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Differential forms and stratifications (1)*

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前書き：

§0. 9月のセミナーで話す予定であった題材は次の二つである。

1° Division properties of diff. forms に関して，[3] への補足。

2°. 我々が [2], [3] で行った議論の基礎をなすある種の stratification についての説明。

9月のセミナーでは，講演の為の原稿の準備不足の為 1° 丈で終了してしまったが，ここでは多論 2° についても議論を加える。

I 1° について：まず若干 abstract な formulation より始める。最初に次の如き datum を考えよう。

(i) 集合 $\mathfrak{X}_1$, $\mathfrak{X}_2$, (ii) 写像 $\mathfrak{D}: \mathfrak{X}_1 \longrightarrow \mathfrak{X}_2$, (iii) '対象' $\mathfrak{E}$.

以下に於いては，写像 $\mathfrak{D}$ をしばしば作用素と呼ぶ事もある。(i), (ii), (iii) に於ける datum $\{\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{D}, \mathfrak{E}\}$ に対して次の写

(*) この題は，9月のセミナーの講演の題とは相異なる。当十月の 'complex manifold' の seminar での筆者の同 tytle の講演参照.

像 $\mathrm{div}_{\mathfrak{z}}$ が定義されているとする。

$$(\star)\qquad \mathrm{div}_{\mathfrak{z}} : \mathfrak{X}_1 \cup \mathfrak{X}_2 \longrightarrow \mathbb{Z}^+ = \{0, 1, 2, \cdots\}.$$

$\mathfrak{X}_1 \cup \mathfrak{X}_2$ の元 x_i $(i=1,2)$ に対して, $\mathrm{div}_{\mathfrak{z}}(x_i) \in \mathbb{Z}^+$ を x_i の division index w.r.t. $\mathfrak{z}$ と呼ぶ。さて,$(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{D}, \mathfrak{z}, \mathrm{div}_{\mathfrak{z}})$ は上記の如きものであるとして,次の如き定義をする。

定義 Datum $(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{D}, \mathfrak{z}, \mathrm{div}_{\mathfrak{z}})$ が division property を持つとは,次の如き条件$(\star)$が満たされる事である。

$(\star)$. $\exists\ s : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ such that $\lim_{n\to\infty} s(n) = \infty$ が存在して,

$$\boxed{\mathfrak{D}(\mathfrak{X}_1^{s(m)}) \supset \mathfrak{X}_2^{m} \cup \mathfrak{D}(\mathfrak{X}_1)}.$$

ここで, $\mathfrak{X}_i^{m'}$ $(i=1,2)$ $= \{x_i \in \mathfrak{X}_i : \mathrm{div}_{\mathfrak{z}}\, x_i \geqq m'\}$.

division properties を有する datum の例

例1.(Artin-Reeの定理). $\mathfrak{O}$ を noetherian ring, $\mathfrak{a}, \mathfrak{b}$ を $\mathfrak{O}$ の ideal として, $\mathfrak{b}$ の basis $(f) = (f_1, \cdots, f_\ell)$ を固定する。この時 $\mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{O}^\ell$, $\mathfrak{z} = \mathfrak{a}$ とし更に $\mathfrak{D} = \mathfrak{D}_{(f)}$ を次の如く定める。

$$\mathfrak{D}_{(f)} : \quad \mathfrak{O}^\ell \ni (x_1, \cdots, x_\ell) \longrightarrow \mathfrak{O} \ni \mathfrak{D}(x_1, \cdots, x_\ell) = \sum_{j=1}^{\ell} x_j f_j.$$

更に $\mathrm{div}_{\mathfrak{a}}$ を次式により定める。

$\mathrm{div}_{\mathfrak{a}}(x_1,\dots,x_\ell) = \min v_{\mathfrak{a}}(x_j),\ x_j\neq 0,\ \mathrm{div}_{\mathfrak{a}}(b) = v_{\mathfrak{a}}(b),\ b\in\mathfrak{b}$.
ここで $v_{\mathfrak{a}}(x_j)$, $v_{\mathfrak{a}}(b)$ は, x_j, b の $\mathfrak{a}$ に関しての order：

$$x_j \in \mathfrak{a}^{v_{\mathfrak{a}}(x_j)}, \quad x_j \notin \mathfrak{a}^{v_{\mathfrak{a}}(x_j)+1},$$ b についても同様.

この時, $s(n) = n - K_0$ (K_0 は $\mathfrak{a}, (f)$ のみによる整数) と選べる。(これは, 勿論, Artin-Rees の定理 (ring theory における) の言い換えに他ならない.)

例2. (division properties of diff. forms [Q]). 今 $\mathfrak{V}_1$ を原点0を含む $\mathbb{C}^n$ の領域, $\mathfrak{V}_2$ を0を含む充分小さな一つの(適当な)領域とし, 更に(f)：$(f_1,\dots,f_\ell)$ を $\mathfrak{V}_1$ で定義された正則函数とする。この($i=1,2$)時 $\Omega(\mathfrak{V}_i) = \mathfrak{X}_i$ を $\mathfrak{V}_i$ 上の hol. diff. forms の全体とする。更に $\mathfrak{Z}=(f)$ $(f_1,\dots,f_\ell)$ とする。亦, $\mathfrak{D} = d$ を外微分作用素 (holomorphic) とする。更に亦, $\mathrm{div}_{\mathfrak{Z}} = \mathrm{div}_{(f)}$ を次の如く定める.

$$\mathrm{div}_{(f)}\varphi_i,\ \varphi_i\in\mathfrak{X}_i = \max_v\{v\in\mathbb{Z}^+ : \varphi_i \equiv 0 \ (f_1^v,\dots,f_\ell^v)\cdot\Omega(\mathfrak{V}_i)\}.$$

この時 $s(n)$ を一次式 $s(n) = C_1 n + C_2$, $C_1, C_2\in\mathbb{Q}$ と取れる。([Q],[3]).

例1は, ideal の completion の理論の基礎をなし, 例2は $\hat{\Omega}^{(f)} = \Omega$ の (f) による完備化の理論で役割を果たす。

例3. $\mathfrak{X}_i = \Omega_{alg}(\mathbb{C}^n)$ $(i=1,2)$ を $\mathbb{C}^n$ 上の regular diff. forms 全体. V を $\mathbb{C}^n$ 内の algebraic variety (affine variety/$\mathbb{C}$) とする. 更に $\mathfrak{D} = d$。は外微分作用素とし, $\mathfrak{Z} = V =$ affine variety in $\mathbb{C}^n$ とする. この時 $\mathfrak{J}_V$ は V の ideal を表わし, $\mathrm{div}_{\mathfrak{Z}(=V)}\varphi$ $(\varphi \in \Omega_{alg}(\mathbb{C}^n))$ は例2と類似に $\varphi \equiv 0\ (\mathfrak{J}_V^{\nu(\varphi)} \cdot \Omega)$, $\varphi \not\equiv 0\ (\mathfrak{J}_V^{\nu(\varphi)+1} \cdot \Omega)$, (但し $\nu(\varphi) = \mathrm{div}_{\mathfrak{Z}(=V)}\varphi$) と定める。この時

$$(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{D}, \mathfrak{Z}, \mathrm{div}_{\mathfrak{Z}}) \text{ が division property を持つ} \Longleftrightarrow H^q(V; \mathbb{C}) \cong 0\ (q \geqq 1).$$

例3 は大域的な一つの結果である。[3]に於いては, かなり漠然とした形で述べたが, 要するに筆者の質問は, 次の通りである。

1° Division property を有する datum $(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{D}, \mathfrak{Z}, \mathrm{div}_{\mathfrak{Z}})$ の例を見出せ。特に $\mathfrak{D}$ が楕円型微分作用素, or Cauchy-Riemann operator 等の代数(or 解析)幾何にしばしば現われたものについて論ぜよ。

2°. 例1, 例2は, 直接的に variety V の代数 or 幾何学的意味と結び付く。(例2に関しては[2], [3]。例1は周知である。) この時, (i) variety (alg. or analytic) の幾何学的性質に結び付き得てかつ (ii) division property を持つ 微分 (or より広範囲の) 作用素は, 外微分作用素 d の他にあるか?

2° に於いて, 幾何学的性質とは, $H^*(V; \mathbb{C})$ or $H^*(V, \mathcal{F})$, $\mathcal{F}$: V 上の

連接層 の意味にこの論説では了解しておく。(勿論より広範囲の対象について論じられればずっと良いと思う。) 微分作用素の議論には不案内であるから臆測は避けるが, 2° の問題は必ずしも肯定的でない感を有するが, 問題 1° は, 一定の興味を有するのではないかと思われる。

(II) ---本論--- Diff. forms and Stratifications : この部分は [4] と照応して頂きたい。[4] 及びここで述べる事は, かなりの部分が, [2], [3] への準備であり, 又いくらかの部分は, [2], [3] の ~~pur~~ parts の拡張である。大雑把に言って [4] は, 代数的部分であり, ここで述べる事柄は(あまり適当でないかも知れないが) 幾何学的と呼んでおく。一般の variety V の研究に於いて, stratification の方法は基本的であり, V の cohomology 群の研究に於いても---その cohomology 群の表示方法をどの様なものにしても---- stratification の方法と cohomology 群の研究の方法を結び付け様とするのは, 自然と思われる。特に我々が用いている analytic de Rham cohomology group と stratification の ideas の幾つかの関係を論じ様とするのが [4], 及びここでの主題である。我々の基本的な立場を明らかにする為に [5] を挿入する。([5] は, 外にも掲載する予定であるが, この論説と [5] を独立に読める様にする為であって寛容をお願いする次第である。) [5] で述べた様に stratifications の idea 幾つかは, 「問題の

局所化' 及び'局所的結果の大域化' という解釈を許す様に思われる。ここで述べる事柄は[5]で述べている(1)に相当する部分である：即ち我々の de Rham cohomology theory への approach に好都合な stratification の type を定義する。そして亦 germ (analy. variety の) に対して、望ましい形での stratification の存在を記述する*。ここで述べる stratification は[ ]での議論での基礎になる。我々がここで ~~定義するもの~~ 行うのは、ある与えられた variety に対して normalized series of prestratified spaces という概念を定義する事である。(この§0の後に掲載する英文 type を参照)

normalized series of prestratified spaces は N.S.S. と略される事があり亦、一般に記号$(\mathfrak{R}, \mathfrak{N})$で表わす。(定義の詳細は、英文 type の §:

(Germann, 大文字)

Normalized series of prestratified spaces を参照されたい)。N.S.S. に関して簡単な説明を付する。代数あるいは解析 variety の理論に於いて、与えられた variety V に対し、適当な ~~射影~~ 写像 $\pi: V \to W = \pi(V)$

(*) タイプ作成の都合上、証明を記載するのは、間に合わなかった。問題の reduction 迄を載せる。残りの部分は、本質的に困難を含まない。(他の場所で詳細を議論する。)

を取り，$\mathfrak{W}$ を $\mathfrak{V}$ より簡単なものに帰着させて，$\mathfrak{V}$ の性質を調べるのは，standard である。特に $\dim \mathfrak{V} = \dim \mathfrak{W}$ である場合，射影 $\pi : \mathfrak{V} \to \mathfrak{W}$ を適当に選び，$\mathfrak{W}$ 及び π の性質より $\mathfrak{V}$ の性質を考察する方法を ramified covering map の方法 と呼ぶ事にする。ramified covering map の方法は，algebraic variety，complex analytic variety の考察のみならず，real analytic variety の考察に於いても ~~基本的な意味~~ 重要である．（cf. S. Łojasiewicz [1]，…）

そして ramified covering map の方法 に於いては，Weierstrass の予備定理及び E. Noether の正規化定理（normalization theorem）が基本的な意味を持つ事も周知である。亦上記 ramified covering map の方法 は勿論多くの技術的な意味での長所を有するが同時に variety に対し幾何学的に直観的（視覚的）な approach を許す という特徴を有すると言えると思う。我々の N.S.S の導入の一つの理由は，次の様に言い表わされると思う。

