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Asymptotic Invariants for Slowly Changing Ordinary Linear Differential Equations

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1. History of the Problem.

If a physical system changes, most of the quantities describing its state also change. Occasionally, one meets invariants with respect to change, and such invariants are the cornerstones of physical theories. I need only mention conservation of mass, energy or momentum to indicate what I am referring to.

I wish to talk about physical quantities that are not invariant in the sense just mentioned, but which change "very little", if the system changes "very slowly". They are called "adiabatic invariants" and have been the subject of many studies during the last 60 years. Let me begin with an example that is more in the nature of an analogy: Take a physical system consisting of a man and a mountain and consider the process of the man climbing the mountain. The man's elevation above sea level is a quantity connected with the system, and so is the rate of his heart beat. The former quantity changes as the man climbs the mountain, no matter how long it takes, but his heart beat depends on his speed. In fact, if the process of climbing is performed very slowly, the rate of heart beat will change hardly at all.

The origin of the term "adiabatic invariant" is the thermodynamics of a system without heat transfer. The entropy of such a system remains unchanged if its state is changed "infinitely slowly".

The simplest instance of an adiabatic invariant is connected with the motion of a frictionless pendulum that oscillates with a small amplitude. The distance $u(\tau)$ of the pendulum from its rest position at time τ satisfies the differential equation

$$\frac{d^2u}{d\tau^2} + \phi^2 u = 0 , \qquad (1.1)$$

where ϕ is the frequency. Let m denote the mass of the pendulum, then, as is well known, the total energy,

$$\frac{1}{2}m[(\frac{du}{d\tau})^2 + \phi^2 u^2] \qquad (1.2)$$

is a strict invariant, i.e., independent of τ .

Now assume that the length of the pendulum is slowly altered. Then $\phi = \phi(\tau)$ is no longer constant. The differential equation

$$\frac{d^2u}{d\tau^2} + \phi^2(\tau)u = 0 \tag{1.3}$$

is still valid, but the quantity (1.2) now changes with time, and this change may be quite substantial, if the length, and therefore $\phi(\tau)$, is changed substantially, no matter how slowly the system is altered. It turns out, however, that the ratio of the expression (1.2) and $\phi(\tau)$ changes very little, if the pendulum is changed slowly. In other words, the quantity

$$A(\tau) = \phi^{-1}(\tau)\left(\frac{du}{d\tau}\right)^2 + \phi(\tau)u^2 \tag{1.4}$$

(we have removed the irrelevant factor $m/2$) is an adiabatic invariant.

The recognition that $A(\tau)$ is an adiabatic invariant has an interesting history. In the early years of this century, which are also the early years of Quantum Theory, the atom was thought of as a vibrating system governed by the laws of classical physics, but subject to the rule that the quotient energy/frequency remained constant over long periods of time and then changed abruptly by some multiple of Planck's constant. One of the mysteries of this behavior was that the continuously changing forces of the surrounding electro-magnetic field ought to change that ratio in a continuous manner. In a short but memorable exchange of opinions between Lorentz and Einstein in 1911 at the first International Solvay Congress Lorentz asked the question above and mentioned the illustration by the pendulum of changing length. He also conjectured an explanation: The quotient in question changes very little if the field changes slowly, and measured by the high frequencies of atomic radiation the changes of the exterior forces are very slow indeed. Then Einstein got up and stated that he had performed calculations showing that the quantity energy/frequency did change infinitesimally little if the data changed infinitely slowly.

It is not known how Einstein obtained his result, but now, 63 years later, there exists a substantial literature on this and related questions. It is true that since 1924 Quantum Mechanics has superseded the older attempts at explaining atomic processes by classical mechanics, but it turned out that many other physically important questions lead to the same mathematical formulation. This is true, in particular, for recent research in the theory of plasmas.

The first step towards a precise mathematical statement of such problems is a mathematical interpretation of statements such as "$\phi(\tau)$ varies very slowly". The simplest model is to replace $\phi(\tau)$ by a function $\phi(\varepsilon\tau)$, where ε is a positive parameter that tends to zero. Then $d\phi/d\tau = \varepsilon d\phi(\varepsilon\tau)/d(\varepsilon\tau)$ tends to zero with ε. It is now natural to measure time in a "compressed" scale by setting

$$\varepsilon\tau = t \qquad (1.5)$$

and to change the differential equation (1.3), accordingly, into

$$\varepsilon^2\ddot{u} + \phi^2(t)u = 0, \quad (\dot{u} = du/dt). \qquad (1.6)$$

