

On the maximally overdetermined system  
of linear differential equations (I)

by

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INTRODUCTION

The purpose of this paper is to present finiteness theorems and several properties of cohomologies of holomorphic solution sheaves of maximally overdetermined systems of linear differential equations. The proof relies on the finiteness theorem for elliptic systems due to T. Kawai [4], as an analytic tool, and on the theory of stratifications of analytic sets introduced by H. Whitney [8] and [9], as a geometric tool.

Our goal is the following theorem.

Theorem (3.1) Let  $\mathcal{M}$  be a maximally overdetermined system on a complex manifold  $X$  and  $X = \cup \alpha U_\alpha$  be a stratification of  $X$  satisfying the regularity conditions of H. Whitney. If the singular support of  $\mathcal{M}$  is contained in the union of conormal projective bundles of strata, then the restriction of  $\text{Ext}_{\mathcal{D}_X}^i(\mathcal{M}, \mathcal{O}_X)$  to each stratum is a locally constant sheaf of finite rank.

We mention that the introduction of differential operators of infinite order is indispensable when we analyze more precisely, the structure of maximally overdetermined systems, but we leave this subject to the next paper and we restrict ourselves in the category of finite order differential operators in this paper.

As for notions on systems of differential equations, we refer Kashiwara [2] and Sato-Kawai-Kashiwara [6].

We will list up the notations used in this paper.

#### Notations

- $X$  : complex manifold  
 $\mathcal{D} = \mathcal{D}_X$  : the sheaf of differential operators of finite order on  $X$   
 $\mathcal{M}$  : a wherent  $\mathcal{D}_X$ -Module  
 $\mathcal{O}_X$  : the sheaf of holomorphic functions on  $X$ , which is a left coherent  $\mathcal{D}_X$ -Module  
 $\mathcal{P}_{Y|X}$  : a left  $\mathcal{D}_X$  Module,  $\mathcal{N}_Y^d(\mathcal{O}_X)$  where  $d = \text{codim } Y$   
 $\Omega_X^n$  : the sheaf of holomorphic  $n$ -forms, which is a right  $\mathcal{D}_X$ -Module if  $n = \dim X$   
 $M$  : real analytic manifold  
 $\mathbb{R}$  : the right derived function in the derived category  
 $f_!$  : the direct image by the map  $f$  with proper support.

## §1. Finiteness theorem for elliptic systems

In this section, we recall the finiteness theorem for elliptic systems due to T. Kawai [4].

Let  $M$  be a real analytic manifold,  $X$  be its complexification. We will denote by  $\mathcal{D}_X$  the sheaf of differential operators of finite order.  $\mathcal{D}_M$  is the restriction of  $\mathcal{D}_X$  to  $M$ . Let  $\mathcal{M}$  be an elliptic system of differential equations on  $M$ , that is, the coherent  $\mathcal{D}_M$ -Module whose singular support does not intersect with  $\sqrt{-1}S^*M = S_M^*X$ . Let  $\Omega$  be a relatively compact open subset in  $M$  whose boundary is a real analytic hypersurface in  $M$ . The following theorem is due to T. Kawai [4].

Theorem (1.1) If  $\mathcal{M}$  is an elliptic system on  $M$  and if the system  $\mathcal{M}_{\partial\Omega}$  induced from  $\mathcal{M}$  to the boundary  $\partial\Omega$  of  $\Omega$  is an elliptic system on  $\partial\Omega$ , then

$$\text{Ext}_{\mathcal{D}_M}^i(\Omega; \mathcal{M}, \mathcal{A}_M) = \text{Ext}_{\mathcal{D}_M}^i(\Omega; \mathcal{M}, \mathcal{B}_M)$$

are finite dimensional vector spaces (where  $\mathcal{A}_M$  (resp.  $\mathcal{B}_M$ ) is a sheaf of real analytic functions (resp. hyperfunctions) on  $M$ ).

This theorem is not sufficient for our study of maximally overdetermined system. We must discuss on the invariantness of the cohomology group by the perturbation of domains.

Let  $\{\Omega_c\}_{c \in \mathbb{R}}$  be a family of relatively

compact open subsets in  $M$  which have real analytic hypersurfaces as their boundaries satisfying the following properties

$$(1.1) \quad \Omega_{c_1} \supset \bar{\Omega}_{c_2} \quad \text{for } c_1 > c_2$$

(1.2) For any  $c_0$ ,  $\Omega_{c_0}$  is a union of  $\Omega_c$  where  $c$  runs over the set of all numbers strictly less than  $c_0$ .

(1.3) For any  $c_0$ ,  $\{\Omega_c; c > c_0\}$  is a neighborhood system of the closure  $\bar{\Omega}_{c_0}$  of  $\Omega_{c_0}$ .

Theorem (1.2) Let  $\{\Omega_c\}$  be a family satisfying the above conditions. If  $\mathcal{M}$  and  $\mathcal{M}_{\partial\Omega_c}$  are elliptic, then the restriction homomorphisms

$$\text{Ext}_{\mathcal{D}_M}^i(\Omega_c; \mathcal{M}, \mathcal{A}_M) \rightarrow \text{Ext}_{\mathcal{D}_M}^i(\Omega_{c'}; \mathcal{M}, \mathcal{A}_M)$$

are isomorphisms for  $c \geq c'$ .

Proof

$$\text{Set } E_c^i = \text{Ext}_{\mathcal{D}}^i(\Omega_c; \mathcal{M}, \mathcal{A}_M).$$

Then there is a canonical map  $E_c^i \rightarrow E_{c'}^i$ , if  $c \geq c'$ . Note the following lemma (we omit its proof).

Lemma (1.3) Let  $\{V_c\}_{c \in \mathbb{R}}$  be a family of finite dimensional vector spaces parametrized by  $c \in \mathbb{R}$ . For  $c \geq c'$ , there are given the homomorphism  $\rho_{c',c}: V_c \rightarrow V_{c'}$ , satisfying the following chain condition:

$$(1.4.1) \quad \rho_{c_1 c_2} \rho_{c_2 c_3} = \rho_{c_1 c_3} \quad \text{for } c_1 \leq c_2 \leq c_3,$$

$$(1.4.2) \quad \rho_{c,c} = \text{identity}$$

Moreover, they satisfy the following property of continuity.

$$(1.5.1) \quad \varprojlim_{c < c_0} V_c \xrightarrow{\sim} V_{c_0} \quad \text{for any } c_0$$

$$(1.5.2) \quad \varinjlim_{c > c_0} V_c \xrightarrow{\sim} V_{c_0} \quad \text{for any } c_0$$

Then all  $\rho_{c,c}$  are isomorphisms.