（I） Ramified covering の方法を多かれ少なかれ組織化する事。

さて定義の詳細は後にゆずるとして，N.S.S. $(\mathfrak{R}, \mathfrak{F})$ を構成する成分の内 $\{\mathcal{U}, \mathcal{V}, \mathcal{S}_0\}$，$\{\mathfrak{F}, \mathfrak{F}\}$ を取上げる。ここで，$\mathcal{U}$，… $\mathfrak{F}$ は概略次の

Germann・大文字

English 大文字 斜体

如し。

(**) $\mathcal{V} = \{\mathcal{V}^1_{*}, \cdots, \mathcal{V}^n_{*}\}$, $V = \{V^1_{*}, \cdots, V^n_{*}\}$, $\mathcal{S}_0 = \{\mathcal{S}^1_0, \cdots, \mathcal{S}^n_0\}$,

$\mathcal{F} = \{\mathcal{F}^1, \cdots, \mathcal{F}^n\}$, $\mathcal{F}' = \{\mathcal{F}'^1, \cdots, \mathcal{F}'^n\}$ の形で、$\mathcal{V}^j$は $\mathbb{R}^j = \{(x_1, \cdots x_j)\}$ の domain, $V^j \subset \mathcal{V}^j$ は proper subvariety of $\mathcal{V}^j$, $\mathcal{S}^j_0$ は pair$(\mathcal{V}^j, V^j)$ の prestratification $(i = 1, \cdots, n)$, 又、$\mathcal{F}^j$, $\mathcal{F}'^j$ は $\mathcal{S}^j_0$ の元 $S^j_{\lambda_j}$ に付随された函数族 $\underbrace{f(S^j_{\lambda_j}),\ f'(S^j_{\lambda_j})}_{全体}$ の寄せ集め：$\mathcal{F}^j = \{f(S^j_{\lambda_j})\,;\, S^j_{\lambda_j} \in \mathcal{S}^j_0\}$, $\mathcal{F}'^j = \{f'(S^j_{\lambda_j}) \cdots\}$.

(詳細は、後に付す表入type参照。)

ここで上記の様な列 $\mathcal{V}$, V, $\mathcal{S}_0$, $\mathcal{F}$, $\mathcal{F}'$, $\cdots$ を考える理由は勿論次の理由である。

(II) ~~[crossed-out text]~~ varietyの次元に関する帰納的な考察を、我々の問題に対しても取れる様にする事。

さて[2], [3]で述べた様に real analytic variety の二つの定量的性質 ：(i) 与えられた subvariety に対しての Poly. growth properties (ii) 与えられた subvariety に対しての division properties ： の考察が我々には重要である。従って、我々の N.S.S. の定義に於いて、次の事柄を考慮する事が望ましい。

(III) Data $\mathfrak{T}, \mathfrak{V}, S_0, \mathfrak{F}, \mathfrak{F}', \cdots$ が与えられた variety の定量的考察に対して有効である事。

さて上記の data $\mathfrak{T}, \mathfrak{V}, S_0, \mathfrak{F}, \mathfrak{F}', \cdots$ の内 variety の定量的考察に特に効いて来るのは函数の寄せ集め $\mathfrak{F} = \{\mathfrak{F}^j(S)\}_{j=1,\cdots n}$ である。$f(S^i_{\lambda_j}) \in \mathfrak{F}^j$ は、$(j - \dim S^i_{\lambda_j})$ 個の monic polynomials $f_1(S^i_{\lambda_j}), \cdots f_{j-\dim S^i_{\lambda_j}}(S^i_{\lambda_j})$ よりなる。(これは勿論 E. Noether の正規化定理の類似を狙ったものであり、筆者の terminology '<u>normalized</u> series of prestr—' はこの事に基づく。)

そして、(II)で述べた帰納的考察を $\mathfrak{F}$ に対して有効に行なう為の条件として導入したのが <u>Higher discriminant condition</u> (cf. 英文タイプ) である。我々が導入した N.S.S. という概念は [2], [3] での基礎的な意味を有する。[2], [3] でなした考察の各々の step と N.S.S. の定義に用いられた data あるいは条件との関連を分析するのは、左程容易でなく余り有意義とも思われない。然しいづれにしろ <u>ramified covering map</u> の

方法が有する技術的長所あるいは上記の方法の underlying ideaで

あろう'視覚的直観的な approach'が [2], [3] での各段階に影響を

有するといいうると思われる。

(注)₁ 我々の N.S.S. の導入に於いて、Łojasiewicz [1] から刺激を受けた。

Łojasiewicz [1] からの影響の他は、N.S.S. の導入はかなり'経験的事実に基づく'。
という表現を取っておく。
(事実、N.S.S. の導入の 'motivation' を手短かに、説明しようと試みたのであるが、
'theoretical' あるいは 'a priori' な説明では、余りうまく出来そうでもない様に思われる)。

(注)₂ 我々は real analytic variety を対象としている。最終的な応用は

complex analytic case に対して行なうのであるが、[2], [3] で述べた事実は

次の様な意味(☆)で実解析的と思われる。

(☆) complex analytic variety V は勿論自然な方法で real analytic

variety の構造を有する。Vに自然な real analytic variety の構造を与えられた

ものを V_{re} と表わす。この時[2], [3]で述べた事実は、complex analytic variety V に対しては、V_{re} に付随された概念を用いずに定式化出来る。他方[2], [3]で述べた事実の証明は複素 variety に対して行なうよりも、実 variety に対して行なう方が容易であり同時に(恐らくは)自然でもある。

実 variety を考察する種々の方法は、既に多くあると思う。注2を挿入した訳は、実 variety に興味を有される方々に、若干の注意をひき起し、亦、意見を知りたいと思った為である。

§1. 以後の論説の説明： 先ず最初に、[5]との関連を明らかにし亦同時に筆者の考えをはっきりさせる為に[4]を掲さいする。([4]は学士院紀要に投稿中で、本質的には announcement であるが[4]に述べられた事実を check する事は左程難しくない)。
次に二つの §§： § Normalized series of prestratifications

及び § Normalized series attached to germs を掲さいする

前者が§4, 後者が§6 となっているのは, 目下作成中の原稿の都合上で少々不体裁であるが, 寛容をお願いする次第である。尚 §6は完結したものでなく, 問題の reduction step 迄で止めているが, 残りは本質的に困難を含まぬ。尚 §5は, §6の準備として real analytic variety の germ に対して elementary arguements をする。§5に含まれる事項の大半は多かれ少かれ周知だと思われるので, 省略させて頂いた。この論文を読む限りでは§5は必要でない。勿論 §5と§6の残りは既に存在するので, どこかにまとめて発表をしたいと思っている。

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De Rham cohomologies and stratifications .

Complex analytic de Rham cohomology III.

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The importance of the idea of stratifying varieties in the study of algebraic and analytic varieties is well known. The investigation of stratification of varieties would involve basically the following steps*:

(1) To stratify varities so that each stratum as well as the relations among the strata ,e.g., incidence relation ,..., are of simple (or typical) forms.

(2) To obtain results of desired nature for each stratum or each series of strata,etc. with respect to a fixed stratification for given varieties.

(3) To piece together results from the step (2) in order to obtain results of a desired sort for given varieti--es and subvarieties,....

The steps $\{(1),(2)\}$ and (3) might reasonably called , respectively, localization steps (for given global problems)

(*) See R.Thom [8] , H.Whitney [9]. The author learned the theories of stratifications in connection with his proposed approach to Complex analytic de Rham cohomology. (Cf. [4],[5].)

and globalization steps (to be applied to local results).

In [5],[7] we investigated certain quantitative proper--ties of real analytic varieties. Results of [5] are used in our study of the complex analytic de Rham cohomology. Our investigations in [7] are carried out using steps (1), (2) and (3). Exact sequences of Mayer- Vietoris type are used repeatedly in our globalization steps.The basis of our arguments used in the globalization steps is algebraic in nature.

The main purpose of the present note is to introduce the notion of cochain complex with incidence relations (C.C.I.) for a prestratified space. (See n.1. and n.2. below.) The arguments used in the study of C.C.I. are gene--ralizations, as well as abstractions , of those in [7]. When C.C.I.'s are related to de Rham cohomologies of certain types, the arguements applicable to C.C.I.'s in general clarify relations between 'local' and 'global'* data in the de Rham cohomologies in question. Actually the author's hope in introducing the notion of C.C.I. is to clarify relations between 'local' and 'global' data in de Rham cohomologies of various types.

(*) The terms 'local' and 'global' in this note should be understood in the sense explained at the beginning of this note.

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The cotents of this note are preliminary in nature. However, the arguments applicable to C.C.I. in general are indispensable in [5] and have certain theoretically pleasa-nt aspects.

n.1. <u>Prestratifications.</u> Let X be a topological space. By a prestratification* Ⓢ of X we mean a collection $Ⓢ = \{S_\lambda\}_{\lambda\in\Lambda}$ of subsets S_λ's of X satisfying the following conditions.

$(1.1)_1$ X is the disjoint union of S_λ's in Ⓢ : $X = \bigcup S_\lambda$, $S_\lambda \in$ Ⓢ.

$(1.1)_2$ For each stratum $S_\lambda \in$ Ⓢ, the dimension : $\dim S_\lambda \in Z^+$ is given.

$(1.1)_3$ Frontier condition : For each stratum $S_\lambda \in$ Ⓢ, $\mathrm{fron}(S_\lambda) = \overline{S_\lambda} - S_\lambda$ is the disjoint union of lower dimensi-onal strata of Ⓢ .

To substantially simplify notations in later arguments we assume that

$(1.1)_4$ Ⓢ is a finite set.**

A pair (X,Ⓢ) consisting of a topolcgical space and its prestratification Ⓢ will be called a <u>prestratified space</u>. Let (X,Ⓢ) be a prestratified space. For Ⓣ ⊂ Ⓢ, let |Ⓣ| denote the

(*) For definitions of stratifications and prestratifi-cations , see J.Mather [2], R.Thom[8] . Our definition of prestratifications is ,for technical reasons, not the same as in [2],[8].

(**) For the case where Ⓢ is locally finite , see [6]

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support of T : $|T| = \bigcup S_\lambda$, $S_\lambda \in T$. Moreover, for $T' < T < S$, $\overline{|T'|}_{T}$ denotes the closure of $|T'|$ in $|T|$. We list certain notations used below. For $T \subset S$, define T_C, T_m by

$$T_C = \{ T' \subset T : \overline{|T'|}_{T} - T' = \phi \text{ or closed} \} ,$$

$$T_m = \{ S_\lambda \in T : l(S_\lambda) \leq m \}.^{*}$$

Moreover, let T_0 denote the collection of series of strata in T:

$$T_0 = \{ S_{\lambda_1, \ldots, \lambda_t} = S_{\lambda_1} < \cdots < S_{\lambda_t} : S_{\lambda_j} \in T (j = 1, \ldots, t) \} .$$

In the above $S_{\lambda_1} < S_{\lambda_2}, \ldots$ means that $S_{\lambda_1} \subset \text{fron}(S_{\lambda_2})$, ... For $T' \subset T$ and a series $S_{\lambda_1, \ldots, \lambda_t}$, $S_{\lambda_j} \in T'$, let $T'_m(S_{\lambda_1, \ldots, \lambda_t})$ den--ote the intersection $T'_m \cap \{ S \in T' : S_{\lambda_t} < S \}$. For $T < S$ define T_{0C} by

$$T_{0C} = \{ T'_m(S_{\lambda_1, \ldots, \lambda_t}) : T' < T , S_{\lambda_j} \in T' , j = 1, \ldots, t \} .$$

Then one easily derives the following fact .

(1.2) If $T \in S_C$, then $T_m \in S_C$ and $T_C, T_{0C} \subset S_C$. Moreover, if $T_1, T_2 \in S_C$ satisfy the relation $T_1 \vee T_2$, then $T_1 \cup T_2 \in S_C$.

