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K.OKUBO: CONNECTION PROBLEMS FOR EQUATIONS OF FUCHSIAN CLASS AND FOR THEIR LAPLACE ADJOINT.

Differential Equations:

(1) $L[y]=t(dy/dt)+(A+tB)y = 0$

(2) $M[x]=(t-B)dx/dt-(A-I)x = 0$

(3) $N[z]=(t-B)dz/dt-Az = 0$

Assumptions. x,y,z n-vectors: A,B n by n matrices.

(A-1) $B=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ $\lambda_1=\lambda_2=\ldots=\lambda_p$, otherwise $|\lambda_j-\lambda_k|>|\lambda_k|>0$.

(A-2) $A=(a_{j,k})$. The initial block $A'=(a_{j,k})$ $1\leq j,k\leq p$ is diagonal matrix.

(A-3) All the diagonal elements $a_{kk}(k=1,2,\ldots,n)$ are non-integral.

(A-4) The eigenvalues $\rho_1,\rho_2,\ldots,\rho_n$ of the matrix A; $\det(\rho_j I-A)=0$ satisfy:

$$\rho_j-\rho_k \not\equiv 0 \pmod{1}$$

except for one set of q eigenvalues. But the system (1) at t=0, the systems (2.),(3) at $t=\infty$ have no logarithmic solution.

(A-5) All the quantities ρ_j-a_{kk} $(j,k=1,2,\ldots,n)$ are non-integral.

(A-6) At least one of ρ_j's is zero. (Just for the sake of simplicity).

Theorem 1. The system (3) has an independent set of solutions:

$$z_k(t) = \sum_{m=o}^{\infty} g_k(m)(t-\lambda_k)^{a_{kk}+m} \qquad (k=1,2,\ldots,n)$$

and we have:

$$\det(z_1(t),z_2(t),\ldots,z_n(t)) = \prod_{k=1}^{n} [(t-\lambda_k)^{a_{kk}}\Gamma(a_{kk}+1)/\Gamma(\rho_k+1)]$$

in a simply connected compact domain containing the origin t=0.

Theorem 2. The monodromy group of the system (3) with respect to the above set of solutions is a free group generated from:

$M_1=T_1^{-1}\mathrm{diag}[e_1,e_2,\ldots,e_p,1,\ldots,1]T_1$,T_1-I contains non-zero elements in the p by n-p block at the upper right corner.

$M_j = T_j^{-1}\mathrm{diag}(1,\ldots ,e_j,\ldots,1)T_j$, T_j-I contains non-zero elements only for j-th row vector whose j-th element is zero.

The group is completely determined if

$$p(p-1) = (n-q-1)(2-n-q) \qquad 1\leq p,q\leq n-1.$$

Theorem 3. The system (3) has a constant vector solution $z(t)=g$, which is the eigenvector for the matrix A corresponding to the eigenvalue 0. In case, we can determine the group completely, we can write down g as a linear combination of the solutions $z_1(y),\ldots,z_n(t)$. The coefficient of this linear combination is the eigenvector for the eigenvalue $\exp(2\pi i)=1$ of the matrix $M_1\cdots\cdots M_m$ and explicitly computable up to a constant multiplier.

By the Laplace transform

$$(4) \qquad y(t)=\int_C \exp(-t\sigma)x(\sigma)d\sigma$$

of a solution $x(\sigma)$ of (2), we have:

$$L[y]= [\exp(-t\sigma)(\sigma-B)x(\sigma)]_C + \int_C \exp(-t\sigma)M_\sigma[x(\sigma)]d\sigma$$

Theorem 4. Let $x(t)$ be the solution near $t=\infty$ corresponding to the eigenvalue -1 of $(A-I)$, then $y(t)$ defined by (4) is the solution of the system (1) corresponding to the characteristic exponent 0 at $t=0$, when C is a cicle of sufficiently large radius.

Theorem 5. Let $x_k(t)$ be the solution at $t=\lambda_k$ of the form:

$$x_k(t)=\sum_{m=o}^{\infty} h_k(m)(t-\lambda_k)^{a_{kk}+m-1} \qquad (h_k(o)=\varepsilon_k)$$

Let us define $y_k(t)$ by:

$$y_k(t) =\int_o^{(\lambda_k)} \exp(-t\sigma)x_k(\sigma)d\sigma.$$

Then we have:

$$L[y_k] = (1-e_k)(-B)x_k(o), \qquad y_k(o)=(1-e_k)a_{kk}{}^{-1}z_k(o).$$

Theorem 6. $$y(t) = \sum_{k=1}^{n} \int_o^{(\lambda_k)} \exp(-t\sigma)(a_{kk}-1)(1-e_k)^{-1}S_k x_k(\sigma)d\sigma$$

$$\left(\sum_{k=1}^{n} S_k z_k(t) = g \right).$$