By virtue of this lemma, it suffices to show

$$\varinjlim_{c > 0} E_c^i = E_0^i, \quad \varprojlim_{c < 0} E_c^i = E_0^i.$$

Lemma (1.4)  $\varinjlim_{c > 0} E_c^i = E_0^i$

Proof

Since  $\mathcal{M}$  is elliptic and  $\mathcal{M}_{\partial\Omega_0}$  is elliptic, we have

$$R\Gamma(\partial\Omega_0; R\Gamma_Z(\mathbb{R}\text{Hom}(\mathcal{M}, \mathcal{O}_M)|_{\partial\Omega_0})) = 0$$

by virtue of Kashiwara-Kawai [3], where  $Z = M - \bar{\Omega}_0$ .

It follows that

$$\begin{aligned} & \varinjlim_V H_{Z \wedge V}^i(V; \mathbb{R}\text{Hom}(\mathcal{M}, \mathcal{O}_M)) \\ &= \varinjlim_V \text{Ext}_{Z \wedge V}^i(V; \mathcal{M}, \mathcal{O}_M) \\ &= 0 \end{aligned}$$

where  $V$  runs over the neighborhood system of  $\partial\Omega_0$ .

There for

$$\varinjlim_{c \downarrow 0} \text{Ext}^i(\Omega_c \text{ mod } \Omega_0; \mathcal{M}, \mathcal{O}_M) = 0$$

The long exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Ext}^i(\Omega_C \text{ mod } \Omega_0; \mathcal{M}, \mathcal{O}_M) &\rightarrow \text{Ext}^i(\Omega_C; \mathcal{M}, \mathcal{O}_M) \\ &\rightarrow \text{Ext}^i(\Omega_0; \mathcal{M}, \mathcal{O}_M) \rightarrow \cdots \end{aligned}$$

implies the desired result. Lastly, we will prove the following

Lemma (1.5)  $\varprojlim_{c < 0} E_c^i = E_0^i$

Proof

Since  $E_c^{i-1}$  is finite dimensional,  $\{E_c^{i-1}\}_{c < 0}$  satisfies the condition (ML). Therefore  $\varprojlim_{c < 0} E_c^i = E_0^i$  (see [1]).

Q.E.D.

This completes the proof of Theorem (1.2).

We will apply Theorem (1.2) to the special case.

Let  $X$  be a complex manifold,  $\mathcal{M}$  be a coherent  $\mathcal{D}_X$ -Module. A real analytic submanifold  $N$  of  $X$  is said to be non characteristic with respect to  $\mathcal{M}$  when the singular support of  $\mathcal{M}$  does not intersect with the conormal bundle of  $N$ .

Theorem (1.6) Let  $\{\Omega_c\}_{c \in \mathbb{R}}$  be a family of relatively compact open subsets of  $X$  with real analytic hypersurfaces as boundary satisfying (1.1), (1.2) and (1.3),  $\mathcal{M}$  be a coherent  $\mathcal{D}_X$  Module. Assume that the boundary  $\partial\Omega_c$  is non-characteristic with respect to  $\mathcal{M}$ . Then  $\text{Ext}_{\mathcal{D}_X}^i(\Omega_c; \mathcal{M}, \mathcal{O}_X)$  are finite dimensional and all the restriction homomorphisms

$$\text{Ext}_{\mathcal{D}_X}^i(\Omega_c; \mathcal{M}, \mathcal{O}_X) \rightarrow \text{Ext}_{\mathcal{D}_X}^i(\Omega_{c'}; \mathcal{M}, \mathcal{O}_X)$$

are isomorphisms for  $c \geq c'$ .

Proof

We will consider  $X$  as real analytic manifold and  $\mathcal{O}_X$  be a solution of Cauchy-Riemann equation. We can take  $X \times \bar{X}$  as a complex neighborhood of  $X$  where  $\bar{X}$  is the complex conjugate of  $X$ .

$$\text{Then } \mathcal{O}_X = \mathbb{R} \text{ Hom}_{\mathcal{D}_{X \times \bar{X}}}(\mathcal{D}_X \hat{\otimes} \mathcal{O}_{\bar{X}}; \mathcal{B}_X).$$

Therefore

$$\text{Ext}_{\mathcal{D}_X}^i(\Omega_c; \mathcal{M}, \mathcal{O}_X) = \text{Ext}_{\mathcal{D}_{X \times \bar{X}}}^i(\Omega_c; \mathcal{M} \hat{\otimes} \mathcal{O}_{\bar{X}}, \mathcal{B}_X).$$

Since  $\mathcal{M} \hat{\otimes} \mathcal{O}_{\bar{X}}$  is an elliptic system on  $X$  and  $\mathcal{M} \hat{\otimes} \mathcal{O}_{\bar{X}}|_{\partial\Omega_c}$  is also elliptic, the theorem follows from Theorem (1.1) and (1.2).

Q.E.D.

## §2. Finitistic sheaves

In order to clarify the main theorem of this paper, we will introduce the following notion. In this section,  $A$  is always a commutative noetherian ring and sheaves are  $A$ -Modules. In the following sections we take  $\mathbb{C}$  as  $A$ .

Definition (2.1) Let  $S$  be an analytic space,  $F$  be a sheaf on  $S$ . We will say that  $F$  is finitistic, if there is a stratification of  $S$  such that  $F$  is locally constant on each stratum and every stalk is an  $A$ -module of finite type.

It is evident that the category of finitistic sheaves is an abelian category, and if there is an exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

of sheaves and two of them are finitistic, then so is the other.

If  $F$  and  $G$  are finitistic, then  $\mathcal{T}or_z^A(F, G)$  is also finitistic.