Here $T_1 \vee T_2$ if , for any $S_{\lambda_i} \in T_i$ $(i = 1, 2)$, $S_{\lambda_1} \not< S_{\lambda_2}$, $S_{\lambda_1} \not> S_{\lambda_2}$.

n.2. <u>Cochain complex with incidence relation (C.C.I).</u>

Let R be a noetherian ring , and let (X, S) be a prestratified space. Moreover, let $C(S)$ be a collection

(*) For $S \in S$, the length $l(S)$ can be defined in an obvious manner (see [6]).

of assignments $\{C'(S), E(S), E'(S)\}$ of the following forms: $C'(S)$: $T \in S_C \to C'(T)$, $E(S)$: $U \in S_0 \to E(U)$ and $E'(S)$: $U \in S_{0C} \to E'(U)$. Here $C'(T)$, $E(U)$ and $E'(U)$ are R-cochain complexes. The collection $C(S) = \{C'(S), E(S), E'(S)\}$ as above is called a <u>cochain complex with incidence relations attached to</u> (X, S) (C.C.I. attached to (X, S)) if $\hat{C}(S)$ is equipped with isomorphisms $i(S)$, $i(T_1, T_2)$, $i(U_1)$ and homomorphisms* $h_k(\hat{T}_m)$, $h_k(\hat{U}_m)$ of the following forms.

$(1.3)_1$ $\quad i(S): 0 \to \hat{C}(S) \to E(S) \to 0$, $S \in S$.

$(1.3)_2$ $\quad i(T_1, T_2): 0 \to C'(T_1 \cup T_2) \to C'(T_1) \oplus C'(T_2) \to 0$, $T_1, T_2 \in S_C$ and $T_1 \vee T_2$.

$(1.3)'_2$ $\quad i(U_1): 0 \to E'(U_1) \to \oplus_{\lambda_{t+1}} E(S_{\lambda_1, \dots, \lambda_{t+1}}) \to 0$, where $U_1 = T_1(S_{\lambda_1, \dots, \lambda_t})$ with $T \in S_C$ and $S_{\lambda_j} \in T$. Moreover, $S_{\lambda_{t+1}} \in U_1$.

$(1.3)_3$ $\quad 0 \to \hat{C}(T_{m+1}) \xrightarrow{h_1(T_m)} C'(T_m) \oplus \{\oplus_{\lambda^{m+1}} C'(S_{\lambda^{m+1}})\} \xrightarrow{h_2(T_m)} \oplus_{\lambda^{m+1}} E'(T_m(S_{\lambda^{m+1}})) \to 0$, where $T \in S_C$ and $S_{\lambda^{m+1}} \in T_{m+1} - T_m$.

$(1.3)'_3$ $\quad 0 \to C'(U_{m+1}) \xrightarrow{h_1(U_m)} E'(U_m) \oplus \{\oplus_{\lambda^{m+1}_{t+1}} E(S_{\lambda_1, \dots, \lambda^{m+1}_{t+1}})\} \xrightarrow{h_2(U_m)} \oplus_{\lambda^{m+1}_{t+1}} E(U_m(S_{\lambda^{m+1}_{t+1}})) \to 0$, where $U_m = T_m(S_{\lambda_1, \dots, \lambda_t})$ and $U_{m+1} = T_{m+1}(S_{\lambda_1, \dots, \lambda_t})$. Moreover, $S_{\lambda^{m+1}_{t+1}} \in U_{m+1} - U_m$.

Postulated conditions of the existence of isomorphisms in $(1.3)_1$, $\{(1.3)_2, (1.3)'_2\}$ and homomorphisms in $(1.3)_3, (1.3)'_3$ will be called '<u>Identification condition</u>', '<u>Disjoint condition</u>'

(*) Isomorphisms and homomorphisms are those of R-cochain complexes.

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and 'Incidence condition' ('Mayer- Vietoris condition') respec--tively. The collection of isomorphisms $i(S)$,... and homomo--rphisms h_k's will be denoted by $K(C(S))$. When we emphasize the role of $K(C(S))$, we say that $C(S)$ is $K(C(S))$-C.C.I. . Let $C(S)$ and $K(C(S))$ be as above . Then isomorphisms $i(S)$,. ...and homomorphisms h_k's of cochain complexes induce corre--sponding isomorphisms $i^*(S)$,...and homomorphisms h_k^*'s of cohomology groups naturally. The collection of $i^*(S)$,...and h_k^*'s will be denoted by $K^*(C(S))$.

Equivalences between C.C.I.'s. Let (X,S) be a prestra--tified space, and let $C^i(S)$ be $K(C^i(S))$-C.C.I. ($i = 1,2$). Moreover, let α^*, β^* and β'^* be families of R-homomorphisms of the following forms: $\alpha^* = \{\alpha^*(T) : H^*(C'^1(T)) \to H^*(C'^2(T)), T \in S_C\}$, $\beta^* = \{\beta^*(U) : H^*(E^1(U)) \to H^*(E^2(U)), U \in S_0\}$ and $\beta'^* = \{\beta'^*(U) : H^*(E'^1(U)) \to H^*(E'^2(U)), U \in S_{0C}\}$. Then we can define the notion of commutativity of $\{\alpha^*, \beta^*, \beta'^*\}$ with $K(C^i(S))$ in an obvious manner ([6]). We say that $C^i(S)$ ($i = 1,2$) are $\{\alpha^*, \beta^*, \beta'^*\}$- equivalent if (i) $\{\alpha^*, \beta^*, \beta'^*\}$ commute with $K(C^i(S))$ ($i = 1,2$) and if (ii) the homomorphism $\beta^*(U)$ is an isomorphism for any $U \in S_0$.

Now , in our investigations, there are reasons for regarding $C'(T)$, $T \in S_C$ as 'global' data and $E(U)$, $U \in S_0$ as 'local' data. The following lemma shows that certain propert-

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-ies of global data are derived from those of local data.

Lemma 1, Let $\mathcal{C}^i(\mathcal{S}) = \{C'^i(S), E^i(S), E'^i(S)\}$ be $\mathcal{K}(\mathcal{C}^i(\mathcal{S}))$-C.C.I. $(i = 0,1,2)$ of a prestratified space $(X, \mathcal{S})$.

(I) If $H^*(E^0(U))$ is a finitely generated $\mathcal{R}$-module for each $U \in \mathcal{S}_C$, them $H^*(C'^0(U))$ is so for each $T \in \mathcal{S}_C$.

(II) Let $\mathcal{C}^i(\mathcal{S})$ $(i = 1,2)$ be $\{\alpha^*, \beta^*, \beta'^*\}$-equivalent. Then $\alpha^*(T)$ is an $\mathcal{R}$- isomorphism for each $T \in \mathcal{S}_C$. Here $\alpha^*, \beta^*, \beta'^*$ are families of $\mathcal{R}$-homomorphisms of the forms given in the be-ginning of n.2.

For the proof of Lemma 1, see [6].

n.3. An exact sequence of Mayer- Vietoris type. Let K be an algebraically closed field of any characteristic. In n.3. every variety in question is assumed to be a reduced K-variety. Let A^n, V and D be an affine space of dimension n, a variety in A and a divisor in A, respectively, such that for each irreducible component V_j of V, $V_j \not\subset D$ and $V_j \cap D \neq \phi$. We denote by W the variety in A characterized by $|W| = |V| \cap |D|$. Now let $\mathcal{O}, I_V, I_W$ and $I_D = \mathcal{O}(h)$ denote respectively the ring $K[x_1, \dots, x_n]$ and the ideals of V, W, and D. The completions $\varprojlim_n \mathcal{O}/I_V^n$ and $\varprojlim_n \mathcal{O}/I_W^n$ are denoted by $\hat{\mathcal{O}}^V$ and $\hat{\mathcal{O}}^W$ respectively. We denote the localizations $\mathcal{O}[h^{-1}]$ and $\hat{\mathcal{O}}^W[h^{-1}]$ by $\mathcal{O}[*D]$ and $\hat{\mathcal{O}}^W[*D]$ respectively. Moreover, let $\hat{\mathcal{O}}^{V-W}$ and $\hat{\mathcal{O}}^{W,V-W}$ be respe-ctively the completions defined by $\varprojlim_n \mathcal{O}[*D]/(\mathcal{O}[*D] \cdot I_V^n)$ and $\varprojlim_n \hat{\mathcal{O}}^W[*D]/(\hat{\mathcal{O}}^W[*D] \cdot I_V^n)$. In the above we regard I_V as contained in $\mathcal{O}[*D]$ and $\hat{\mathcal{O}}^W[*D]$ in a natural fashion. For the graded ring Ω of K-differential forms over A, let $\hat{\Omega}^{V}, \hat{\Omega}^{W}, \hat{\Omega}^{W,V-W}$ respectively denote $\Omega \otimes \hat{O}^{V}, \ldots, \Omega \otimes_{O} \hat{O}^{W,V-W}$. Then we have :

Lemma 2. For the rings $\hat{\Omega}^{V}, \ldots, \hat{\Omega}^{W,V-W}$, the exact sequence

$$\text{(I)} \qquad 0 \to \hat{\Omega}^{V} \xrightarrow{\hat{\rho}^{V}_{W} \oplus \hat{\rho}^{V}_{V-W}} \hat{\Omega}^{W} \oplus \hat{\Omega}^{V-W} \xrightarrow{\hat{\rho}^{W}_{W,V-W} \oplus \hat{\rho}^{V-W}_{W,V-W}} \hat{\Omega}^{W,V-W} \to 0,$$

holds. In (I) the continuous homomorphisms $\hat{\rho}^{V}_{W}, \ldots, \hat{\rho}^{V-W}_{W,V-W}$ are determined naturally from topologies of $\hat{\Omega}^{V}, \ldots \hat{\Omega}^{W,V-W}$.

For the proof of Lemma 2 , see [6]. The exact sequence (I) relates cohomology groups $H^{*}(\hat{\Omega}^{V})$, $H^{*}(\hat{\Omega}^{V-W})$ and $H^{*}(\hat{\Omega}^{W,V-W})$ to the cohomology group $H^{*}(\hat{\Omega}^{V})$. For a pair (V,W) of smooth varieties (defined over the complex number field $\mathbb{C}$), the idea of relating the cohomology groups of W,V-W and N(W)-W (*) to that of V may be regarded as one of the basic ideas in the classical (analytic) theories of residues. (Cf. J.Leray[2], P.A.Griffith[1],..) . The sequence (I) might be regarded as a generalization in an alge-braic direction of the idea explained above. Moreover, Lemma 2 enables us to attach the algebraic de Rham C.C.I. to the pre-stratified space (V,S=(W,V-W)) in a natural manner.

Remarks about results untouched here. In this note, we have spent several pages explaining ideas used in defining C.C.I.'s. For arguements on C.C.I.'s untouched here, see [6]. In particular , [6] contains examples of C.C.I.'s such

(*) N(W) is a suitable neighbourhood of W in V.

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as the singular , the C^{∞}- de Rham , and the P.G.(polynomial growth) de Rham C.C.I.'s, as well as an application of the sequence (I).

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§4. Normalized series of prestratified spaces.

4.1. Normalized series of prestratified spaces. Let $R^n(x)$ be an n-dimensional euclidean space with coordinates $(x) = (x_1,\dots,x_n)$. By a series R of prestratified spaces in $R^n(x)$, we mean a collection $R = \{R, (U,V,S), (U',V',S')\}$, of a series R of euclidean spaces $U(U')$ of bounded connected domains, $V(V')$ of closed analytic varieties and $S(S')$ of prestratifications, where $R \dots S$ are of the following forms.

Germ

Here ...

$(4.1)_1$ $R = \{R^1(y_1), R^2(y_2), \dots, R^n(y_1,\dots,y_n)\}$, where $(y_1,\dots,y_n)$ is a system of coordinates of $R^n(x)$.