A particular solution can be identified by initial conditions such as, e.g.,

$$u = u_0, \quad du/d\tau = \varepsilon du/dt = u_1, \quad \text{at} \quad t = 0. \qquad (1.7)$$

The aim is to calculate asymptotically, as $\varepsilon \to 0+$, the quantity in (1.4), which has now the form

$$A(t, \varepsilon) = \phi(t)u^2(t, \varepsilon) + \varepsilon^2\phi^{-1}(t)\dot{u}^2(t, \varepsilon). \qquad (1.8)$$

The early work on this problem suffers from two drawbacks:

(i) It was assumed that the length of the pendulum — and hence $\phi(t)$ — was constant outside some finite time interval $t_1 \leqq t \leqq t_2$;

(ii) No precise smoothness conditions on $\phi(t)$ were explicitly stated. The first condition may be deplored more by mathematicians than by physicists, but the second one should concern the physicists, as well, since the total change of $A(t, \varepsilon)$ turns out to depend in a very sensitive way on the smoothness of $\phi(t)$.

Einstein's assertion on $A(t,\varepsilon)$ reads, in our present notation,

$$\lim_{\varepsilon \to 0+} [A(t_2, \varepsilon) - A(t_1, \varepsilon)] = 0 . \tag{1.9}$$

In 1959 the Physicist A. Lenard [4] obtained the more precise and striking result that

$$A(t_2, \varepsilon) - A(t_1, \varepsilon) = O(\varepsilon^N), \quad \text{for all} \quad N > 0.$$

An inspection of his proof showed that he had not only used the assumption that $\phi(t)$ was constant outside the interval $t_1 \leqq t \leqq t_2$, but also — without stating it specifically — the essential hypothesis that $\phi(t)$ was infinitely differentiable, not only in $[t_1, t_2]$, but for <u>all</u> real t.

In 1963 Littlewood [6] proved that Lenard's result could be extended to positive functions $\phi(t)$ that were not necessarily constant outside some interval, provided the limits $\phi(\pm\infty)$ existed and were positive, and all derivatives of ϕ were integrable on $-\infty < t < \infty$. The result must now be written

$$A(\infty, \varepsilon) - A(-\infty, \varepsilon) = O(\varepsilon^N), \quad \text{for all} \quad N > 0. \tag{1.10}$$

(The existence of $A(\pm\infty, \varepsilon)$ can be proved.)

On the other hand, it is not difficult to prove that, in general,

$$A(t_2, \varepsilon) - A(t_1, \varepsilon) = O(\varepsilon) \, , \qquad (1.11)$$

and no better.

It is remarkable that the vibrating system should have this kind of memory: After an infinite time interval the quantity $A(t, \varepsilon)$ settles down to a value much closer to its initial value than the ones it had assumed in between. This is, however, a mathematical property whose physical significance may be an illusion, in as much as (1.10) is no longer true, as soon as the stringent smoothness requirements on $\phi(t)$ are relaxed.

Ignoring this question of physical relevance, several papers have been written — some by mathematicians, some by physicists — endeavouring to replace the right hand member of (1.10) by a more explicit asymptotic expression. (See, e.g., [2], [1], [3].) In the next section I shall give a brief account of a new proof of Littlewood's result which can be used as a starting point for a more precise analysis of the total change of $A(t, \varepsilon)$. (See [12], [13].) The method has points of resemblance with that developed by R. E. Meyer ([7], [8]) simultaneously and independently.

2. The Analytic Case.

Y. Sibuya [9] has described a very general method for the reduction of linear differential equations depending in a singular manner on a parameter. When that method is applied to the simple differential equation (1.6) it amounts to a transformation to the Riccati equation

$$\varepsilon\dot{p} = 2i\phi p + \psi - \varepsilon^2\psi p^2 \tag{2.1}$$

by means of a change of dependent variable described by

$$\frac{\dot{u}}{u} = \frac{i\phi}{\varepsilon}\,\frac{\varepsilon p-1}{\varepsilon p+1}\ . \tag{2.2}$$

Here

$$\psi = \dot{\phi}/2\phi\ . \tag{2.3}$$

Under Littlewood's hypotheses ψ and all its derivatives are in $L^1(-\infty, \infty)$.

The Riccati equation (2.1) has the pleasant property that some of its solutions possess asymptotic power series expansions. By a slight modification of Sibuya's argument one can, in fact, prove the following theorem.