Firstly, we will prove the following

Lemma (2.2) Let  $F$  be a sheaf on  $S$ ,  $S = \cup X_\alpha$  be a stratification which satisfies the regularity conditions of Whitney, and  $F$  is locally constant on each stratum. Then, for every  $x_0 \in X_\alpha$ , there is sufficiently small open neighborhood  $U$  of  $x_0$  such that

$$F(U) \longrightarrow F_x$$

is an isomorphism for every point  $x$  in  $X_\alpha$  sufficiently



near  $x_0$ , and  $H^k(U, F) = 0$  for every  $k > 0$ .

Proof.

We choose a local coordinate. It is sufficient to show that any  $\epsilon$ -neighborhood of  $x_0$  satisfies the condition of the lemma for any sufficiently small  $\epsilon$ . Let  $X_\beta$  be an open stratum in the support of  $F$  and  $j$  be the inclusion map.

Then,

$$0 \rightarrow j_!(F|_{X_\beta}) \rightarrow F \rightarrow F|_{S-X_\beta} \rightarrow 0$$

is an exact sequence in a neighborhood of  $X_\alpha$ . Since the support  $j_!(F|_{X_\beta})$  is contained in  $X_\beta$  and that of  $F|_{S-X_\beta}$  is contained in  $S-X_\beta$ , we can assume that the support of  $F$  is concentrated onto only one stratum  $X_\beta$  dominating  $X_\alpha$ . Then, if  $U$  is an  $\epsilon$ -neighborhood and  $x$  is a point in  $X_\alpha$  sufficiently near  $x_0$ , then  $V \cap X_\beta$  is a deformation retract of  $U \cap X_\beta$  for sufficiently small open ball  $V$  with center  $x$ . It follows that  $H^k(U, F) \cong H^k(V, F)$ , which shows this lemma. Q.E.D.

Lemma (2.3) Let  $S$  be an analytic set,  $F$  be a finitistic sheaf on  $S$ . Let  $W$  be the largest open set in  $S$  such that  $F|_W$  is locally constant on  $W$ . Then  $T = S - W$  is an analytic subset of  $S$  with condimension  $\geq 1$ .

Proof

It is evident that  $T$  is nowhere dense. So, it suffices to show that  $T$  is an analytic set. Let  $S = \cup X_\alpha$  be a stratification which satisfies the regularity conditions of Whitney of  $S$  such that  $F$  is locally constant on each stratum. It suffices to show that  $W \cap X_\alpha \neq \emptyset$ , then  $W$  contains

$X_\alpha$ , because this implies that  $T$  is a union of strata. Since  $X_\alpha$  is connected, it suffices to show that  $W \cap X_\alpha$  is a closed subset. Let  $x_0$  be a cluster point of  $W \cap X_\alpha$  in  $X_\alpha$ . By the induction of codimension of strata, we may assume that  $W$  contains all strata dominating  $X_\alpha$ , which implies that  $F$  is locally constant on  $U - X_\alpha$ , where  $U$  is a sufficient small neighborhood of  $x_0$ . By the preceding lemma, we can assume that

$$F(U) \rightarrow F_x$$

is an isomorphism for any  $x \in X_\alpha$  sufficiently near  $x_0$ . Since  $F$  is locally constant on  $W$ ,  $F(U) \rightarrow F_x$  is an isomorphism for some point  $x$  in  $W \cap X_\alpha \cap U$ . It implies  $F$  is locally constant in a neighborhood of  $x_0$  because we can assume that  $U \cap X_\beta$  is connected for any stratum  $X_\beta$ . It implies that  $x_0$  is contained in  $W$ . Q.E.D.

By using above lemmas, we can show that finitisticity is a local property.

Proposition (2.3) Let  $F$  be a sheaf on  $S$ . If there is an open covering of  $S$  such that  $F$  is finitistic on each over, then  $F$  is finitistic.

Proof

Let  $W$  be the largest open subset of  $S$  on which  $F$  is locally constant. Set  $T = S - W$ .  $T$  is a closed analytic set of  $S$ . Then, by the induction, we can assume that  $F|_T$  is finitistic. Therefore  $F$  is finitistic. Q.E.D.

We will introduce the following notion.

Definition (2.4) Let  $F$  be a sheaf on  $S$ . The stratification  $S = \cup_{\alpha} U_{\alpha}$  of  $S$  is said to be regular with respect to  $F$  if it satisfies the regularity conditions of Whitney and  $F$  is locally constant on each stratum.

We will prove that "finitisticity" is invariant under "cohomology".

Proposition (2.45) Let  $S = \cup_{\alpha} U_{\alpha}$  be a stratification which satisfies the regularity conditions of Whitney,  $T$  be a closed analytic subset of  $S$  which is a union of strata and  $j$  be the inclusion map  $S-T \hookrightarrow S$ .

Let  $F$  be a finitistic sheaf on  $S-T$  locally constant on any stratum. Then  $R^k j_{*}(F)$  and  $R^k j_{!}(F)$  are finitistic sheaves on  $S$  and locally constant on each stratum. Moreover, if  $S$  is a manifold and  $Z$  is a submanifold of  $S$  transversal to each stratum; then

$$R^k j_{*}(F)|_Z = R^k j_{*}(F|_Z)$$

$$R^k j_{!}(F)|_Z = R^k j_{!}(F|_Z)$$

on  $Z$ .

Proof

By the long exact sequence

$$\cdots \rightarrow R^k j_{!}(F) \rightarrow R^k j_{*}(F) \rightarrow R^k j_{*}(F)|_T \rightarrow \cdots$$

it is sufficient to show the statement on the direct image. We can suppose without loss of generality that  $S$  is a manifold and the support of  $F$  consists of only one stratum  $X_\beta$ . Let  $X_\alpha$  be the stratum contained in  $T$  and dominated by  $X_\beta$ . Let  $d$  be the codimension of  $X_\alpha$ ,  $H$  be the submanifold of dimension  $d$  transversal to  $X_\alpha$ . Since  $(X_\beta, X_\alpha)$  is locally isomorphic to  $((X_\beta \wedge H) \times X_\alpha, X_\alpha)$ ,  $R^k j_*(F)|_{X_\alpha}$  is locally constant. Since  $X_\beta \wedge H$  is triangulated by finite cell, the stalk of  $R^k j_*(F)$  is an  $A$ -module of finite type.