$(4.1)_2$ $U = \{U^j\}_{j=1,\dots,n}$ $(U' = \{U'^j\}_{j=1,\dots,n})$

where U^j (U'^j) is a connected bounded domain in $R^j(y_1,\dots,y_j)$ $(j = 1,\dots,n)$.

$(4.1)_3$ $V = \{V^j\}_{j=1,\dots,n}$ $(V' = \{V'^j\}_{j=1,\dots,n})$, where V^j (V'^j) is a closed analytic variety in $U^j(U'^j)$ such that $\dim V^j$ $(\dim V'^j) \leqq j - 1$ $(j = 1,\dots,n)$.

$(4.1)_4$ $S_0 = \{S_0^j\}_{j=1,\dots,n}$ $(S_0' = \{S_0'^j\}_{j=1,\dots,n})$, where S_0^j $(S_0'^j)$ is a prestratification of the pair (U^j, V^j) (U'^j, V'^j) $(j = 1,\dots,n)$.

~~For a series $R = R, (U,V,S_0), (U',V',S'_0)$ of prestrati~~

Let $(R)=\{R, (U, V, S_0), (U', V', S_0')\}$ be a series of prestrat-ified spaces in $R^n(x)$. We then call the series $S_0 = \{S_0^j\}_{j=1,..,n}$ ($S_0' = \{S_0'^j\}_{j=1,..,n}$) a <u>series of prestratifications of the pair</u> $U = \{U^j\}_{j=1,..,n}$, $V = \{V^j\}_{j=1,..,n}$ ($(U' = \{U'^j\}_{j=1,..,n}$, $V' = \{V'^j\}_{j=1,..,n})$). Moreover, let S^j (S'^j) denote the <u>induced</u> prestratification of V^j (V'^j) from S_0^j ($S_0'^j$), $j = 1, \dots, n$,. We mean by the <u>induced series S (S') of prestratifi-cations of V (V')</u> the series $\{S^j\}_{j=1,..,n}$ ($\{S'^j\}_{j=1,..,n}$) of prestratifications of V^j's (V'^j's).

Let $R = \{R, (U, V, S_0), (U', V', S_0')\}$ be a series of presrra-tified spaces in $R^n(x)$. For the series $R = \{R^j(y_1, \dots, y_j)\}_{j=1,\dots,n}$, ..., $S_0' = \{S_0'^j\}_{j=1,..,n}$, we call $R^j(y_1, \dots, y_j)$, ..., $S_0'^j$ the <u>j-th component</u> of $R, \dots, S_0'$ respectively. Also for the induced series $S = \{S^j\}_{j=1,..,n}$ ($S' = \{S'^j\}_{j=1,..,n}$) of V (V') from S_0^j (S_0'), we call S^j (S'^j) the j-th component of S (S').

Let $R = \{R, (U, V, S_0), (U', V', S_0')\}$ be a series of prestra-tified spaces in $R^n(x)$. We say that R is <u>admissible</u> if the following conditions $(4.2)_{1\sim 3}$ and $(4.3)_{1\sim 3}$ are valid.

$(4.2)_1$ Each stratum $S_\lambda^j \in S_0^j - S^j$ ($S_\lambda'^j \in S_0'^j - S'^j$), $j = 1, \dots, n$, is a connected component of $U^j - V^j$ ($U'^j - V'^j$). Here S_0^j ($S_0'^j$) is the j-th component of S (S'), and S^j (S'^j)

is the induced prestratification of $V^j (V'^j)$ from S_0^j $(S_0'^j)$.

$(4.2)_2$ The j-th component U^j (U'^j) of U (U') , $j = 1,..,n$, satisfies the (d) - regularization condition. Moreover, the j -th components $(U'^j, V'^j, S_0'^j)$ of the triple (U', V', S_0') is a (d)-envelop of the j-th components (U^j, V^j, S_0^j) of (U, V, S_0).

$(4.2)_3$ The j-th component S_0^j of S_0 satisfies the (δ)-separation condition , $j = 1,...,n$.

$(4.3)_1$ Between components U'^j (U^j) and $U'^{j-1} (U^{j-1})$, $j = 2,..,n$, commutative diagrams of the following forms are valid.

$$\begin{array}{ccc} U'^j & \xrightarrow{\varphi'^j} & U'^{j-1} \times I^0 \\ \downarrow \pi_{j-1,j} & & \downarrow pr_1^j \\ U'^{j-1} & \xrightarrow{id'^j} & U'^{j-1} \end{array}, \qquad \begin{array}{ccc} U^j & \xrightarrow{\varphi^j} & U^{j-1} \times I'^0 \\ \downarrow \pi_{j-1,j} & & \downarrow pr_2^j \\ U^{j-1} & \xrightarrow{id^j} & U^{j-1} \end{array}.$$

Here φ'^j is a C^∞-diffeomorphism and φ^j is the restriction of φ'^j to U^j. Moreover, $pr_1^j (pr_2^j)$ denotes the projection from the product $U'^{j-1} \times I^0$ $(U^{j-1} \times I'^0)$ to the first factor $U'^{j-1} (U^{j-1})$, and id'^j (id^j) denotes the identity map on U'^j (U^j).

$I'^0 \subset I^0$.

$(4.3)_2$ The restriction $\pi_{j-1,j}(y)|V'^j$ of $\pi_{j-1,j}(y)$ to V'^j is comatible with prestratifications S'^j of V'^j and $S_0'^j$ of U'^j, $j = 2,...,n$.

$(4.3)_3$ For each atratum S'^j of the induced prestratification $S_\lambda'^j$ of V'^j, the map $\pi_{j-1,j}(y) : S_\lambda'^j \to \pi_{j-1,j}(y) (S_\lambda'^j)$ is <u>real analytically biholomorphic</u> , $j = 2,...,n$.

Remark 4.1. Let $\mathfrak{R} = \{R, (U, V, S_0), (U', V', S_0')\}$ be an admi--ssible series of prestratifications in $R^n(x)$. Then the conditions $(4.2)_{1\sim 3}$ and $(4.3)_{1\sim 3}$ imply the following.

$(4.4)_1$ The restriction $\pi_{j-1,j}|V^j$ of $\pi_{j-1,j}(y)$ to V^j is compatible with prestratifications S^j of V^j and S_0^j of U^{j-1}, $j = 2,\dots,n$. Here V^j, S^j, S_0^j, U^{j-1} are j-th components of V, the induced series S of prestratifications of V from S_0, and the (j-1)-th component of U respectively.

$(4.4)_2$ For each stratum $S^j_\lambda \in S^j$ the projection $\pi_{j-1,j}(y)$: $S^j_\lambda \longrightarrow \pi_{j-1,j}(y)\ (S^j_\lambda)$ is (real analytically) biholomor--phic, $j = 2,\dots,n$.

$(4.4)_3$ If S^1 (S'^1) consists of strata which are connected, then any stratum of S_0^j $(S_0'^j)$, $j = 1,2,\dots,n$, is a connected analytic manifold. S^1: con

Let $\mathfrak{R} = \{R, (U, V, S_0), U', V', S_0')\}$ be an admissible series of prestratifications in $R^n(x)$. Moreover, let S (S') denote the induced series of prestratifications of V (V') from S_0 (S_0') . Let $S'^j_\lambda \in S'^j$, $j = 2,\dots,n$. By a <u>representation datum of</u> S'^j_λ , we mean a collection $\mathfrak{f}(S'^j_\lambda) = \{f(S'^j_\lambda), \bar{f}(S'^j_\lambda)\}$ of sets $f(S'^j_\lambda)$, $\bar{f}(S'^j_\lambda)$ of analytic functions, where $f(S'^j_\lambda)$ Here and $\bar{f}(S'^j_\lambda)$ are of the following forms.

d ~~$(4.5)_1$ $f(S'^j) = f(S'^j;\ x_1, \dots, y_{n'j}, y_{n'+1})$~~

$(4.5)_1$ $f(S_\lambda'^j) = \{f_t(S_\lambda'^j : y_1, \ldots, y_{n_\lambda'^j}, y_{n_\lambda'^j+t})_{t=1,\ldots,n_\lambda''^j}\}$

, where $f_t(S_\lambda'^j)$, $t = 1,\ldots,n_\lambda''^j$ is a <u>monic polynomial</u> in $y_{n_\lambda'^j+t}$ such that (i) coefficients of $f_t(S_\lambda'^j)$ are analytic functio-ns in $U'^{n_\lambda'^j} \subset R(y_1,\ldots,y_{n_\lambda'^j})$ and (ii) $f_t(S_\lambda'^j)$ vanishes on $S_\lambda'^j$. In the above $R^{n_\lambda'^j}(y_1,\ldots,y_{n_\lambda'^j})$ is the $n_\lambda'^j$-th component of R^n and $U'^{n_\lambda'^j}$ is the $n_\lambda'^j$-th component of U_λ^n.

$(4.5)_2$ $f(S_\lambda'^j) = \{f_t(S_\lambda'^j : y_1,\ldots,y_j)\}_{t=1,\ldots,k_\lambda'^j}$,

where $k_\lambda'^j \geqq j - n_\lambda'^j$ ans $f_t'(S_\lambda'^j)$ vanishes on $S_\lambda'^j$.

For a representation datum $f(S_\lambda'^j) = \{f(S_\lambda'^j), f(S_\lambda'^j)\}$, $S_\lambda'^j \in S'^j$, let $\Delta(f(S_\lambda'^j))$ denote the closed analytic variety in U'^j defined to be the zero locus of the following functio-ns.

$(4.5)_1'$ $f'(S_\lambda'^j)$, $J\left(\dfrac{f^I(S_\lambda'^j)}{y_{n_\lambda'^j+1},\ldots y_j}\right)$, where $I = (i_1,\ldots,i_{n_\lambda''^j})$

and $f^I(S_\lambda'^j)$ is the abbriviation of $\{f_{i_1}(S_\lambda'^j),\ldots,f_{i_{n_\lambda''^j}}(S_\lambda'^j)\}$

In the above we write $j - n_\lambda'^j$ as $n_\lambda''^j$.

Moreover, for $m \in Z^{+n''^j}$, let $D_m(f(S_\lambda'^j))$ denote the <u>locally closed</u> analytic variety in U'^j defined as follows.

$(4.5)_2'$ $D_m(f(S_\lambda'^j)) = \left\{Q'^j \in U'^j : \left(\partial^{k_s} f_s(S'^j) \big/ \partial y_s^{k_s}\right)(Q'^j) = 0,\ 0 \leqq k_s \leqq m_s - 1,\ \left(\partial^{m_s} f_s(S'^j) \big/ \partial y_s^{m_s}\right)(Q'^j) \neq 0,\ s = n'+1,\ldots,n''^j\right\}$.

The collection $F^j = \{F^j, F'^j\}$ of $F^j = \{f^j(S_\lambda'^j),\ S_\lambda'^j \in S'^j\}$

$\boxed{F^j(S)}$

$n \geq j \geq 2$

and $F'^{j} = \{f'(S'^{j}_{\lambda}), S'^{j}_{\lambda} \in S'^{j}\}$, where $f(S'^{j}_{\lambda}), f'(S'^{j}_{\lambda})$ is a rep-resentation datum of S'^{j}, will be called a <u>representation datum</u> of S'^{j}. Moreover, by a <u>representation datum of S'</u> we mean a collection of series $F = \{F, F'\}$, where $F = \{F^{j}\}_{j=2,\ldots,n}$ and $F' = \{F'^{j}\}_{j=2,\ldots,n}$ consist of representation data (F^{j}, F'^{j}) $(j = 2,\ldots,n)$. For a representation datum $F = (F, F')$ as above $F^{j} = (F^{j}, F'^{j})$ will be called the <u>j-th component of F</u>.

