\cdots,\gamma$.

<u>Theorem 1</u>. There exist a set $\Omega_0 \in \mathcal{B}$ with $P(\Omega_0) = 1$ and a function $\mathcal{H}(\gamma, \omega)$, $-\infty < \gamma < \infty$, $\omega \in \Omega_0$, satisfying the following:

(i) For each $\omega \in \Omega_0$, $\mathcal{H}(\gamma, \omega)$ is a probability distribution function on $(-\infty, \infty)$. For each $\gamma \in (-\infty, \infty)$, $\mathcal{H}(\gamma, \omega)$ is a random variable.

(ii) For each $\omega \in \Omega_0$ and for each choice of admissible boundary elements $\delta(\partial\Lambda, \omega)$ and $\pi(\partial\Lambda, \omega)$, $\Lambda \in \mathcal{L}$,

$$(7) \qquad \lim_{\substack{\Lambda \in \mathcal{L}, L^{(i)}(\Lambda)\to\infty \\ i=1,2,\cdots,\nu}} \frac{\mathcal{N}^{\delta,\pi}(\gamma ; \Lambda)(\omega)}{|\Lambda|} = \mathcal{N}(\gamma, \omega)$$

at any continuous point γ of $\mathcal{N}(\gamma, \omega)$. In particular when the process $X(a) = (m(a), q(a))$ is metrically transitive, $\mathcal{N}(\gamma, \omega)$ does not depend on ω ; $\mathcal{N}(\gamma, \omega) = \mathcal{N}(\gamma)$, $\gamma \in (-\infty, \infty)$, $\omega \in \Omega_0$.

Let us denote by S_ω the operator H_ω with domain $C_0(Z^\nu)$. It is clear that, for each $\omega \in \Omega$, S_ω is a symmetric operator on $L^2(Z^\nu ; m) = \{u ; (u, u)_{Z^\nu,m} < \infty\}$, where $(u, v)_{Z^\nu,m} = \sum_{a \in Z^\nu} u(a)v(a)m(a, \omega)$.

<u>Lemma 1</u>. Assume that

$$(8) \quad \inf_{a \in Z^\nu} m(a, \omega) > 0.$$

Then S_ω is essentially self-adjoint and its self-adjoint extension coincides with the following operator A_ω :

$$(9) \quad \begin{cases} \mathcal{D}(A_\omega) = \{u \in L^2(Z^\nu ; m) ; H_\omega u \in L^2(Z^\nu ; m)\} \\ A_\omega u(a) = H_\omega u(a), \quad u \in \mathcal{D}(A_\omega), \ a \in Z^\nu. \end{cases}$$

Under the assumption (8), $-A_\omega$ can be expressed as $-A_\omega = \int_{-\infty}^{\infty} \gamma \, dE_\gamma^\omega$ by a unique spectral family $\{E_\gamma^\omega, -\infty < \gamma < \infty\}$. We then put

$$(10) \quad \rho(\gamma, \omega) = (E_\gamma^\omega I_0, I_0)_{Z^\nu,m} ,$$

I_a being the indicator function : $I_a(a') = \delta_{aa'}$.

Theorem 2. If (8) is true for almost every $\omega \in \Omega$, $\mathcal{H}(\gamma, \omega) = E(\rho(\gamma, \cdot)/\mathcal{J})$, a.e., where the right hand side denotes the conditional expectation with respect to the probability P. In particular when $X(a) = (m(a), q(a))$ is metrically transitive,

(11) $\mathcal{H}(\gamma) = E(\rho(\gamma, \cdot))$.

In order to prove Theorem 1 and 2, we have to use a version of the Birkhoff ergodic theorem for the case of several automorphisms. One of the most general theorems of this kind is the following one which is essentially due to A. Zygmund [6].

Lemma 2. Let $S_1, S_2, \cdots, S_\nu$ be a family of mutually commuting, measure preserving one to one transformations on a probability space $(\Omega, \mathcal{B}, P)$. If a random variable $f(\omega)$ satisfies

(12) $E(|f|\{\log^+|f|\}^{\nu-1}) < \infty$,

then

(13)
$$\lim_{\substack{\ell_i+m_i\to\infty \\ \ell_i, m_i\geq 0 \\ i=1,2,\cdots,\nu}} \frac{1}{\prod_{i=1}^{\nu}(\ell_i+m_i+1)} \sum_{\substack{-\ell_i\leq n_i\leq m_i \\ i=1,2,\cdots,\nu}} f(S_1^{n_1}S_2^{n_2}\cdots S_\nu^{n_\nu}\omega)$$
$$= E(f \mid \mathcal{J})(\omega), \qquad \text{a.e.,}$$

where $\mathcal{J} = \{B \in \mathcal{B} ; P(S_i B \ominus B) = 0, i = 1,2,\cdots, \nu\}$.

Incidentally, R.T. Smythe [7] recently found that the condition (12) is also necessary for (13) provided that S_i's are generated by independent, identically distributed random variables with index set Z^ν.

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