$(X_\beta \wedge Z, Z \wedge X_\alpha)$  being local isomorphic to  $((X_\beta \wedge H) \times (Z \wedge X_\alpha), Z \wedge X_\alpha)$ , we obtain  $R^k j_*(F)|_Z = R^k j_*(F|_Z)$ .

Q.E.D.

As a corollary of this proposition we obtain the following

Proposition (2.6) If  $F$  is a finitistic sheaf on  $S$  and  $T$  is a closed analytic set in  $S$ , the  $\mathcal{N}_T^k(F)$  is also a finitistic sheaf on  $T$ . More precisely, if  $S = \cup X_\alpha$  is a stratification of  $S$  regular with respect to  $F$  and  $T$  is a union of strata, then it is a stratification regular with respect to  $\mathcal{N}_T^k(F)$ .

Proof

Let  $j$  be the inclusion map  $S \supset T \hookrightarrow S$ . Then the exact sequences

$$0 \rightarrow \mathcal{H}_T^0(F) \rightarrow F \rightarrow j_* j^{-1} F \rightarrow \mathcal{H}_T^1(F) \rightarrow 0$$

$$\mathcal{N}_T^k(F) = R^{k-1} j_* j^{-1} F \quad (k > 1)$$

and Proposition (2.5) implies the proposition.

In the same reasoning, we obtain the following

Proposition (2.7) Let  $X$  be a complex manifold  $Y$  be a submanifold,  $T$  be an analytic subset of  $X$  and  $F$  be a finitistic sheaf on  $X$ . Assume that there exists a stratification of  $X$  regular with respect to  $F$  such that  $Y$  is transversal to each stratum and  $T$  is a union of strata, then

$$\mathcal{N}_T^k(F)|_Y \xrightarrow{\cong} \mathcal{H}_{T \cap Y}^k(F|_Y).$$

Corollary (2.8) Under the assumption as above, the support of  $\mathcal{H}_T^k(F)$  is a locally closed analytic set of codimension  $\geq k/2$  in  $S$ .

In fact, if  $X_\alpha$  is a stratum of codimension  $< k/3$  we can take  $Y$  whose dimension  $< k/2$ . Therefore

$\mathcal{H}_{T \cap Y}^k(F|_Y) = 0$  because the real dimension of  $Y$  is less than  $k$ .

§3. Finiteness theorem for  
maximally overdetermined system

Let  $X$  be a complex manifold of dimension  $n$ . We will show, in this section, the following main theorem.

Theorem (3.1) If  $\mathcal{M}$  is a maximally overdetermined system on  $X$ , then  $\text{Ext}_{\mathcal{O}_X}^i(\mathcal{M}, \mathcal{O}_X)$  are finitistic.

Remember that a maximally overdetermined system on  $X$  is, by the definition, a coherent  $\mathcal{O}_X$  Module whose singular support is of codimension  $n$  in  $P^*X$  at each point in it.

From now,  $\mathcal{M}$  is always a maximally overdetermined system.

Firstly, we will discuss the Lagrangean analytic set. Lagrangean analytic set is by the definition the involutory analytic subset in  $P^*X$  of dimension  $(n-1)$  (a fortiori, purely  $(n-1)$  dimensional). An analytic set of pure codimension  $n$  is Lagrangean if and only if the fundamental 1-form on  $P^*X$ -vanishes on the tangent cone of it at any point (equivalently on the non singular locus). The singular support of a maximally overdetermined system is always Lagrangean.

Lemma (3.2) Let  $\Lambda$  be a Lagrangean closed analytic set in  $P^*X$ . Then there is a stratification  $X = \cup X_\alpha$  of  $X$  satisfying the regularity conditions a) and b) of Whitney [ ] such that

$$(3.1) \quad \Lambda \subset \cup_{X_\alpha} P_{X_\alpha}^* X$$

where  $P_{X_\alpha}^* X$  is the conormal projective bundle of  $X_\alpha$  in  $X$ .

Proof

Let  $\pi$  be the projection from  $P^*X$  onto  $X$ . Set  $X_0 = X - \pi(\Lambda)$ .  $X_0$  is a dense open set in  $X$  because  $\dim \pi(\Lambda)$  is less than  $(n-1)$ . Let  $X_1^!$  be a set of non singular point of  $\pi(\Lambda)$ . Note the following fact.

Sublemma (3.3) Let  $\Lambda$  be a Lagrangean closed analytic set in  $P^*X$ . Assume that  $Y = \pi(\Lambda)$  be a non singular set. Then  $\Lambda$  contains  $P_Y^*X$  and  $\pi(\overline{\Lambda - P_Y^*X})$  is an analytic subset of  $Y$  with codimension  $\geq 1$ .

Proof

The question being local in  $Y$ , we can assume that  $Y$  is defined by  $x_1 = \dots = x_r = 0$ . Since  $\Lambda$  is involutory  $\Lambda$  is invariant by the infinitesimal transformation  $\partial/\partial\xi_1, \dots, \partial/\partial\xi_r$  where  $\xi$  is cotangent vector. Therefore  $\Lambda$  contains  $\{(x, \xi); x_1 = \dots = x_r = 0, \xi_{r+1} = \dots = \xi_n = 0\}$ , which equals to  $P_Y^*X$ . Since  $\overline{\Lambda - P_Y^*X}$  is also Lagrangean, and  $\overline{\Lambda - P_Y^*X}$  does not contain  $P_Y^*X$ ,  $\pi(\overline{\Lambda - P_Y^*X})$  is not equal to  $Y$ . Q.E.D.

Set  $\Lambda_1 = \overline{\Lambda - P_{X_1^!}^*X}$ ,  $X_1 = X_1^! - \pi(\Lambda_1)$ . Then  $X = X_0 \cup X_1 \cup \pi(\Lambda_1)$ . By the induction we define  $X_j, X_j^!$  and  $\Lambda_{j+1}$  as follows.  $X_j^!$  is a non singular locus of  $\pi(\Lambda_j)$ .

$$\begin{aligned}\Lambda_{j+1} &= \overline{\Lambda_j - P_{X_j^!}^*X} \\ X_j &= X_j^! - \pi(\Lambda_{j+1}).\end{aligned}$$

Then, since  $\dim X_j$  is strictly decreasing,  $X_{n+1} = \phi$ , and therefore  $\Lambda_n = \phi$ . It is clear that  $\{X_j\}$  is a stratification

of  $X$  and satisfies (3.1). By Whitney [ ], there exists a refinement of  $\{X_j\}$  which is a stratification satisfying the regularity conditions. It is evident that this stratification satisfies (3.1).