Let $R^{n}(x)$ be an n-dimensional euclidean space with coordinates (x). By a <u>normalized series of prestratified spaces in $R^{n}(x)$</u>, we mean a pair (R, F) of an admissible series R of prestratifies spaces in $R^{n}(x)$ and a representat-tion datum F of R satisfying the following conditions (4.6)

(4.6)$_1$ For each stratum $S'^{j}_{\lambda} \in S'^{j}$, $j = 2,\ldots,n$, the zero locus $V(f(S'^{j}_{\lambda}))$ of $f(S'^{j}_{\lambda})$ is expressed as the union of strata in S'^{j} of dimension at most n'^{j}_{λ}. Moreover, for any stratum $S'^{j}_{\lambda'} \subset V(f(S'^{j}))$ of dimension n'^{j}_{λ}, $f'(S'^{j}_{\lambda'}) = f'(S'^{j}_{\lambda})$.

(4.6)$_2$ <u>Ramification conditions</u>. For each stratum $S'^{j}_{\lambda} \in S'^{j}$, $j = 2,\ldots,n$,

$$S'^{j}_{\lambda} \cap \Delta(f'(S'^{j}_{\lambda})) = \phi.$$

(4.6)$_3$ <u>Higher discriminalt conditions</u>. For any pair $(S'^{j}_{\mu}, S'^{j}_{\lambda})$ of strata in S'^{j} satisfying $S'^{j}_{\mu} \prec S'^{j}_{\lambda}$,

Now our position is to define a notion of n.s. of pre...:

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Such that the following condition $(4.6)_{1\sim3}$ are satisfied.

for (4.6)

there exists a (uniquely determined) element $m \in Z^{+n''j}$, $n = j - d$
in such a manner that

$$S'^{j}_{\mu} \in D_m(f(S'^{j}_{\lambda}))$$

holds.

Of course, S'^{j}, $f(S'^{j}_{\lambda})$ ~~$f'(S'^{j}_{\lambda})$ in (4.6)~~ are the j-th component of S' and the representation datum of the stratum $S'^{j}_{\lambda} \in S'^{j}$ respectively.

Let $R^{n}(x)$ be a euclidean space of dimension n, and let $\{R, F\}$ be a normalized series of prestratifications in $R^{n}(x)$. We write the admissible series R as $R = \{B, U, V, S), U', V', S'\}$ Moreover, let S (S') denote the induced series of prestratifications of V (V') from S_0 (S'_0). Let S^{n}_{λ} (S'^{n}_{λ}) be a stratum of the n-th component S^{n} (S'^{n}) of S (S'). We say that the normalized series (R, F) satisfies the <u>differentiablity condition</u> for S^{n} (S'^{n}) if the following condition is val--id.

(4.7) For any $j > \dim(S^{n}_{\lambda})$ $(j' > \dim(S'^{n}_{\lambda}))$, the length $l(S^{j}_{\lambda})$ $(l(S'^{j'}_{\lambda}))$ of $S^{j}_{\lambda} = \pi_{jn}(S^{n}_{\lambda})$ $(S'^{j'}_{\lambda} = \pi_{j'n}(S'^{n}_{\lambda}))$ in S^{j} $(S'^{j'})$ is equal to one. Here S^{j} and $S'^{j'}$ are j-th components of S and the j'-th component of S' respectively.

Note that if (R, F) satisfies the differentiablity cond--ition for $S^{n}_{\lambda} \in S^{n}$ $(S'^{n}_{\lambda} \in S'^{n})$, then $l(S^{n}_{\lambda})$ in S^{n} ($l(S'^{n}_{\lambda})$ in S'^{n}) is equal to one. We are using the terminology of 不要

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'differentiablity condition' for the condition (4.7), becau-
-se the condition (4.7) assures the differentiablity of
sertain maps generally' on S^n_λ (S'^n_λ). (Cf [C]... Also see the later arguements of this paper.) C.A.D.R.

Let (U,V) be a pair of a bounded domain U in a euclide-an space $R^n(x)$ and a closed analytic variety in U. Moreover let $W = \{W_t\}$ be a finite set of closed analytic subvarieties W_t's of V. We mean by a <u>normalized series (R,F) of prestratified spaces attached to (U,V,W)</u> a normalized series $(R,F) = (R, (U,V,S), \{(U,V,S)\})$ of prestratified spaces in $R^n(x)$ satisfying the following conditions.

$(4.8)_1$ $(U,V) = (U^n, V^n)$, where U^n and V^n are n-th components of U and V respectively.

$(4.8)_2$ Each subvariety W_t is the union of strata of S^n where S^n is the n-th component of the induced series S from S_0.

Let $R^n(x)$ and (U,V,W) be as above. Moreover, let X be an equidimensional subvariety of V. A normaliz-ed series $(R,F) = (R, (U,V,S), (U',V',S'), (F,F))$ will be said to satisfy the <u>differentiablity condition for X</u> if the following is valid.

$(4.7)'$ X is the union of strata of S^n. Moreover, for each stratum $S^n_\lambda \subset X$ such that $\dim S^n_\lambda = \dim X$, (R,F) satisf-ies the differentiablity condition.

Remark 4.2. When there are no dangers of confusions, we shall do the following abbriviations of terminologies: (i) The terminology 'a normalized series of prestratifications attached to ....' will be abbriviated as 'a normalized series attached to ...'. (ii) The terminology 'the induced series of prestratifications of V from S_0,...' will be abbriviated as 'the induced series from S_0,...'.

文字に入れる。

Remark 4.3. Let $(R,F) = ((R, U, V, S, U', V', S')), (F, F'))$ be a normalized series of prestratifications in a euclidean space $R^n(x)$. We denote by S and S' the induced series from S_0 and S'_0 respectively. Let S'^j_λ be a stratum of the j-th component S'^j of S'. Then we use the symbol $\mathrm{fron}(S'^j_\lambda, F'^j)$ for the closed set $V(f'(S'^j_\lambda)) - S'^j_\lambda$. Here $f'(S'^j_\lambda)$ is in the j-th component F'^j of F'. Moreover, assume that S'^j_λ and the j-th component U^j of U has a common point, and let S^j_λ denote the intersection $S'^j_\lambda \cap U^j$. Then we use the following symbols: (i) The symbols $f(S^j_\lambda)$ and $f'(S^j_\lambda)$ mean the sets $f(S'^j_\lambda)$ and $f'(S'^j_\lambda)$ of analytic functions respectively. (ii) The symbol $V(f'(S^j_\lambda))$ denotes the intersection $V(f'(S'^j_\lambda)) \cap U^j$. (iii) The symbol $\mathrm{fron}(S^j_\lambda, F'^j)$ denotes the closed set $V(f'(S^j_\lambda)) - S^j_\lambda$.

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In the sequel of this section we will mainly be concer--ned with investigations of properties of normalized series. We begin by defining manifolds of certain types for strata in question.

4.2. <u>Level manifolds</u> $T'_\sigma(S^j_\lambda)$,...,. Let $(R,F) = ((R,(U,V,S),(U',V',S')),(F,F'))$ be a normalized series of prestratifications in a euclidean space $R^n(x)$. For the series $R = R^1(y_1),\ldots,R^n(y_1,\ldots,y_n))$, projections $\pi_{j,j'}(y)$ will be abbriviated as $\pi_{j,j'}$, $1 \leqq j \leqq j' \leqq n$. Moreover, ~~by~~ let $S = \{S^j\}$ and $S' = \{S'^j\}$ $(j = 1,\ldots,n)$ we denote mean induced series from S_0 and S'_0 respectively. Now let S^j_λ be a stratum of S^j, $j = 2,\ldots,n$. We first remark that the condition $(4.6)_2$ implis that the coordinates $(y_1,\ldots,y_{n'^j_\lambda})$ are local para--meters at any point on S^j. For a point $P^j_\lambda \in S^j_\lambda$ and a couple define an $(\alpha - n'^j - 1)$-dimensional (real analytic) manifold $T'_{\sigma}(P'^j_\lambda)$ by

$$T'_{\sigma}(P'^j_\lambda) = \Big\{ Q'^j \in R^j \; : \; y_t(Q^j) = y_t(P^j_\lambda), t = 1,\ldots,n^j_\lambda$$

and

$$\text{and } t = j \; , \; |y_t(Q^j) - y_t(P^j_\lambda)| < \sigma \cdot d(P^j_\lambda, \mathrm{fron}(S^j_\lambda)),\; t = n'^j_\lambda + 1,\ldots,j-1 \Big\}$$

Moreover, for a couple , define a (j-1)-dimensional manifold $T_\sigma(S^j_\lambda)$ by

(4.9) $$T_\sigma(S^j_\lambda) = \bigcup_{P^j_\lambda} T_\sigma(P^j_\lambda) \text{ , where } P^j_\lambda \in S^j_\lambda \cdot$$

~~We call the above manifolds T(P'~~

We call the manifolds $T'_{\sigma}(P^j_\lambda)$ and $T'_{\sigma}(S^j_\lambda)$ as above the $\sigma(y_1,\ldots,y_{n'j},y_j)$ - level manifold of $P^j_\lambda(\in S^j_\lambda)$ ~~and the~~ with the size σ $\sigma(y_j)$-level manifold of S^j_λ respectively. Let $T'_{\sigma}(P^j_\lambda)$ be the $\sigma(y_1,\ldots,y_{n'j},y_j)$ - level manifold of $P^j_\lambda \in S^j_\lambda$. For a point $Q^j_\lambda \in T'_{\sigma}(P^j_\lambda)$ and a couple σ', define a one dimensional manifold $T''_{\sigma}(Q^j_\lambda)$ by

$$T''_{\sigma}(Q^j_\lambda) = \Big\{ Q''_j \in R^j(y_1,\ldots,y_j) : y_t(Q''_j) = y_t(Q_j)$$
$$t = 1,\ldots,j-1, \left|y_j(Q''_j) - y_j(Q_j)\right| < \sigma' d(Q^j, \text{from } \{S^{\sigma}_\lambda\})\Big\}$$

Moreover, for couples σ', σ'' define a neighbourfood $T_{\sigma',\sigma''}(S^j_\lambda)$ by

(4, 10) $$T_{\sigma',\sigma''}(S^j_\lambda) = \bigcup_{Q^j_\lambda} T''_{\sigma''}(Q^j_\lambda) \text{ , where } Q^j_\lambda \in T'_{\sigma'}(S^j_\lambda) .$$

We call the neighbourhood $T_{\sigma',\sigma''}(S^j_\lambda)$ as above the T-neighbourhood of the size (σ',σ'').

The manifolds $T'_{\sigma}(S^j_\lambda), T_{\sigma',\sigma''}(S^j_\lambda)$,...as above are conv--enient for inductive arguements of the normalized series (R), (F)) on $j = 1,\ldots,n$.

We shall define one another type of manifolds, denoted by $U_{\sigma,c}(S^j_\lambda)$, for S^j_λ : Let $T'_{\sigma}(S^j_\lambda)$ be the $\sigma(y_j)$- level manifold of S^j_λ and $(c) = (c_1,c_2)$ be a couple of positive numbers(c_1,c_2). We note that we are doing an exceptional use of the term 'couple' for (c) from the rule §3. 1 . Let $Q^j_\lambda \in T'_{\sigma}(S^j_\lambda)$, and let P^j_λ be the point in S^j_λ satisfying $\boxed{F \setminus (R^j)}$

the condition : $Q^j_\lambda \in T'_\sigma(P^j_\lambda)$. Define a one dimensional manif--old $U_c(Q^j_\lambda)$ by

$$U_c(Q^j_\lambda) = \{Q'^j_\lambda \in R^j(y_1,..,y_j) : y_t(Q'^j_\lambda) = y_t(Q^j_\lambda), t = 1,..,j-1, |y_j(Q'^j_\lambda) - y_j(Q^j_\lambda)| < c_1 d(Q^j_\lambda, P^j_\lambda)\}$$

Moreover, for a couple σ and a couple $c = (c_1,c_2)$ of ~~the~~ positive numbers, define a manifold $U_{\sigma,c}(S^j_\lambda)$ of dimen-sion j by

$$(4.9)_3 \qquad U_{\sigma,c}(S^j_\lambda) = \bigvee_{Q^j_\lambda} U_c(Q^j_\lambda) \text{ , where } Q^j_\lambda \in T'_\sigma(S^j_\lambda) \text{ .}$$

The manifolds $U_{\sigma,c}(S^j_\lambda)$'s as above will be also conven--ient in inductive arguements of the series (R,F) .

