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On a mixed Hodge structure of an isolated singularity

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§0. Introduction.

0.1. In this note, a normal isolated singularity $\mathfrak{X}=(X,p)$ is by definition an equivalence class of a germ of an analytic space that X determines at p, where X is a normal analytic space smooth outside a point p.

Let $\mathfrak{X}=(X,p)$ be a normal isolated singularity. Let $K=\mathbf{Z}, \mathbf{R}$, or $\mathbf{C}$. We define a K-module H_K^* by the formula

$$H_K^* = \varinjlim_V H^*(V-p, K),$$

where V runs through neighborhoods of p in X. This of course depends only on $\mathfrak{X}$. We call H_K^* the cohomology group of $\mathfrak{X}$ with coefficients in K.

Note that H_K^* could also be described as follows; assume that X is realized as an analytic subspace of a domain D of some $\mathbf{C}^N$. Let $\mathbf{K}=X\cap S$ be the intersection of X

with a sufficiently small sphere S around p in $\mathbb{C}^N$. Then $H^* = H^*(\mathbf{K}, \dot{K})$.

0.2 The concept of a mixed Hodge structure is introduced in [1].

Definition 0.2.1. A $\mathbb{Z}$-module H is said to have a mixed Hodge structure, if the following conditions are satisfied; there exist i) a finite increasing filtration W on $H_\mathbb{Q} = H \otimes_\mathbb{Z} \mathbb{Q}$, and ii) a finite decreasing filtration F on $H_\mathbb{C} = H \otimes_\mathbb{Z} \mathbb{C}$ which satisfies the next properties; let $\bar{F}$ be the filtration on $H_\mathbb{C}$ conjugate to F, W denote also the induced filtration on $H_\mathbb{C}$, $H_n = Gr^n_{W_\mathbb{C}}(H_\mathbb{C}) = W_n(H_\mathbb{C})/W_{n-1}(H_\mathbb{C})$, and F_n (resp. $\bar{F}_n$) the filtration induced on H_n by F (resp. $\bar{F}$). Then we have a direct sum decomposition

$$H_n = \bigoplus_{p+q=n} H^{p,q}, \quad H^{p,q} = F^p_n(H_n) \cap \bar{F}^q_n(H_n).$$

0.3 Then our result is.

Theorem. Let $\mathfrak{X} = (X, p)$ be a normal isolated singularity, and $H^*_\mathbb{Z}$ be a cohomology group of $\mathfrak{X}$ with coefficients in $\mathbb{Z}$. Then $H^*_\mathbb{Z}$ has a natural mixed Hodge structure.

§ 1. Complex cohomology at infinity

1.1. We denote by (X, A) a pair consisting of an (equidimensional) complex manifold X and a divisor A with only normal crossings in X. Let (X_i, A_i), $i=1,2$, be such pairs. Then a morphism $f: (X_1, A_1) \to (X_2, A_2)$ of (X_1, A_1) to (X_2, A_2) is a morphism $f: X_1 \to X_2$ with $f^{-1}(A_2) \subseteq A_1$.

Let $\mathcal{C}$ be the category whose objects are pairs (X, A) and whose morphisms are morphisms of pairs.

1.2. Hereafter in §1 and §2, we choose and fix a $(X, A) \in \mathcal{C}$. Then for any point $p \in A$, there exists a neighborhood $U \ni p$ in X with a local coordinate system $(z_1, \cdots, z_n)$, $n = \dim X$, such that A is defined in U by the equation

$$(1) \qquad z_1 \cdots z_r = 0,$$

for some r, $1 \leq r \leq n$.

Definition 1.3. The logarithmic de Rham complex $\Omega_X^*\langle A\rangle$ is a complex of $\mathcal{O}_X$-modules defined locally by

$$\Omega_X^0\langle A\rangle := \mathcal{O}_X$$

$$\Omega_X^1\langle A\rangle := \left\{ \sum_{i=1}^{r} a_i \, dz_i/z_i + \sum_{j=r+1}^{n} a_j \, dz_j \;;\; a_k \in \mathcal{O}_X,\ k=1, \cdots, n \right\}$$

$$\Omega_X^p\langle A\rangle := \Lambda^p \Omega_X^1\langle A\rangle,$$

where the differential is the usual exterior differentiation.

The formation of $\Omega^*\langle\ \rangle$ is a contravariant functor on $\mathcal{C}$.

Note that $\Omega_X^p\langle A\rangle$ are locally free $\mathcal{O}_X$-modules.

Dually we make the following

Definition 1.4. $\Sigma^*_X\langle A\rangle$ is a (locally free) complex of $\mathcal{O}_X$-modules defined by

$$\Sigma^*_X\langle A\rangle := \mathcal{H}om(\Omega^{n-*}_X\langle A\rangle, \Omega^n_X),$$

differential being induced by that of $\Omega^*_X\langle A\rangle$.