Theorem 2.1. *The solution* $p = p(t, \varepsilon)$ *of equation* (2.1) for which $p(-\infty, \varepsilon) = 0$ possesses an asymptotic expansion of the form

$$p(t, \varepsilon) \sim \sum_{r=0}^{\infty} p_r(t)\varepsilon^r \ , \quad \text{as} \quad \varepsilon \to 0+ \ , \qquad (2.4)$$

which is uniformly valid in $-\infty \leq t \leq \infty$ and may be indefinitely differentiated. Moreover, the limit

$$\lim_{t\to\infty} p(t, \varepsilon)\exp[-\frac{2i}{\varepsilon}\ \Phi(t)] \ , \qquad (2.5)$$

where

$$\Phi(t) = \int_0^t \phi(s)ds \ , \qquad (2.6)$$

exists and is $O(\varepsilon^N)$ for all $N > 0$.

Returning from equation (2.1) to the original differential equation one can calculate u, $\dot{u}$ and A. After some calculation one finds, in particular, the formula

$$A(\infty, \varepsilon) - A(-\infty, \varepsilon) = 2\,\mathrm{Re}\,\{k\int_{-\infty}^{\infty} e^{-\frac{2i}{\varepsilon}\Phi(t)}\ \psi(t)(1-\varepsilon^2 p^2(t,\varepsilon))dt\}(1+O(\varepsilon)), \qquad (2.7)$$

which expresses the total change of $A(t, \varepsilon)$ in terms of $p(t,\varepsilon)$. Here k is an explicitly known constant depending only on the initial values u_0 and u_1.

Littlewood's result (1.10) follows immediately from (2.7) by repeated integrations by part and the facts that ψ and all its derivatives vanish at $\pm\infty$, while all derivatives of p

remain bounded.

The details of the arguments just sketched can be found in [12].

Formula (1.10) cannot be improved unless conditions more stringent than those of Littlewood are imposed on $\phi^2(t)$. One plausible such hypothesis is that $\phi^2(t)$ should be an analytic function holomorphic in a parallel strip of the complex t-plane including the real axis, and, also, that the function $\psi(t)$ should remain integrable along parallels to the real t-axis. With such properties of $\phi^2(t)$ one may hope that $p(t, \varepsilon)$ will remain bounded in such a strip and that the integral in formula (2.7) can be replaced by the integral, over the same integrand along a parallel to the real t-axis on the side where $\text{Im}\, \Phi(t) < 0$. It is not difficult to prove that this is really true, and then a glance at the integral shows that

$$A(\infty, \varepsilon) - A(-\infty, \varepsilon) = O(e^{-c/\varepsilon}) \tag{2.8}$$

with some positive constant c.

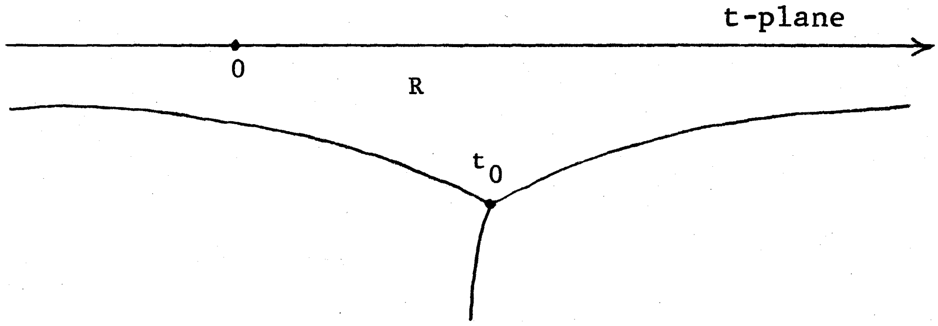
It is much more difficult to replace the right member of (2.8) by an explicit asymptotic formula. One must expect that the upper bound of admissible values for c in (2.8) will depend on still more specific properties of $\phi^2(t)$. In [13] I introduced some fairly natural condition on $\phi^2(t)$, whose description requires a few preparatory sentences: In the asymptotic theory of differential equations such as (1.6) the zeros of $\phi^2(t)$, the so-called "turning points", play a central role. If

$$\phi^2(t_0) = 0, \quad \text{but} \quad [d(\phi^2)/dt]_{t=t_0} \neq 0 , \tag{2.9}$$

then t_0 is a "simple" or "first order" turning point. The value of $\operatorname{Im} \Phi(t)$, where $\Phi(t)$ is defined as in (2.6), largely, determines the asymptotic behavior of the solutions of equation (1.6). The curves defined by

$$\operatorname{Im} \Phi(t) = \operatorname{Im} \Phi(t_0) \tag{2.10}$$

in the complex t-plane, often called "Stokes curves", are therefore important. It is easy to see that at the branch point $t = t_0$ of $\Phi(t)$ three such Stokes curves meet and form equal angles there. I now add the condition that two of these curves can be extended to infinity and bound, together with the real axis, a region R in which $\phi^2(t)$ is holomorphic and has no zeros. (See the figure below.)