Remark to 4.2. Similar arguements as above for strata S'^j_λ 's of $(S)'^j$ are ,of course ,possible. However, arguements in 4.2. suffice for the later arguements.

Let (R,F) be a normalized series(in $R^n(x)$) of the form 4.2.. Moreover, let the symbols $(S), (S)', (S)^j, (S)'^j, \ldots$ have the same meanings as in §4.2. Our next task is to examine relations between prestratifications $(S)^j$ and $(S)_0^{j-1}$, $j = 2,..,n$ Here $(S)_0^{j-1}$ means,of course, the (j-1)-th component of $(S)_0$. Our arguements will be divided into several pieces. Before we enter into detailed arguements, we shall do certain remarks on materials concerned below.

(i) Let S^j_λ be a stratum of $\mathfrak{S}^j$, $j = 2,\ldots,n$. Then, by a simple computation, we have the following relation.

$\{N_\delta(S^j_\lambda)_\cap \mathcal{T}^j\} \sim \{T_{\sigma,\delta}(S^j_\lambda)\}$, where δ, σ and $\sigma\delta$ ex-haust all the couples obeying the rule in 3.3..

(ii) Let $S'^j_\lambda \in \mathfrak{S}'^j$, and let $S^j_\lambda = S'^j_\lambda \cap U^j$, $j = 2,\ldots,n$. Moreover, let D'^j be an analytic variety in U'^j defined to be the zero locus of an analytic function g in U'^j. Here U^j and U'^j are the j - th components of $\mathfrak{U}$ and $\mathfrak{U}'$ respective-ly. By the inequality of Lojasiewicz, $d(P^j_\lambda, V(\mathfrak{f}'(S'^j_\lambda)))$ $\sim d(P^j_\lambda, D'^j)$ for any $P^j_\lambda \in S^j_\lambda$. Here $\mathfrak{f}'(S'^j_\lambda)$ is in the j-th component $\mathfrak{F}'^j$ of $\mathfrak{F}'$. Assume that $D'^j \cap S'^j_\lambda = \phi$. Then from the obvious relation $D'^j \cap V(\mathfrak{f}'(S'^j_\lambda)) = D'^j \cap \{(V(\mathfrak{f}'(S'^j_\lambda)) - S'^j_\lambda\}$ and the condition $(4.2)_3$, we have

(4.10) $\quad d(P^j_\lambda, \mathrm{fron}(S^j_\lambda)) \lesssim d(P^j_\lambda, D'^j)$ for any $P^j_\lambda \in S^j_\lambda$.

(ii)' Let S'^j_λ, S'^j_μ be strata in $\mathfrak{S}'^j$ such that $S'^j_\lambda < S'^j_\mu$. Moreover, let $m = (m_1,\ldots,m_{n''^j_{\lambda\mu}})$ be the element in $Z^{+n''^j_\lambda}$ such that $S'^j_\mu \subset D_m(\mathfrak{f}(S'^j_\lambda))$, where $\mathfrak{f}(S'^j_\lambda) \in F^j$. Let D'^j_m be the variety in U'^j defined to be the zero locus of the function $\partial^{m_s} f_s(S'^j_\lambda)/\partial y^{m_s}_{n'+s}$, $s = 1,\ldots,n''^j_\lambda$. Then from the basic conditions $(4.6)_4$ and (4.10) the following fact is valid.

(4.10)' $\quad d(P^j_\lambda, \mathrm{fron}(S^j_\lambda)) \lesssim d(P^j_\lambda, D'^j_m)$,for any $P^j_\lambda \in S^j_\lambda$.

Now we begin by arguements by investigating the inverse ~~image $\pi^{-1}_{j-1,j}(S^j)$ of S^j's , S^j~~

images $\pi^{-1}_{j-1,j}(S^{j-1}_{\lambda})$'s, where $S^{j-1}_{\lambda} \in \mathfrak{S}^{j-1}$.

4.3. <u>The inverse image $\pi^{-1}_{j-1,j}(S^{j-1}_{\lambda})$, $S^{j-1}_{\lambda} \in \mathfrak{S}^{j-1}_{0}$, $j=2,\ldots,n$.</u>
Let $S^{j-1}_{\lambda} \in \mathfrak{S}^{j-1}_{0}$. Define the subset $\mathfrak{S}^{j}(S^{j-1}_{\lambda})$ of $\mathfrak{S}^{j}$ by $\mathfrak{S}^{j}(S^{j-1}_{\lambda}) = \{S^{j}_{\lambda'} \in \mathfrak{S}^{j} : \pi_{j-1,j}(S^{j}_{\lambda'}) = S^{j-1}_{\lambda}\}$. For a stratum $S^{j}_{\lambda'} \in \mathfrak{S}^{j}(S^{j-1}_{\lambda})$ let $y^{j}_{\lambda'}(S^{j-1}_{\lambda})$ denote the <u>single valued</u> analytic function on S^{j-1}_{λ} defined by

$$y^{j}_{\lambda'}(S^{j-1}_{\lambda} : P^{j-1}_{\lambda}) = y^{j}(P^{j}_{\lambda'}) \quad \text{for } P^{j-1}_{\lambda} \in S^{j-1}_{\lambda},$$

where $P^{j}_{\lambda'}$ is the point in $S^{j}_{\lambda'}$ satisfying $\pi_{j-1,j}(P^{j}_{\lambda'}) = P^{j-1}_{\lambda}$. Take a point $P^{j-1}_{0} \in S^{j-1}_{\lambda}$. By a suitable change of suffices, we can assume the following.

$$y^{j}_{\lambda'_1}(P^{j-1}_{0}) < y^{j}_{\lambda'_2}(P^{j-1}_{0}) < \cdots < y^{j}_{\lambda'_k}(P^{j-1}_{0}).$$

Then from the connectedness of S^{j-1}_{λ} and the very basic condition of the disjointness of strata (cf. § 1), we know that the following series of inequalities

$$(4.11)' \quad y^{j}_{\lambda'_1}(P^{j-1}_{\lambda}) < y^{j}_{\lambda'_2}(P^{j-1}_{\lambda}) < \cdots < y^{j}_{\lambda'_k}(P^{j-1}_{\lambda})$$

holds for any $P^{j-1}_{\lambda} \in S^{j-1}_{\lambda}$.

From the above observations, we can define an order : the set

$$(4.11) \quad S^{j}_{\lambda'_1} < S^{j}_{\lambda'_2} \quad < S^{j}_{\lambda'_k},$$

in an accordance with the series (4.11)'.

In the above arguement we made an essential use of the existence of a '<u>natural</u>' order in the fibers $\pi^{-1}_{j-1,j}(P^{j-1}_{\lambda})$, $P^{j-1}_{\lambda} \in S^{j-1}_{\lambda}$. The existence of the 'natural' order (4.11) in $\mathfrak{S}^{j}(S^{j-1}_{\lambda})$ is led by a very simple observation, but makes later arguements on quantitative prperties of <u>real</u> analytic varieties smooth.

Let $\tilde{\mathfrak{S}}^j(S^{j-1}_\lambda)$ denote the set compsed of connected components of $\pi^{-1}_{j-1,j}(S^{j-1}_\lambda)_{|U^j} - |\mathfrak{S}^j(S^{j-1}_\lambda)|$, where $\pi^{-1}_{j-1,j}(S^{j-1}_\lambda)_{|U^j}$ denotes the intersection $\pi^{-1}_{j-1,j}(S^{j-1}_\lambda) \wedge U^j$. Then from the existence of the order (4.11) and the condition $(4.3)_1$ a commutative diagram of the following form ~~(4.12)~~ is valid for each $\tilde{S}^j_{\lambda'} \in \tilde{\mathfrak{S}}^j(S^{j-1}_\lambda)$.

(4.12)
$$\begin{array}{ccc} \tilde{S}^j_{\lambda'} & \xrightarrow{\tilde{\varphi}^j_\lambda} & S^{j-1}_\lambda \times I^0 \\ \downarrow & & \downarrow pr_1 \\ S^{j-1}_\lambda & \xrightarrow{id} & S^{j-1}_\lambda \end{array}$$

In the above $\tilde{\varphi}^j_\lambda$ is a C^∞ diffeomophism, and pr_1, id.. are the projection to the first factor S^{j-1}_λ and the identity map on S^{j-1}_λ respectively. Also ,from the order (4.11), we can define an order

$(4.11)'$ $\tilde{S}^j_{\lambda_1} < \tilde{S}^j_{\lambda_2} \cdots < \tilde{S}^j_{\lambda_\ell}$

in the set $\tilde{\mathfrak{S}}^j(S^{j-1}_\lambda)$ in a natural and obvious manner. Moreover, for each $\tilde{S}^j_{\lambda'} \in \tilde{\mathfrak{S}}^j(S^{j-1}_\lambda)$ at most two strata $S^j_{\lambda''} \in \mathfrak{S}^j(S^{j-1}_\lambda)$ are contained in $\bar{\tilde{S}}^j_{\lambda'} - \tilde{S}^j_{\lambda'}$. A stratum $S^j_{\lambda''} \in \mathfrak{S}^j(S^{j-1}_\lambda)$ contained in $\bar{\tilde{S}}^j_{\lambda'} - \tilde{S}^j_{\lambda'}$ will be called a horizontal component of the boundary of $\tilde{S}^j_{\lambda'}$.

4.4. The going down property. Comparisons of frontiers and distances.

and distances . As before, let (R),(F) be a normalized series of the form given in 4.2, and let (S)^j, (S)$_0^j$ have the same meanings as in §4.2. First we show the following

Proposition 4.1. (GOING DOWN THEOREM FOR NORMALIZED SERIES). Being (R),(F), (S)$_0$, (S), (S)$_0^j$, (S)^j, ...as above, any triplet $(S^{j-1}_{\mu}, S^{j-1}_{\lambda}, S^j_{\lambda'}) \in (S)_0^{j-1} \times (S)_0^{j-1} \times (S)^j$, $j = 2,\ldots,n$, satisfying the following diagram

(a)
$$\begin{array}{ccc} & & S^j_{\lambda'} \\ & & \downarrow \\ S^{j-1}_{\mu} & < & S^{j-1}_{\lambda} \end{array}$$

can be completed to a quatlet $(S^{j-1}_{\mu}, S^{j-1}_{\lambda}, S^j_{\mu'}, S^j_{\lambda'}) \in (S)_0^{j-1} \times (S)_0^{j-1} \times (S)^j \times (S)^j$ satisfying a diagram of the following form.

(b)
$$\begin{array}{ccc} S^j_{\mu'} & < & S^j_{\lambda'} \\ \downarrow & & \downarrow \\ S^{j-1}_{\mu} & < & S^{j-1}_{\lambda} \end{array}$$

In the aboves diagrams (a),(b) should be understood respectively as follows: (a)' $S^{j-1}_{\mu} < S^{j-1}_{\lambda}$. $\pi_{j-1,j}(S^j_{\lambda'}) = S^{j}_{\lambda}$, (b)' $S^{j-1}_{\mu} < S^{j-1}_{\lambda}$, $S^j_{\mu'} < S^j_{\lambda'}$, $\pi_{j-1,j}(S^j_{\lambda'}) = S^{j-1}_{\lambda}$, $\pi_{j-1,j}(S^j_{\mu'}) = S^{j-1}_{\mu}$.

Remark to Proposition 4.1. The stratum $S^j_{\mu'}$ satisfying the diagram (b) may not be unique.

In Proposition 4.1. we borrow the terminology of Going down theorem from the ring theory, (See M.Nagata [1] Local rings)

Note, however, that the context of Proposition 4.1. corresponds to that of the Going up theorem rather than to that of Going down theorem in the ring theory .