We see easily the following fact:

(1.5.) $\Sigma^*_X\langle A\rangle$ is naturally the subcomplex of the usual Poincaré complex Ω^*_X, generated locally as an $\mathcal{O}_X$-algebra by the elements

(2) $\quad z_{i_1}\cdots z_{i_q}\, dz_{i_{q+1}}\cdots dz_{i_r}, \qquad \{i_1,\dots,i_r\} = \{1,\dots,r\}.$

The formation of $\Sigma^*\langle\ \rangle$ is also a contravariant functor from the category $\mathcal{C}$, as is seen from (2).

1.6. The importance of $\Omega^*_X\langle A\rangle$ comes from the following lemma which is proved in [1] and [2].

Lemma 1.6.1. Let $U=X-A$ and $j: U\hookrightarrow X$ be the inclusion. Then the complex cohomology $H^*(U,\mathbb{C})$ of U can be calculated as a hypercohomology of the complex $\Omega^*_X\langle A\rangle$;

$$H^*(U,\mathbb{C}) = \mathbb{H}^*(X, \Omega^*_X\langle A\rangle),$$

where the right side denotes the hypercohomology. See [1].

1.7. As for $\Sigma^*_X\langle A\rangle$, the next proposition holds.

Proposition 1.7. We have

$$H^*_\Phi(U,\mathbb{C}) = \mathbb{H}^*(X, \Sigma^*_X\langle A\rangle),$$

where H^*_Φ denotes the cohomology with support in a closed set of X contained in U.

This immediately follows from the lemma below, since then $\Sigma^*_X\langle A\rangle$ is a resolution of $\mathbb{C}_U$, $\mathbb{C}_U$ being the constant sheaf $\mathbb{C}$ on U extended by 0 to X.

Lemma 1.8. The complex $\Sigma^*_X\langle A\rangle$ is exact.

Proof is attained quite analogously to that of the classical Dolbeault lemma for the complex Ω^*_X, in view of (2).

Remark 1.9. From the exact sequence

$$0 \to \Sigma^*_X\langle A\rangle \to \Omega^*_X\langle A\rangle \to \Omega^*_X\langle A\rangle/\Sigma^*_X\langle A\rangle \to 0$$

we have the cohomology exact sequence

$$\cdots \to H^q_\Phi(U, \mathbb{C}) \to H^q(U, \mathbb{C}) \to \mathbb{H}^q(X, \Omega^*_X\langle A\rangle/\Sigma^*_X\langle A\rangle) \to \cdots.$$

Then using five lemma we can get an isomorphism.

$$H^*_\infty(U, \mathbb{C}) \underset{\text{def}}{=} \varinjlim_{V,\, V\supset A} H^*(V\setminus A, \mathbb{C}) = \mathbb{H}^*(X, \Omega^*_X\langle A\rangle/\Sigma^*_X\langle A\rangle),$$

where in the middle term V runs through a n.b.d. of A in X.

§2. Mixed Hodge structure at infinity.

2.0. Let $A = \bigcup A_i$ be the decomposition of A into irreducible components A_i. We use the following notation. $A^{(p)} = \bigcup_{i_1<\cdots<i_p} A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_p}$, $U_{(p)} = A^{(p)} - A^{(p+1)}$, and $\tau_p : \tilde{A}^{(p)} \to X$ be the normalization composed with the inclusion $i_p : A^{(p)} \hookrightarrow X$.

Further for simplicity we assume that each irreducible component A_i is compact and nonsingular.

2.1. We set $K^* = K_X^*\langle A\rangle = \Omega_X^*\langle A\rangle / \Sigma_X^*\langle A\rangle$. Note that this is naturally a complex of $\mathcal{O}_A$-modules.

We define an increasing filtration W on $\Omega_X^*\langle A\rangle$ by the formula:

$$W^s(\Omega_X^*\langle A\rangle):$$

$$= \Big\{ \sum_{i_1<\cdots<i_s} \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_s}}{z_{i_s}} \wedge \alpha_{i_1\cdots i_s},\ 1 \leq i_1 < \cdots < i_s \leqq r,\ \alpha_{i_1\cdots i_s} \in \Omega_X^* \Big\},\ \text{if } s \geqq 0$$

$$= \Big\{ \sum_{i_1<\cdots<i_t} z_{i_1}\cdots z_{i_u} dz_{i_{u+1}} \wedge \cdots \wedge dz_{i_t} \wedge \alpha_{i_1\cdots i_t},\qquad \{i_1,\cdots,i_t\} \subseteq \{1,\cdots,r\}$$

$$\text{and } \alpha_{i_1\cdots i_t} \in \Omega_X^* \Big\} \qquad \text{if } t = -s > 0.$$

The induced filtration on K^* is still denoted by the same letter W. The formation of $W(K_X^*\langle A\rangle)$ also defines a functor. As for the associated gr. we have

$$(3)\begin{cases} Gr^s(K^*) \\ \cong \tau_{s*}\Omega^*_{\tilde{A}^{(s)}}[-s] & s \geqq 0. \quad (\text{Poincaré residue } [\ ,\ 3.1.5]. \\ \cong \Sigma^*_{\tau_t^{-1}(A^{(t+1)})}\langle \tilde{A}^{(t)}\rangle & t = -s > 0. \end{cases}$$

In fact, $W^s(\Omega_X^*)$ coincides with the kernel of the restriction map. $r_s : \Omega_X^* \longrightarrow \Omega^*_{A^{(s+1)}}$.