We remark that the regularity condition (a) of Whitney is equivalent to say that  $UP_{X_\alpha}^* X$  is a closed analytic set of  $P^*X$ . Since the stratification which satisfies (3.1) appears frequently, we introduce the following notion.

Definition (3.4) Let  $\mathcal{M}$  be a maximally overdetermined system on  $X$ . The stratification of  $X$  is said to be regular with respect to  $\mathcal{M}$  if it satisfies the regularity conditions a) b) of Whitney and the singular support of  $\mathcal{M}$  is contained in the union of the conormal projective bundle of strata.

Now we will prove the following refined form of Theorem (3.1).

Theorem (3.5) Let  $\mathcal{M}$  be a maximally overdetermined system on  $X$ ,  $X = \cup X_\alpha$  be a stratification of  $X$  which is regular with respect to  $\mathcal{M}$ . Then  $Ext_{\mathcal{O}_X}^i(\mathcal{M}, \mathcal{O}_X)|_{X_\alpha}$  is a locally constant  $\mathbb{C}$ -Module of finite rank.

We will prove firstly the following preparatory lemma.

Lemma (3.6) Let  $x_0$  be a point in  $X_\alpha$ , and choose a local coordinate near  $x_0$ . Then  $(x, (\bar{x}-\bar{y})^\infty)$  does not belong to  $SS(\mathcal{M})$  for  $x \in X$ ,  $y \in X_\alpha$  such that  $|x-x_0| \ll 1$ ,  $|y-x_0| \ll 1$



and  $x \neq y$ . (where  $\bar{x}$  is the complex conjugate of  $x$ )

Proof

We may assume  $x \in X_\beta$ , where  $X_\beta$  is a stratum whose closure contains  $X_\alpha$ . If the lemma is false there is a sequence  $X_\beta \ni x_n$  and  $X_\alpha \ni y_n$  which converge to  $x_0$  and  $(x_n, (\bar{x}_n - \bar{y}_n)^\infty) \in SS(\mathcal{M})$ . We may assume that  $(TX_\beta)_{x_n}$  tends to  $T$  and  $a_n(x_n - y_n)$  tends to a non zero vector  $v$  in  $T$ , where  $a_n$  is a sequence in  $\mathbb{C}^X$ . By the assumption  $\bar{a}_n(\bar{x}_n - \bar{y}_n) \in (T_{X_\beta}^* X)_{x_n}$ , which is an orthogonal vector space of  $(TX_\beta)_{x_n}$ . It follows that  $\bar{v}$  belongs in the orthogonal of  $T$ , which implies  $\langle v, \bar{v} \rangle = 0$ . This is a contradiction.

Q.E.D.

Now we can prove Theorem (3.5). Let  $x_0$  be a point in  $X_\alpha$ . We chose a local coordinate of  $X$  such that  $x_0$  is the origin and  $X_\alpha$  is a vector space. Suppose that  $(x, (\bar{x} - \bar{y})^\infty) \notin SS(\mathcal{M})$  for  $y \in X_\alpha$ ,  $x \in X$  such that  $|x| \leq c$ ,  $|y| \leq c$ ,  $x \neq y$ . Set  $\mathcal{P}(t, x, y) = |x - (1-t)y|^2 - t^2 c^2 / 2$ . Then  $\partial_x \mathcal{P}(t, x, y) \neq 0$  and

$$(x, \partial_x \mathcal{P}(t, x, y)) \notin SS(\mathcal{M})$$

$$\text{for } 0 < t \leq 1, |y| < c/2, \mathcal{P}(t, x, y) = 0.$$

It implies that the boundary of  $\Omega_{t,y} = \{x; \mathcal{P}(t, x, y) < 0\}$  is non characteristic to  $\mathcal{M}$ .

$$\Omega_{t_1, y} \subset \Omega_{t_2, y} \quad \text{if } t_1 \leq t_2$$

and  $\Omega_{1,y}$  is independent on  $y$ . By Theorem (1.6), the

restriction homomorphism

$$\text{Ext}^i(\Omega_{1,y}; \mathcal{M}, \mathcal{O}_X) \rightarrow \text{Ext}^i(\Omega_{t,y}; \mathcal{M}, \mathcal{O}_X)$$

are isomorphisms. Since  $\{\Omega_{t,y}\}_{t>0}$  is a neighborhood system of  $y$ , we have

$$\text{Ext}^i(\Omega_{1,0}; \mathcal{M}, \mathcal{O}_X) \cong \text{Ext}^i(\mathcal{M}, \mathcal{O}_X)_y.$$

for every  $y$ . It follows that  $\text{Ext}^i(\mathcal{M}, \mathcal{O}_X)|_{X_\alpha}$  is locally constant. Since  $\text{Ext}^i(\Omega_{1,0}; \mathcal{M}, \mathcal{O}_X)$  is finite dimensional,  $\text{Ext}^i(\mathcal{M}, \mathcal{O}_X)_y$  is finite dimensional. It completes the proof of Theorem (3.5).

Let  $Y$  be a complex submanifold of  $X$  of codimension  $d$ .  $\mathcal{B}_{Y|X}$  is, by the definition, a left  $\mathcal{O}_X$ -Module  $\mathcal{N}_Y^d(\mathcal{O}_X)$ . Remark that  $\mathcal{N}_Y^k(\mathcal{O}_X)$  vanishes for any  $k$  except  $d$ .  $\mathcal{B}_{X|X}$  is nothing but  $\mathcal{O}_X$ .