Proof of Proposition 4.1. Let $(S^{j-1}_{\mu}, S^{j-1}_{\lambda}, S^{j}_{\lambda'}) \in \mathfrak{S}^{j-1}_{a} \times \mathfrak{S}^{j-1}_{0} \times \mathfrak{S}^{j}$ be a triplet, $j = 2,\dots,n$, such that the diagram

$$\begin{array}{ccc} & & S^{j}_{\lambda'} \\ & & \downarrow \\ S^{j-1}_{\mu} & < & S^{j-1}_{\lambda} \end{array}$$

holds. For the stratum $S'^{j}_{\lambda'} \in \mathfrak{S}'^{j}$ satisfying $S'^{j}_{\lambda} \cap U^{j} = S^{j}_{\lambda}$, let $V(f_{n''j}(S'^{j}_{\lambda'}))$ denote the zero locus in U'^{j} of the monic polynomial $f_{n''j}(S'^{j}_{\lambda'}) \in \mathfrak{f}(S'^{j}_{\lambda'})$. Take a point $P^{j-1}_{\lambda} \in S^{j-1}_{\lambda}$. Then $\pi^{-1}_{j-1,j}(P^{j-1}_{\lambda}) \cap V(f_{n''j}(S'^{j}_{\lambda'}))$ consists of finite points $P^{j}_{\lambda'_s}$, $s = 1,\dots,s_0$. Take neighbourhoods $N(P^{j}_{\lambda'_s})$'s of $P^{j}_{\lambda_s}$'s in U^{j} and a neighbourhood $N(P^{j-1})$ suitably. Then we can assume that (i) $N(P^{j}_{\lambda'_s}) \cap N(P^{j}_{\lambda'_{s'}}) = \phi$, if $s \neq s'$ and that (ii) $\pi^{-1}_{j-1,j}(N(P^{j-1}_{\lambda})) \cap V(f_{n''j}(S'^{j}_{\lambda})) \subset \bigcup_s N(P^{j}_{\lambda_s})$. Moreover, we can assume that if $P_{\lambda'_s} \notin V^{j}$, then $N(P^{j}_{\lambda'_s}) \cap V'^{j} = \phi$. Take a sequence $\{Q^{j-1}_{t}\}_{t = 1,\dots,\dots}$ of points in S^{j-1}_{λ} such that $\lim_t Q^{j-1}_{t} = P^{j-1}_{\lambda}$. Let Q^{j}_{t} denote the point characterized by $\pi^{-1}_{j-1,j}(Q^{j-1}_{t}) \cap S^{j}_{\lambda'}$. Then we can choose a subsequence $\{Q^{j}_{i_t}\}$ $t = 1,\dots$ of $\{Q^{j}_{t}\}$ which converges to one of the points $P^{j}_{\lambda_s}$. Because $Q^{j}_{i_t}$'s are in $S^{j}_{\lambda'}$, the limit : $\lim_t Q^{j}_{i_t} = P^{j}_{\lambda_{\tilde{s}}}$ is in $\overline{S}^{j}_{\lambda'}$. Then clearly the stratum $S^{j}_{\mu'} \in \mathfrak{S}^{j}$ containing $P^{j}_{\lambda_{\tilde{s}}}$ satisfies the diagram

$$\begin{array}{ccc} S^{j}_{\mu'} & < & S^{j}_{\lambda'} \\ \downarrow & & \downarrow \\ S^{j-1}_{\mu} & < & S^{j-1}_{\lambda} \end{array} .$$

Q.E.D.

Now let $S^{j-1}_{\lambda}, S^{j-1}_{\mu}, S^{j}_{\lambda'} \in \mathfrak{S}^{j-1}_{a} \times \mathfrak{S}^{j-1}_{0} \times \mathfrak{S}^{j}$, $j = 2,\dots,n$, be a a triplet satisfying the diagram

$$\begin{array}{ccc} & & S^j_{\lambda'} \\ & & \downarrow \\ S^{j-1}_{\mu} & \prec & S^{j-1}_{\lambda} \end{array}.$$

Let $S^j(S^{j-1}_{\lambda})_{\bar{S}^j_{\lambda}}$ denote the subset of $S^j(S^{j-1}_{\lambda})$ composed of strata contained in $\bar{S}^j_{\lambda}$. Proposition 4.1. assures that $S^j(S^{j-1}_{\lambda}) \neq \phi$. By a simple observation we know that

(4.13)' $|S^j(S^{j-1}_{\lambda})_{\bar{S}^j_{\lambda'}}| = \pi_{j-1,j}(\mathrm{fron}(S^{j-1}_{\lambda})) \cap \bar{S}^j_{\lambda'}$.

Let $(S^{j-1}_{\lambda}, S^j_{\lambda'}) \in S^{j-1}_0 \times S^j$ such that $\pi_{j-1,j}(S^j_{\lambda'}) = S^{j-1}_{\lambda}$. Then from (4.13)' we know easily that

(4.13) $\qquad \mathrm{fron}(S^j_{\lambda'}) = \pi_{j-1,j}(\mathrm{fron}(S^{j-1}_{\lambda})) \cap \bar{S}^j_{\lambda'}$.

Let $S^{j-1}_{\lambda}, S^j_{\lambda'}$ be as above. Moreover, let S'^{j-1}_{λ} be the stratum in S'^{j-1}_0 such that $S'^{j-1}_{\lambda} \cap U^j = S^{j-1}_{\lambda}$. Define an analytic function $g(S'^{j-1}_{\lambda})$ in U'^{j-1} by

$$g(S'^{j-1}_{\lambda}) = \prod_{\mu} f'(S'^{j-1}_{\mu}), \text{ where } S'^{j-1}_{\mu} \subset \mathrm{fron}(S'^{j-1}_{\mu}).$$

Then, by the inequality of Lojasiewicz, we have

$$g(S'^{j-1}_{\lambda} : P^{j-1}_{\lambda}) \sim d(P^{j-1}_{\lambda}, V(g(S'^{j-1}_{\lambda})))$$

$$\tilde{d}(P^{j-1}_{\lambda}, V(g(S'^{j-1}_{\lambda}))) \sim V(f)'(S'^{j-1}_{\lambda} : \bar{S}'^{j}_{\lambda}) \text{ for any } P^{j-1}_{\lambda} \in S^{j-1}_{\lambda}.$$

In the above $V(g(S'^{j-1}_{\lambda}))$ denotes the zero locus of $g(S'^{j-1}_{\lambda})$ in U'^{j-1}. On the otherhand, from $(4.6)_1$ and $(4.2)_3$ we know that

$$\tilde{d}(P^{j-1}_{\lambda}, \mathrm{fron}(S^{j-1}_{\lambda})) \sim \tilde{d}(P^{j-1}_{\lambda}, V(g(S'^{j-1}_{\lambda})) \cap V(f)'(S'^{j-1}_{\lambda})))$$

for any $P^{j-1}_{\lambda} \in S^{j-1}_{\lambda}$.

Therefore we have

(4.14)' $\quad g(P^{j-1}_{\lambda}) \sim \tilde{d}(P^{j-1}_{\lambda}, \mathrm{fron}(S^{j-1}_{\lambda}))$ for any $P^{j-1}_{\lambda} \in S^{j-1}_{\lambda}$.

Now define an analytic function $\tilde{g}$ in U'^{j} be $\tilde{g}(P^{j}) = g(\pi_{j-1,j}(P^{j}))$, where $P^{j} \in U'^{j}$. Then, for the stratum $S'^{j}_{\lambda'}$ satisfying $S'^{j}_{\lambda'} \cap U^{j} = S^{j}_{\lambda'}$, we have

$$\tilde{g}(P^{j}_{\lambda'}) \sim d(P^{j}_{\lambda'},\ V(f'(S'^{j}_{\lambda'})) \cap \pi^{-1}_{j-1,j}(V(g(S'^{j-1}))))$$

for any $P^{j}_{\lambda'} \in S^{j}_{\lambda'}$.

But the right side of the above is equivalent to $d(P^{j}_{\lambda'}, \mathrm{fron}(S^{j}_{\lambda'}))$ for any $P^{j}_{\lambda'} \in S^{j}_{\lambda'}$. Thus we have

(4.14)" $\quad \tilde{g}(P^{j}_{\lambda'}) \sim \tilde{d}(P^{j}_{\lambda'}, \mathrm{fron}(S^{j}_{\lambda'}))$ for any $P^{j}_{\lambda'} \in S^{j}_{\lambda'}$.

From (4.14)' and (4.14)" we have the following <u>comparison of distances</u>.

(4.14) $\quad d(P^{j}_{\lambda'}, \mathrm{fron}(S^{j}_{\lambda'})) \sim d(P^{j-1}_{\lambda}, \mathrm{fron}(S^{j-1}_{\lambda}))$,where

$P^{j}_{\lambda} \in S^{j}_{\lambda'}$,and $P^{j-1}_{\lambda} = \pi_{j-1,j}(P^{j}_{\lambda'})$.

~~Let (R,F), S_0, S, etc., ... be as in 4.2. 4.4. Now we examine strata S^{j}'s in $S^{j}_{0} - S^{j}$, j = 2,...,n.~~

~~4.5. Strata S^{j}'s in $S^{j}_{0} - S^{j}$.~~

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§ 6. Normalized series attached to germs of varieties.

6.1. Formulations of problems . Let $R^n(x)$ be a euclid-ean space with coordinates $(x) = (x_1,..,x_n)$ and 0^n the orig-in of $R^n(x)$. By an admissible pair of germs of varieties at 0^n , we mean a pair (V,W) of germs of varieties at 0^n such that the following condition is valid.

$$(6.1) \qquad 1 \leqq \dim V \leqq n-1 , \quad \text{and} \quad W \subset V .$$

An admissible pair (V,W) of germs of varieties at 0^n will be called an admissible pair (V,W) of germs at 0^n, when there is no fear of confusions.

Let (V,W) be an admissible pair of germs at 0^n. By a normalized series of prestratified spaces attached to (V,W) we mean a normalized series $(R)(F) = ((R),((U),(V),(W)),((U)',(V)',(W)'))$, $((F),(F)'))$ in $R^n(x)$ satisfying the following conditions.

$(6.2)_1$ $\quad U^n \ni 0^n$.

$(6.2)_2$ $\quad$ For each stratum $S^n_\lambda \in (S)^n_0$ $({S'}^n_\lambda \in (S')^n)$, $S^n_\lambda \ni 0^n$ $({S'}^n_\lambda \ni 0^n)$.

$(6.2)_3$ $\quad$ Each component V_τ of V is an irreducible component of the germ of V^n at 0^n.

$(6.2)_4$ $\quad$ Each component V_τ of V and each component W_μ of W are expressed as the union of germs of strata S^n_λ's of $(S)^n$ at 0^n.

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$(6.2)_5$ For each $S^j_\lambda \in Ⓢ^j$, the set $ⓕ(S^j_\lambda) = \{f_t(S^j_\lambda)\}_t \in Ⓕ^j$ consists of Weierstrass polynomials $f_t(S^j_\lambda)$ in $y_{n^j_\lambda+t}$ with variables $(y_1,\ldots,y_{n^j_\lambda+t})$, $t = 1,\ldots,j - n^j_\lambda$ $(j = 2,\ldots,n)$. ?

In the above we write dim S^j_λ as n^j_λ.

In $(6.2)_{1,\ldots,5}$, $Ⓤ^n$, $Ⓥ^n$ and $Ⓢ^n_0$ denote the n-th component of Ⓤ, Ⓥ, $Ⓢ_0$ respectively. Moreover, $Ⓢ^n$ denotes the n-th component of the series Ⓢ of prestratifications of Ⓥ induced from $Ⓢ_0$, and $(y_1,\ldots,y_n)$ denotes the system of coordinates of $R^n(x)$ defining the series Ⓡ.

When there is no fear of confusions, we call a normalized series (Ⓡ,Ⓕ) of prestratified spaces attached to (V,W) simply a <u>normalized series (Ⓡ,Ⓕ) attached to (V,W)</u>.