Since Φ assumes conjugate values in conjugate points, it is no loss of generality if we stipulate that

$$\operatorname{Im} \Phi(t_0) < 0 \ . \tag{2.11}$$

It is now possible to replace the path of integration in (2.7) by the union of the two Stokes curves that bound R and to evaluate it asymptotically. The details of these rather long arguments can be found in [13]. They involve the asymptotic calculation of three different fundamental systems of solutions for the differential equation (1.6). Two of these are found by methods similar to those leading to Theorem 2.1. The third one is based on the theory of simple turning points as developed, e. g., in [11], sections 29, 30. Then, connection formulas between these fundamental systems are established, $p(t, \varepsilon)$ is asymptotically calculated along the whole new path of integration, and, finally, the integral in (2.7), or rather its leading term, is determined. Thus the following result is eventually obtained.

Theorem 2.2. Let

$$[\phi^{1/2} u + i\phi^{-1/2} \frac{du}{d\tau}]_{\tau=0} = r_0 e^{i\theta_0} \ .$$

Then

$$A(\infty, \varepsilon) - A(-\infty, \varepsilon) = r_0^2 \operatorname{Re}\{\kappa e^{-2i[\frac{\Phi(t_0)}{\varepsilon}+\theta_0]}\} + \cdots ,$$

where

$$\kappa = -\frac{4}{3}\int_{-\infty}^{\infty} \frac{e^{-2is}}{s}\, \rho^{1/2} \frac{Ai(\rho)Ai'(\rho)}{[\rho^{\frac{1}{2}}Ai(\rho) - Ai'(\rho)]^2}\, ds ,$$

$\rho = (\frac{3}{2}is)^{2/3}$, $Ai(\rho)$ is Airy's function.

3. Brief Remarks on Related Problems.

It is clearly impossible in a short expository article even to mention all the numerous existing investigations on adiabatic invariants. A list of a few problems and results related to the work already described here will have to suffice. I must leave aside the most important question, namely that of extending the theory to nonlinear differential equations. More work in that direction from the point of view of the mathematician would be very desirable.

As was mentioned before, Littlewood's hypotheses on $\phi^2(\tau)$ in the differential equation (1.3) are probably a poor mathematical model of physical reality. Of greater interest is the case when $\phi^2(\tau)$ is allowed to have a finite number of jumps in some derivative, the first or second, say. It is not difficult to modify the method of [12] so as to yield explicit formulas for the variation of $A(t, \varepsilon)$ in that case.

Can the results of [12] and of [13] be extended to more general linear differential equations? Such an extension is proved in a recent article by A. Leung and Kenneth Meyer [5]. They consider linear systems of differential equations of the form

$$\frac{dy}{d\tau} = H(\tau)y \ , \tag{3.1}$$

where $H(\tau)$ is a 2n-by-2n <u>Hamiltonian</u> matrix function. If I_n is the identity matrix of order n and J_{2n} is the matrix

$$J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} ,$$

a real $2n \times 2n$ matrix H is called "Hamiltonian" if JH is a symmetric matrix. Most linear differential equations of Mechanics are of this type. If we set

$$y = \begin{pmatrix} u \\ du/d\tau \end{pmatrix} ,$$

equation (1.3), for instance, becomes

$$\frac{dy}{d\tau} = \begin{bmatrix} 0 & 1 \\ -\phi^2(\tau) & 0 \end{bmatrix} y \ , \tag{3.2}$$

which is, indeed, Hamiltonian.

Leung and Meyer adopt Littlewood's hypotheses for the entries of $H(\tau)$. In addition, they need the following two restrictive conditions, which are satisfied by (3.2): The eigenvalues of $H(\tau)$ are distinct throughout (even at $\tau = \pm\infty$), and they are pure imaginary. Since $H(\tau)$ is real, the eigenvalues occur, therefore in pairs of the form $i\lambda(\tau)$, $-i\lambda(\tau)$, ($\lambda(\tau)$ positive).

Next, a satisfactory explanation is needed of what is to be called an "adiabatic invariant" for the system (3.1). I propose the following definition, which differs from the one in [5] and even more from some others in the literature.