Theorem (3.7) Let  $X = \cup X'_\alpha$  be a regular stratification with respect to a maximally overdetermined system  $\mathcal{M}$ , and  $Y$  be a complex submanifold of  $X$  which is a union of strata. Then  $\text{Ext}_{\mathcal{O}_X}^i(\mathcal{M}, \mathcal{B}_{Y|X})$  is finitistic on  $Y$  and locally constant on each stratum.

This is a corollary of Theorem (3.5) and Proposition (2.6) because  $\text{Ext}_{\mathcal{O}_X}^i(\mathcal{M}, \mathcal{B}_{Y|X})$  is equal to  $\mathcal{N}_Y^{i+d}(\mathbb{R} \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X))$ .

Theorem (3.8) Under the same assumption as above,

$$\text{Ext}_{\mathcal{O}_X}^i(\mathcal{M}, \mathcal{B}_{Y|X})|_Z \cong \text{Ext}_{\mathcal{O}_Z}^i(\mathcal{M}_Z, \mathcal{B}_{Y \cap Z|Z})$$

for any  $i$  and any submanifold  $Z$  transversal to every stratum, where  $\mathcal{M}_Z$  is the induced system of  $\mathcal{M}$  onto  $Z$ .

This is a corollary of Theorem (3.7) and Proposition (2.7).

Lastly, we will remark the following propositions.

$\mathcal{B}_{Y|X}^f$  is, by the definition, a maximally overdetermined  $\mathcal{O}_X$  Module  $\varinjlim_k \text{Ext}_{\mathcal{O}_X}^r(\mathcal{O}_X/\mathcal{I}^k; \mathcal{O}_X)$  where  $r$  is the codimension of  $Y$  and  $\mathcal{I}$  is the defining ideal of  $Y$ . If  $Y$  is defined by  $x_1 = \dots = x_r = 0$  for a local coordinate,  $\mathcal{B}_{Y|X}^f = \mathcal{O}_X / \mathcal{O}_X^{x_1} + \dots + \mathcal{O}_X^{x_r} + \mathcal{O}_X^{D_{r+1}} + \dots + \mathcal{O}_X^{D_n}$ .

Proposition (3.9) Let  $\mathcal{M}$  be a maximally overdetermined system whose support is contained in an submanifold  $Y$  and whose singular support is contained in  $P_Y^*X$ , then  $\mathcal{M}$  is locally isomorphic to the direct sum of finite copies of  $\mathcal{B}_{Y|X}^f$ .

Proof

We choose a local coordinate  $(x_1, \dots, x_n)$  such that  $Y = \{(x_1, \dots, x_n); x_1 = \dots = x_r = 0\}$ .

Let  $u$  be a non zero section of  $\mathcal{M}$ . Since  $\mathcal{O}u$  is a coherent sheaf whose support is contained in  $Y$ , there is  $k$  such that  $x_1^k u = \dots = x_r^k u = 0$ . Therefore, there is a non zero section  $u$  of  $\mathcal{M}$  such that  $x_1 u = \dots = x_r u = 0$ . Since  $\text{Ext}^1(\mathcal{B}_{Y|X}^f, \mathcal{B}_{Y|X}^f) = 0$ , if we prove the proposition

for  $\mathcal{D}_u$  and  $\mathcal{M}/\mathcal{D}_u$ , then the proposition is true for  $\mathcal{M}$ . Since  $\mathcal{M}$  is noetherian, it suffices to show it for  $\mathcal{D}_u$ . Let  $\mathcal{D}_u = \mathcal{D}/\mathcal{J}$ .  $\mathcal{J}$  contains  $x_1, \dots, x_r$ . If  $\mathcal{J}$  contains  $P(x'', D) = \sum P_\alpha(x'', D'') D'^\alpha$  (where  $x'' = (x_{r+1}, \dots, x_n)$ ,  $D' = (D_1, \dots, D_r)$ ,  $D'' = (D_{r+1}, \dots, D_n)$ ) then  $\mathcal{J}$  contains all  $P_\alpha(x'', D'')$  since  $[x_i, D_i] = 1$ . It follows that  $\mathcal{D}_u = (\mathcal{D}'/\mathcal{D}'_{x_1} + \dots + \mathcal{D}'_{x_r}) \hat{\otimes} (\mathcal{D}''/\mathcal{J}'')$ , where  $\mathcal{D}' = \mathcal{D}_{\mathbb{C}^r}$ ,  $\mathcal{D}'' = \mathcal{D}_{\mathbb{C}^{n-r}}$ . Since the singular support of  $\mathcal{D}''/\mathcal{J}''$  is a void set,  $\mathcal{D}''/\mathcal{J}''$  is a finite sum of  $\mathcal{O}_{\mathbb{C}^{r-r}}$  (See Kashiwara [2]). It follows the proposition. Q.E.D.

Corollary (3.10) If  $\text{Ext}_{\mathcal{D}_X}^i(\mathcal{M}, \mathcal{O}_X) = 0$  for every  $i$ , then  $\mathcal{M} = 0$ .

Because  $\text{Ext}_{\mathcal{D}_X}^r(\mathcal{B}_{Y|X}^f, \mathcal{O}_X) = \mathbb{C}_Y$  where  $r$  is the codimension of  $Y$ .

This corollary means that the functor  $\mathcal{M} \mapsto \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$  is a faithful functor. This will be investigated more precisely in the next paper.

§4 Several properties of cohomologies of holomorphic solutions of maximally overdetermined systems

Let  $\mathcal{M}$  be a maximally overdetermined system on a complex manifold  $X$  of dimension  $n$ . We fix a stratification  $X = \cup X_\alpha$  of  $X$  which is regular with respect to  $\mathcal{M}$ . The support  $\text{Supp Ext}^i(\mathcal{M}, \mathcal{O}_X)$  is a set of  $x$  where  $\text{Ext}^i(\mathcal{M}, \mathcal{O}_X)_x \neq 0$ . Since  $\text{Ext}^i(\mathcal{M}, \mathcal{O}_X)$  is locally constant on the strata, its support is a union of strata, which implies that the support is a locally closed analytic set (and its closure is also an analytic set).

Theorem (4.1) The support of  $\text{Ext}^i(\mathcal{M}, \mathcal{O}_X)$  is a locally closed analytic set of codimension  $\geq i$ .