Let (V,W) be an admissible pair of germs at 0^n. We say that a normalized series (Ⓡ,Ⓕ) = (Ⓡ, (Ⓤ,Ⓥ,Ⓢ), ...) in $R^n(x)$ is attached <u>properly</u> to (V,W) if, in addition to $(6.2)_{1\sim5}$, the following condition is valid.

$(6.2)'$ $V = V^n$, where V^n is the germ of the n-th component V^n of Ⓥ at 0^n.

Let (V,W) be as above, and let X be an irreducible germ of a variety at 0^n such that $X \subset V$. We say that a 不明確

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M = ($M_2, \ldots, M_n$) of positive monomials M_j's in one variable
(R,F) is of type M

Now our position is to formulate our basic problem in this section: As previously , let $R^n(x)$ be a Euclidean space with coordinates (x) and O^n the origin of $R^n(x)$. Then our problem is to prove the following

Theorem 6.1. For an admissible pair (V,W) of germs at O^n, there exists a normalized series (R,F) attached prop--erly to (V,W) . Moreover, we can assume that (R,F) satisfi--es the following conditions.

(I) The series (R,F) is of monomial type.

(II) For any component V of V , the series (R,F) satisfies the differentiablity condition. ?

We will prove Theorem 6.1. in a little later. Here we give a corollary to Theorem 6.1.

Corollary 6.1. Let (V,W) be as in Theorem 6.1. Then there exists a finite set $\{(R^s, F^s)\}_s$ of normalized series (R^s,F^s) attached to (V,W) such that the following are valid.

(I)' Each series (R^s,F^s), s = 1,.., is of monomial type.

(II)' Each (R^s,F^s), s = 1,..., satisfies the differnti--ablity condition for any irreducible component V_τ of V.

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~~$V = D(R^S, F^S)$, where $D(R^S, F^S)$ is the ger of $D(R^S, F^S)$~~.

(III)' $\bigcap_s D(R^s, F^s) \subset V$.

Remark to Corollary 6.1. Let $\tilde{D}_\mu$ be an irreducible component of $\bigcap_s D(R^s, F^s)$. Then the conditions (II)', (III)' imply the following fact.

(I)' There exists an irreducible component V_τ of V such that $D \subsetneq V_\tau$.

Proof of Corollary 6.1. By Theorem 6.1, we can choose a normalized series (R, F) attached properly to (V,W) such that (I), (II) hold. We write the irreducible decomposition of $D(R, F)$ as $\bigcup_\mu D_\mu$. Then, between any D_μ and any irreducible comp-onent V_λ of V , there is no relation of the form $V_\lambda \subset D_\mu$.
Next define germs V^1, W^1 by $V^1 = V \cup D(R, F)$ and $W^1 = W$.
By Theorem 6.1. there exist normalized series (R^1, F^1) of monomial type attached properly to (V^1, W^1) such that the relation $V^1_\tau \subset D(R^1, F^1)$ holds for any ireeducible component V^1_τ of V^1. (Here $D(R^1, F^1)$ is ,of course, the germ of $D(R^1, F^1)$ at 0^n). Therefore, for any irreducible component D_μ of $D(R, F)$ satisfying $D_\mu \not\subset V$, we have the following relation.

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$$D_\mu \cap D(R^1,F^1) \subsetneq D_\mu .$$

If the intersection $D^1 = D(R,F) \cap D(R^1,F^1)$, then the proof is completed. Otherwise, define germs V^2, W^2 by $V^2 = V \cup D^1$ and $W^2 = W^1$. We apply Theorem 6.1. to the pair (V^2, W^2) and do ?
similar arguements for (V^2, W^2) as above. Clearly repeating these procedures we attain the proof of Corollary 6.1.

Now we enter into discussions of the proof of Theorem 6.1. We want to do discussions of the proof of Theorem 6.1 inductively on the dimension of V. To do inductive arguements of the proof of Theorem 6.1, smooth, it is convenient to consider an another problem instead of considering directly Theorem 6.1. The another problem considered here is as follows: Let $R^n(x)$ denote a euclidean space with coordinates $(x) = (x_1, \ldots, x_n)$ and O^n the origin O^n of $R^n(x)$. By an <u>admissible series of germs at O^n</u>, we mean a collection $R = \{R, V, V', W\}$ of a series R of euclidean spaces and series V, V', W of germs of varieies. Here R, V, V', W should be of the form (6.5) and should satisfy conditions (6.6). ?

$$(6.5)_1 \quad R \; \{ R^{k+1}(y'_1, \ldots, y'_{k+1}) \;, \ldots, \; R^n(y'_1, \ldots, y'_{k+1}, \ldots, y'_n) \}$$

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where $(y') = (y'_1, \ldots, y'_n)$ is a system of coordinstes of $R^n(x)$.

$(6.5)_2$ $(V) = \{V^j\}$, $(V)' = \{V'^j\}$ and $(W) = \{W^j\}$, $j = k+1, \ldots, n$. Here V^j, V'^j and W^j are germs of varieites at $0^j \in R^j(y'_1, \ldots, y'_j)$, $j = k+1, \ldots, N$.

the origin

$(6.6)_1$ The integer k should satisfy the relation: $1 \leqq k \leqq n - 1$. Moreover, $1 \leqq \dim V^j \leqq k$, $\dim V'_j \leqq k$, and $V^j \cup V'^j \supset W^j$, $j = k + 1, \ldots, n$.

$(6.6)_2$ Germs V^j and V'^j have no common irreducible components.

$(6.6)_3$ Any irreducible component $V^{j'}_\lambda$ of $V^{j'}$ ($V'^{j'}_\tau$ of $V'^{j'}$) is integral over $\pi^{an}_{jj'}(y')(V^{j'})$ ($\pi^{an}_{jj'}(y')(V'^{j'})$), $k + 1 \leqq j \leqq j' \leqq n$.

$(6.6)_4$ $\pi^{an}_{jj'}(y')(V^{j'}) \subset V^j$, and for each irreducible component $V^{j'}_\lambda$ of $V^{j'}$, $\pi^{an}_{jj'}(y')(V^{j'})$ is an irreducible component of V^j , $k + 1 \leqq j \leqq j' \leqq n$.

$(6.6)_5$ $\pi^{an}_{jj'}(y')(V'^{j'}) \subset V^j \cup V'^j$, $\pi^{an}_{jj'}(y')(W'^{j'}) \subset W^j$, $k + 1 \leqq j \leqq j' \leqq n$.

Remarks to (6.5) and (6.6). (i) The condition $(6.6)_1$ implies that $V^j \neq \phi$ for $j = k+1, \ldots, n$. However, V'^j, W^j may be empty, $j = k + 1, \ldots, n$.

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(ii) Also note that we are not assuming the following: $\pi^{an}_{jj'}(y')(V'^{j'}) \subset V'^{j}$.

The integer k in (6.6),(6.6) will be called the <u>rank</u> of the admissible series R. Also by the <u>dimension</u> of R (dim R) we mean $\max_j \{V^j \cup V'^j\}$, $j = k+1, \ldots, n$.

Let R = {R, V, V', W} be an admissible series of rank k of germs of varieties at 0^n. We write {R, V, V', W} explicitly as follows: $R = \{R^{k+1}(y'_{k+1},\ldots,y'_n),\ldots,R^n(y'_1,\ldots,y'_n)\}$, $V = \{V^j\}$, $V' = \{V'^j\}$, $W = \{W^j\}$, $j = k+1,\ldots,n$. Then each pair $(V^j \cup V'^j, W^j)$ is admissible at $0^j \in R^j(y'_1,\ldots,y'_j)$, $j = k+1, \ldots, n$.
the origin

By a <u>normalized series (of prestratified spaces) attached to R</u>, we mean a normalized series $((R, U, V, S_0), (U', V', S'_0), (F, F'))$ in $R^n(x)$ such that the following conditions are valid.

$(6.7)_1$ The coordinates $(y) = (y_1,\ldots,y_n)$ defining the series R are of the following forms: $y_{k+2} = y'_{k+2},\ldots,y_n = y'_n$ and $(y_1,\ldots,y_{k+1}) = (y'_1,\ldots,y'_{k+1})A + a$, where $A \in GL(k+1, \mathbb{R})$ and $a \in R^{k+1}$.

$(6.7)_2$ The j-th normalized series (R^j, F^j) derived from (R,F) is attached to $(V^j \cup V'^j, W^j)$.

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Let $\{(Ⓡ,Ⓥ,Ⓥ',Ⓦ)\}$ be an admissible series of rank k of germs at O^n, where Ⓥ,Ⓥ',Ⓦ are explicitly of the forms: Ⓥ $=\{V^j\}$, Ⓥ' $=\{V'^j\}$ and Ⓦ $=\{W^j\}$, $j = k+1,..,n$. Let (Ⓡ,Ⓕ) be a normalized series attached to (Ⓡ,Ⓥ,Ⓥ',Ⓦ). Then the j-th normalized series $(Ⓡ^j,Ⓕ^j)$ derived from (Ⓡ,Ⓕ) is attached to $(Ⓥ^j,Ⓥ'^j,Ⓦ^j)$. We add the following definitions:

(i) We say that (Ⓡ,Ⓕ) is attached properly to (Ⓡ,Ⓥ,Ⓥ', Ⓦ) if $(Ⓡ^j,Ⓕ^j)$ is attached properly to $(V^j \cup V'^j, W^j)$, $j = k+$ $,\dots,n$.

(ii) We say that (Ⓡ,Ⓕ) satisfies the differentiablity condition for (Ⓡ,Ⓥ,Ⓥ',Ⓦ) if $(Ⓡ^j,Ⓕ^j)$ satisfies the differen-tiablity condition for each irreducible component of $V^j \cup V'^j$, $j = k+1$ $,\dots,n$.

(iii) We say that (Ⓡ,Ⓕ) is of monomial type if the series (Ⓡ,Ⓕ), regarded as attached to $(Ⓥ^n, Ⓥ'^n, Ⓦ)$,is of monomial type.

Remark 6.1. Let (Ⓡ,Ⓥ,Ⓥ',Ⓦ) be an admissible series of germs at O^n. We assume that the rank of (Ⓡ,Ⓥ,Ⓥ',Ⓦ) $= n - 1$. Then (Ⓥ,Ⓥ',Ⓦ) is of the form (Ⓥ,Ⓥ',Ⓦ), where V,V',W are germs of varieties at O^n. Moreover, assume that $V' = \phi$.

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Then the problem of finding a normalized series attached to the admissible series $(R, V, V', W) = (R^n(y'), V, V', W)$ is entire--ly same as that of finding a normalized series attached to the admissible pair (V,W) of germs at 0^n. Now our present problem is to prove the following Lemma for each $\overset{d}{k} = 1,2,\ldots$

Lemma $6.1._{k\,d}$ Let $R = \{R, V, V', W\}$ be an admissible series of ~~rank k~~ dimension d of germs of varieties at the origin $0^n \in R^n(x)$. Then there exists a normalized series (R, F) attached proper--ly to R. Moreover, for (R, F) we can assume the following.

(I'') (R, F) satisfies the differentiablity condition for R.

(II'') (R, F) is of monomial type.

It is clear from Remark 6.1. that the validity of Lemma $6.1_{k\,d}$ for each $\overset{d}{k} = 1,2,\ldots$ implies the validity of The--orem 6.1. Now for the proof of Lemma $6.1_{k\,d}$ for each $\overset{d}{k} = 1,2,$ .... it is sufficient to prove the following two facts.

(A) Lemma 6.1_1.

(B) That the validity of Lemma $6.1_{k'\,d'}$, $\overset{d'}{k'} = 1,\ldots,\overset{d}{k} - 1$, implies the validity of Lemma $6.1_{k\,d}$, where $\overset{d}{k} \geq 2$.

The proof of (A) does not contain difficulties. We pos--tpone the proof of (A) for a little while. We first prove (B).

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