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ON THE COSINE PROBLEM OF S. CHOWLA

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Let A be a set of $n \geqq 1$ distinct integers a_1, a_2, ..., a_n. There will be no loss in generality in assuming that the a_j are non-negative, or positive, integers. Define for real x

$$E(x) = \sum_{j=1}^{n} e(a_j x) \qquad [e(x) = e^{2\pi i x}],$$

$$C(x) = \frac{1}{2}\{E(x) + E(-x)\} = \sum_{j=1}^{n} \cos 2\pi a_j x ,$$

and write

$$I(A) = \int_0^1 |E(t)|dt ,$$

$$M(A) = -\min_{0 \leqq x < 1} C(x) .$$

We have

$$M(A) \geqq \frac{1}{2}\int_0^1 |C(t)|dt ;$$

indeed,

$$\int_0^1 |C(t)|dt = \int_0^1 \{|C(t)| - C(t)\}dt \leqq 2M(A) .$$

Also, since

$$|C(x)| = \frac{1}{2} |\{E(-x) + E(x)\}e(ax)|$$

for any integer a, we find

$$M(A) \geqq \frac{1}{4} I(A_1) ,$$

where A_1 is a set of $2n$ distinct (positive, or non-negative) integers.

The Cosine Problem of Ankeny-Chowla (cf. [2]) is to prove that to an arbitrary positive integer K there corresponds an integer $n_0 = n_0(K)$ such that

$$M(A) > K$$

for all $n > n_0$, where $A = \{a_1, \ldots, a_n\}$ is *any* set of n different integers.

This problem is, as we have seen above, closely related to the

Problem of J. E. Littlewood (cf. [9]): Is it true or not that

$$I(A) > B \log n$$

for all sets A of n distinct positive integers $a_1, \ldots, a_n$? Here, and in what follows, B stands for an (unspecified) absolute constant > 0. Littlewood's conjecture $I(A) > B \log n$ is, if true, substantially the best possible.

In 1960 P. J. Cohen [5] and H. Davenport [6] succeeded in proving that for some $B > 0$ and all $n \geqq 3$

$$I(A) > B\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{4}},$$

so that

$$M(A) > B\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{4}},$$

which gives a solution to the Cosine Problem of Ankeny-Chowla.

In 1963 S. Chowla (cf. [3], [4]) revised the Cosine Problem and stated the conjecture: For any set A of n distinct integers one has

$$M(A) > B\, n^{\frac{1}{2}} .$$

This is, if true, essentially the best possible.

The Cosine Problem of S. Chowla (thus revised) seems to be still unsolved, and the best result known so far is the one due to K. F. Roth [17] who proved that

$$M(A) > B\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{2}}.$$

The main purpose of this article is to provide a list of papers published up to 1973, on the Cosine Problem and Littlewood's problem, as well as some allied ones. The list given below is, of course, not exhaustive.

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