Corollary (4.2) If  $s$  is a section of  $\text{Ext}^i(\mathcal{M}, \mathcal{O}_X)$ , then the support of  $s$  is analytic set of codimension  $\geq i$ . Firstly we introduce the notation of the modified singular support  $\widehat{SS}(\mathcal{M})$ .  $\widehat{SS}(\mathcal{M}) = T^*X \cap SS(\mathcal{M} \hat{\otimes} \mathcal{D}_{\mathbb{C}} / \mathcal{D}_{\mathbb{C}} t)$ , where  $t$  is a coordinate of  $\mathbb{C}$  and we embed  $T^*X$  into  $P^*(X \times \mathbb{C})$  in the following way:  $(x, \xi) \mapsto (x, 0; (\xi, 1) \infty)$ .  $\widehat{SS}(\mathcal{M})$  is a closed analytic set in the cotangent bundle  $T^*X$ . It is clear that  $\widehat{SS}(\mathcal{M})$  is a cone, that is, invariant by the multiplication of complex number. Note that the image of the map  $\widehat{SS}(\mathcal{M}) - X \rightarrow P^*X$  coincides with  $SS(\mathcal{M})$ .  $\widehat{SS}(\mathcal{M}) = \emptyset$  if and only if  $\mathcal{M} = 0$  and  $\widehat{SS}(\mathcal{M})$  is contained in the zero section if and only if  $\mathcal{M}$  is locally isomorphic to

the finite number of direct sum of  $\mathcal{O}_X$ . If  $\mathcal{M} = \cup \mathcal{M}_k$  is a good filtration of  $\mathcal{M}$ , then  $\widehat{SS}(\mathcal{M})$  is the support of the coherent sheaf on  $T^*X$  corresponding to the graded Module  $\bigoplus (\mathcal{M}_k / \mathcal{M}_{k-1})$ . (See Kashiwara [2]).

Proposition (4.3) In a neighborhood of  $x$  in  $X$ , we have  $\text{Ext}_{\mathcal{D}_X}^i(\mathcal{M}, \mathcal{O}_X) = 0$  for  $i > \dim(\widehat{SS}(\mathcal{M}) \cap T_x^*X)$  for any coherent  $\mathcal{D}_X$ -Module  $\mathcal{M}$  (not necessarily maximally overdetermined).  $T_x^*X$  is a fiber on  $x$ .

Proof

Set  $d = \dim(\widehat{SS}(\mathcal{M}) \cap T_x^*X)$ . If  $d = -\infty$ , then this is clear. Suppose that  $d \geq 1$ . Then there is a  $d$ -dimensional submanifold  $Y$  through  $x$  such that  $P^*X \cap \widehat{SS}(\mathcal{M}) = \emptyset$  in a neighborhood of  $x$  which means that  $Y$  is non characteristic with respect to  $\mathcal{M}$ . By Kashiwara [2], we have

$$\text{Ext}_{\mathcal{D}_X}^i(\mathcal{M}, \mathcal{O}_X)_x = \text{Ext}_{\mathcal{D}_Y}^i(\mathcal{M}_Y, \mathcal{O}_Y)_x$$

where  $\mathcal{M}_Y$  is the induced system of  $\mathcal{M}$ . Since the global dimension of  $\mathcal{D}_{Y,x}$  is  $d = \dim Y$  (See Kashiwara [2]),  $\text{Ext}_{\mathcal{D}_Y}^i(\mathcal{M}_Y, \mathcal{O}_Y)_x = 0$  for  $i > d$ . Since  $\dim(\widehat{SS}(\mathcal{M}) \cap T_x^*X)$  is upper semi-continuous, the proposition follows. Q.E.D.

Corollary (4.4)  $\text{Ext}^i(\mathcal{M}, \mathcal{O}_X)|_{X_\alpha} = 0$  for  $i > \text{codim } X_\alpha$ .

This corollary immediately implies Theorem (4.1). Let  $Y$  be a submanifold of  $X$  of codimension  $c$ . By taking a refinement of the regular stratification of  $Y$ , we may assume

that  $Y$  is a union of strata. By replacing  $\mathcal{O}_X$  with  $\mathcal{B}_{Y|X}$ , we obtain the same type of the preceding theorems.

Theorem (4.7) The support of  $\text{Ext}^i(\mathcal{M}, \mathcal{B}_{Y|X})$  is a (locally closed) analytic subset of  $Y$  of codimension  $\geq (i-c)$  in  $Y$ .

Corollary (4.8) The support of a global section of  $\text{Ext}^i(\mathcal{M}, \mathcal{B}_{Y|X})$  is an analytic subset of  $Y$  of codimension  $\geq (i-c)$ .

Since this theorem can be proved in the same way as before, we do not repeat it.

## §5. Duality

Let  $\mathcal{M}$  be a maximally overdetermined system on complex manifold  $X$  of dimension  $n$  and  $x_0$  be a point in  $X$ .

Then there exists a free resolution

$$(5.1) \quad 0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X^{r_0} \xleftarrow{P_0(x,D)} \mathcal{D}_X^{r_1} \xleftarrow{P_1(x,D)} \cdots \xleftarrow{P_N(x,D)} \mathcal{D}_X^{r_N} \leftarrow 0$$

in a neighborhood of  $x_0$  where  $P_i(x,D)$  is an  $(r_{i+1} \times r_i)$  matrix of differential operators. Then  $\text{Ext}_{\mathcal{D}}^i(\mathcal{M}, \mathcal{O})_{x_0}$  is an  $i$ -th cohomology of

$$(5.2) \quad \mathcal{O}_{X,x_0}^{r_0} \xrightarrow{P_0(x,D)} \mathcal{O}_{X,x_0}^{r_1} \rightarrow \cdots \xrightarrow{P_N(x,D)} \mathcal{O}_{X,x_0}^{r_N}$$