2.2. Hodge filtration F on K^* is defined also as that induced by the Hodge filtration (still denoted by F) on $\Omega_X^*\langle A\rangle$. Here if we put $T^*_{(p)} = F^p(\Omega_X^*\langle A\rangle)$, then

$$K^s_{(p)} = 0 \qquad s < p$$
$$K^s_{(p)} = \Omega^s_X\langle A\rangle \qquad s \geq p.$$

2.3. Now with the filters W and F defined, K^* becomes a doubly filtered complex (K^*, W, F), functorial with respect to $(X, A) \in \mathcal{C}$. Then as usual we have various spectral sequences associated with this complex. In particular we consider the one arising from the filter W. By virtue of (3) in 2.1 we get

Lemma 2.3.1. The $E_1^{p,q}$ term is given by

$$E_1^{p,q} := H^{2p+q}(\tilde{A}^{(p)}, \mathbb{C}) \qquad \text{if } p \geq 0.$$
$$= H_c^{2p+q}(U^{(p)}, \mathbb{C}), \quad H_c\text{: cohomology with compact supp. if } p < 0.$$

2.4. On each term $E_r^{p,q}$ of the spectral sequence F induces three kinds of filtrations, the first direct filtration F_d, the second direct filtration F_{d^*} and the recursive filtration F_r. The key point in the proof of Theorem 0.3. is

Lemma 2.4.1. (i) For every $r \geq 1$, the differential d_r of the above spectral sequence is strictly compatible with the recursive filtration on $E_r^{p,q}$.

(ii) On each term $E_r^{p,q}$, (r may be ∞), the 3 kinds of filtration coincides, and the filter F on $H^*_\infty(U, \mathbb{C})$ is compatible with the recursive filtration F_r on $E_\infty^{p,q}$.

Remark. 2.4.2 (a) (ii) is a consequence of (i). [1, Th.1.3.16. Cor. 1.3.17].
$d_r^{p,q}=0$ if $\begin{cases} p \geq 0. \\ r \geqq 2 \end{cases}$ [1, Lemma 3.2.10].

(b) Proof of (i) will be omited. But as an explanation we note the following facts. Put $\Omega_A^* = \Omega_X^+/\Sigma_X^*\langle A\rangle$. (Note that this is different from that defined by Grothendieck. Grauert-Kerner). Then since $W^0(\Omega_X^*\langle A\rangle)=\Omega_X^*$, we have an exact sequence of $\mathcal{O}_A$-modules.

$$(4)\ 0 \to \Omega_A^* \to K_X^*\langle A\rangle \to \Omega_X^*\langle A\rangle/\Omega_X^* \to 0 .$$

By virtue of Lemma 1.6.1 and of Lemma 1.8. we have isomorphisms.

$$(5)\quad \begin{cases} \mathbb{H}^*(A,\Omega_A^*) \simeq H^*(A,\mathbb{C}) \\ \mathbb{H}^*(\Omega_X^*\langle A\rangle/\Omega_X^*) \simeq H_A^*(X,\mathbb{C}), \end{cases}$$

and the long exact sequence of hypercohomology corresponding to (4) is nothing but the local cohomology exact sequence.

$$(6)\ \to H^s(A,\mathbb{C}) \to H_\infty^s(V,\mathbb{C}) \to H_A^s(X,\mathbb{C}) \to \cdots .$$

On the other hand, (4) is obviously the sequence compatible with the filte W. Hence on each term of (6) we have a filtration induced by F and it is natural to expect (6) is the sequence of mixed Hodge structures w.r.to. these W and F. As for H_A^s, this (= that H_A^s has mixed Hodge str') is essentially contained in [1]. And for $H^s(A,\mathbb{C})$. the spectral seq associated with W is nothing but the spectral sequence associated to the increasing sequence of closed subspaces $A^{(p)}$ $p=0,1,\cdots r$, of A. The corresponding statement to (i) of Lemma 2.4.1. may be then proved by induction on r.

(c). From the sequence (6), we can deduce easily that the filter W on each term of (6) arises from that on $H^*(\ ,\mathbb{Q})$.

§3. Case of an isolated singularity.

3.1. Let $\mathfrak{X} = (X, P)$ be as in §0 and $f: \tilde{\mathfrak{X}} = (\tilde{X}, A) \to X$ be a resolution of $\mathfrak{X}$. Since $H_c^* = H_c^*(\mathfrak{X}) \cong H_\infty^*(U)$. $U = \tilde{X} - A$.,

by Lemma 2.4.1 and Remark 2.4.2 (c), we conclude that $H^*_{\mathfrak{X}}$ has a mixed Hodge structure. Finally we have to show that this structure does not depend on the resolution chosen above. But this follows from [1, Théorème 1.2.10] by the same argument as in [1, 3.2.11.C].

References.

[1]. Deligne, P., Théorie de Hodge II, Publ. Math. I.H.E.S. 1971. No. 40. P5~58.

[2] Deligne, P., Equations différentielles a points singuliers réguliers, Lecture note in Math. No. 163. Springer. 1970