Definition 3.1: A scalar function $I(y, \tau)$ is called an adiabatic invariant of the differential equation

$$\frac{dy}{d\tau} = H(\tau)y \qquad y = \mu_\varepsilon(t)$$

if the solution of the initial value problem

$$\varepsilon\frac{dy}{dt} = H(t)y\ , \qquad y = y_0 \quad \text{for} \quad t = 0, \tag{3.3}$$

satisfies the relation

$$\lim_{\varepsilon\to 0+} \{I(\mu_\varepsilon(t_2), t_2) - I(\mu_\varepsilon(t_1), t_1)\} = 0$$

for all t_1, t_2 (possibly depending on ε) and for all y_0 independent of ε.

The essence of the method of Leung and Meyer is to construct a matrix function $P(t, \varepsilon)$ such that the transformation

$$y = P(t, \varepsilon)z \tag{3.4}$$

takes the differential system in (3.3) into a system

$$\varepsilon \dot{z} = D(t, \varepsilon) z \tag{3.5}$$

with a diagonal coefficient matrix. This can be achieved in many ways. The procedure in [5] has two very important features:

(a) $D(t, \varepsilon)$ shares with $H(t)$ the property that its eigenvalues occur in pairs that are negatives of each other, i.e.,

$$D(t,\varepsilon) = \mathrm{diag}\{d_1(t,\varepsilon),\cdots,d_n(t,\varepsilon),-d_1(t,\varepsilon),\cdots,-d_n(t,\varepsilon)\}; \tag{3.6}$$

(b) The asymptotic relation

$$P(t,\varepsilon) - P(t,0) = \begin{cases} O(\varepsilon), & \text{for all } t \\ O(\varepsilon^N), & \text{for all } N > 0, \text{ if } t = \pm\infty \end{cases} \tag{3.7}$$

holds.

One sees easily that

$$\mu_\varepsilon(t) = P(t, \varepsilon) e^{\frac{1}{\varepsilon}\int_0^t D(s,\varepsilon)ds} P^{-1}(0, \varepsilon) y_0 . \tag{3.8}$$

Therefore, the components of the vector

$$e^{-\frac{1}{\varepsilon}\int_0^t D(s,\varepsilon)ds} P^{-1}(t,\varepsilon)\mu_\varepsilon(t) \tag{3.9}$$

are independent of t, and are, in this sense, absolute invariants. This is, however, a rather trivial fact, which, by itself, does not lead to adiabatic invariants according to our definition, because the expression (3.9) depends explicitly on ε. The special properties (3.6) and (3.7), on the other hand, imply that the n scalar function of y and τ alone:

$$\{P^{-1}(\tau, 0)y\}_j \{P^{-1}(\tau, 0)y\}_{n+j}\ , \quad j = 1,2,\cdots,n, \qquad (3.10)$$

are, indeed adiabatic invariants. Here, the notation $\{v\}_k$ signifies the k^{th} component of the vector v.

This result reduces to the one obtained by Littlewood, when applied to the system (3.2). It is therefore its natural generalization.

We conclude with a very brief reference to another recent contribution to the theory of Lorentz's adiabatic invariant (1.4).

A C^∞-function $\phi(t)$ is analytic, if the sequence of its successive derivatives $\{d^k\phi/dt^k\}$ does not grow so fast that its Taylor series diverges. It is to be expected that the order of magnitude of the quantity $A(\infty, \varepsilon) - A(-\infty, \varepsilon)$ for infinitely differentiable but not necessarily analytic functions $\phi(t)$ will be related to the rate of growth of that sequence of derivatives. G. Stengle, in a paper to be published shortly, [10], has proved an interesting theorem in this direction. His arguments are too intricate to explain in a few paragraphs. Only the principal result will be described: If ϕ satisfies Littlewood's hypotheses, but is not analytic, let $\|\phi^{(k)}\|$ be the $L_1(-\infty, \infty)$-norm of $d^k\phi/dt^k$ and assume that

$$\|\phi^k\| = O(g(k)) , \quad \text{as} \quad k \to \infty .$$

Set

$$h(k) = \log g(k)$$

and interpolate h so that $h(g)$ is defined for all $y \geq 0$.

Define

$$h^*(x) = \max_y \{ xy - h(y) \} .$$

h^* is sometimes called the convex conjugate of h. Then

$$A(\infty,\varepsilon) - A(-\infty,\varepsilon) = O(e^{-h^*(c \log \varepsilon)}), \quad \text{as} \quad \varepsilon \to 0+$$

for all $c > -1$.

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