$\text{Tor}_i^{\mathcal{D}}(\mathcal{B}_{\{x_0\}|X}^{(n)}, \mathcal{M})$  is an  $i$ -th homology of

$$(5.3) \quad \mathcal{B}_{\{x_0\}|X}^{(n)r_0} \leftarrow \mathcal{B}_{\{x_0\}|X}^{(n)r_1} \leftarrow \cdots \leftarrow \mathcal{B}_{\{x_0\}|X}^{r_N}$$

where  $\mathcal{B}_{\{x_0\}|X}$  is a left  $\mathcal{D}$ -Module defined by  $\mathcal{K}_{\{x_0\}}^n(\mathcal{O}_X)$  and  $\mathcal{B}_{\{x_0\}|X}^{(n)}$  a right  $\mathcal{D}_X$  Module defined by

$$\Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{B}_{\{x\}|X} = \mathcal{H}_{\{x_0\}}^n(\mathcal{O}_X)$$

Since  $\mathcal{O}_{X,x_0}$  is an (DFS) topological vector space and



$\mathcal{B}_{\{x_0\}}^{(n)}|_X$  is a (FS) topological vector space and they are dual to each other. Since the cohomology of (5.2) is finite dimensional,  $\text{Ext}_{\mathcal{D}}^i(\mathcal{M}, \mathcal{O}_{X,x_0})$  is a dual vector space of  $\text{Tor}_i^{\mathcal{D}}(\mathcal{B}_{\{x_0\}}^{(n)}|_X, \mathcal{M})$ . This duality is obtained by the canonical cup product

$$(5.4) \quad \text{Ext}_{\mathcal{D}}^i(\mathcal{M}, \mathcal{O}_{X,x_0}) \times \text{Tor}_i^{\mathcal{D}}(\mathcal{B}_{\{x_0\}}^{(n)}|_X, \mathcal{M}) \\ \rightarrow \mathcal{B}_{\{x_0\}}^{(n)}|_X \otimes_{\mathcal{D}} \mathcal{O}_{X,x_0} = \mathbb{C}.$$

Setting  $\mathcal{M}^* = \text{Ext}_{\mathcal{D}}^n(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{n \otimes (-1)}$ , we call it the adjoint system of  $\mathcal{M}$ .  $\mathcal{M}^*$  is also a maximally overdetermined system and  $(\mathcal{M}^*)^* = \mathcal{M}$ , and  $\mathcal{M} \mapsto \mathcal{M}^*$  is a contravariant exact functor from the category of maximally overdetermined systems into itself. (See Kashiwara [2], Sato-Kawai-Kashiwara [6]). Remark that the existence of such a functor immediately implies that a stalk of maximally overdetermined system is a  $\mathcal{D}$ -module of finite length. Note that

$$\mathcal{O}_X^* = \mathcal{O}_X \quad \text{and} \quad \mathcal{B}_{\{x_i\}}^*|_X = \mathcal{B}_{\{x_0\}}|_X.$$

Since  $\text{Tor}_i^{\mathcal{D}}(\mathcal{B}_{\{x_0\}}^{(n)}|_X, \mathcal{M}) = \text{Ext}_{\mathcal{D}}^{n-i}(\mathcal{M}^*, \mathcal{B}_{\{x_0\}}|_X)$ , we obtain the following

Proposition (5.1)  $\text{Ext}_{\mathcal{D}}^{n-i}(\mathcal{M}^*, \mathcal{B}_{\{x_0\}}|_X)$  and  $\text{Ext}_{\mathcal{D}}^i(\mathcal{M}, \mathcal{O}_{X,x_0})$  are dual vector space of each other.

We will generalize this proposition. Let  $Y$  be a submanifold of  $X$  of codimension  $d$ ,  $x$  be a point in  $Y$ .

$\mathcal{B}_{Y|X}$  is by the definition the left  $\mathcal{D}_X$ -Module  $\mathcal{N}_Y^d(\mathcal{O}_X)$  and  $\mathcal{B}_{Y|X}^{(n)}$  is the right  $\mathcal{D}_X$ -Module  $\mathcal{N}_Y^d(\Omega_X^n)$ .

Since  $\text{Tor}_i^{\mathcal{D}}(\mathcal{B}_{Y|X}^{(n)}, \mathcal{O}_X) = \mathbb{C}_Y$  for  $i = n-d$  and  $0$  for  $i \neq n-d$ , there is a cup product

$$(5.5) \quad \text{Ext}_{\mathcal{D}}^i(\mathcal{M}, \mathcal{O}_X)_x \times \text{Tor}_{n+i-d}(\mathcal{B}_{Y|X}^{(n)}, \mathcal{M})_x \\ \rightarrow \text{Tor}_{n-d}(\mathcal{B}_{Y|X}^{(n)}, \mathcal{O}_X)_x = \mathbb{C}.$$

Theorem (5.2) Let  $X = \bigcup X_\alpha$  be a regular stratification with respect to  $\mathcal{M}$ . Let  $X_\alpha$  be a stratum of codimension  $d$  and  $x$  be a point in  $X_\alpha$ . Then, there is a canonical perfect pairing

$$(5.6) \quad \text{Ext}_{\mathcal{D}}^i(\mathcal{M}, \mathcal{O})_x \times \text{Ext}_{\mathcal{D}}^{d-i}(\mathcal{M}^*, \mathcal{B}_{X_\alpha|X})_x \rightarrow \mathbb{C}$$

that is, the two cohomologies are dual to each other.

Proof.

Since  $\text{Ext}^{d-i}(\mathcal{M}^*; \mathcal{B}_{X_\alpha|X}) = \text{Tor}_{n-d+i}(\mathcal{B}_{X_\alpha|X}^{(n)}; \mathcal{M})$ , the pairing (5.6) is induced from (5.5). Let  $Z$  be a  $d$ -dimensional submanifold transversal to  $X_\alpha$  and through  $x$ . Then, by Theorem (3.8)

$$\text{Ext}_{\mathcal{D}_X}^i(\mathcal{M}, \mathcal{O}_X)_x = \text{Ext}_{\mathcal{D}_Z}^i(\mathcal{M}_Z, \mathcal{O}_Z)_x$$

$$\text{Ext}_{\mathcal{D}_X}^{d-i}(\mathcal{M}^*, \mathcal{B}_{X_\alpha|X})_x = \text{Ext}_{\mathcal{D}_Z}^{d-i}(\mathcal{M}_Z^*; \mathcal{B}_{X_\alpha \wedge Z|Z})_x.$$

Therefore the theorem is a corollary of Proposition

(5.1).

q.e.d.

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