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Special arithmetic groups

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Let V be a finite dimensional vector space defined over the rational number field $\mathbb{Q}$ and let B be a fixed $\mathbb{Q}$-basis of $V_{\mathbb{Q}}$. We take the functorial point of view that V_k is defined for any over-field k of $\mathbb{Q}$, and for a subring A of such a field k, and more generally for any commutative k-algebra A, let V_A be the A-span of B. Denote by $GL(V)$ the group of all linear endomorphisms of V and let G be an algebraic subgroup of $GL(V)$ defined over k (i.e., G is the zero locus in $GL(V)$ of certain polynomial functions on $\mathrm{End}(V)$ with coefficients in k). We call G a linear algebraic group defined over k, and if A is a subring of k or a commutative k-algebra, let

$$G_A = \{g \in G \mid g.V_A = V_A\},$$

the group of matrices in G having coefficients in A and $\det \in A^{\times}$. Let G be defined over $\mathbb{Q}$. A subgroup of $G_{\mathbb{Q}}$ commensurable with $G_{\mathbb{Z}}$ is called "arithmetic", and is called maximal arithmetic if it is not properly contained in a larger arithmetic group. The problem we consider is whether there is some natural class of maximal arithmetic groups in $G_{\mathbb{Q}}$.

Let p be a prime in $\mathbb{Q}$, $\mathbb{Q}_p$ = p-adic completion of $\mathbb{Q}$, $\mathbb{Z}_p$ = closure of $\mathbb{Z}$ in $\mathbb{Q}_p$ = unique maximal compact subring of $\mathbb{Q}_p$.

A subgroup Λ of $V_{\mathbb{Q}}$ commensurable with $V_{\mathbb{Z}}$ is called a lattice in V; for each p, let $\Lambda_p = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \subset V_{\mathbb{Q}_p}$. Then Λ_p is an open compact $\mathbb{Z}_p$-submodule of $V_{\mathbb{Q}_p}$. If Λ' is another lattice in V, then $\Lambda'_p = \Lambda_p$ $\forall'$ p. Conversely, given for each p an open compact $\mathbb{Z}_p$-submodule Λ''_p of $V_{\mathbb{Q}_p}$, i.e., a "local lattice", such that $\Lambda''_p = \Lambda_p$ $\forall'$ p, then there exists exactly one lattice Λ''' in V such that $\Lambda'''_p = \Lambda''_p$ for all p. If K_p is the stabilizer of Λ_p in $G_{\mathbb{Q}_p}$, then K_p is compact and open in $G_{\mathbb{Q}_p}$; conversely, any compact subgroup of $G_{\mathbb{Q}_p}$ stabilizes some local lattice in $V_{\mathbb{Q}_p}$. A subgroup P of G defined over k and containing a maximal, connected triangulizable subgroup B of G is called a k-parabolic subgroup of G. Let K_p be a maximal open compact subgroup of $G_{\mathbb{Q}_p}$. K_p is called "special" if for one (and hence every) minimal $\mathbb{Q}_p$-parabolic subgroup P of G we have

$$G_{\mathbb{Q}_p} = K_p \cdot P_{\mathbb{Q}_p} .$$

If Λ_p is a local lattice, it is called special if its stabilizer in $G_{\mathbb{Q}_p}$ is special, maximal compact. We assume G to be connected, simply connected, and defined over $\mathbb{Q}$. Let Γ be an arithmetic subgroup of $G_{\mathbb{Q}}$; then its closure Γ_p in $G_{\mathbb{Q}_p}$ is open and compact $\forall$ p and is known (Hijikata, et al.) to be special, maximal compact

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Notation: $\forall'$ means "for almost all", $\forall$ means "for all".

(SMC) $\forall' p$. If Γ_p is special, maximal compact $\forall p$, then Γ is called a special arithmetic group, it is easy to see, granted the statements of the preceding sentence, that special arithmetic groups (SAG) always exist (for a given G). If Λ is a lattice in V, it is called special if the stabilizer of Λ_p in $G_{\mathbb{Q}_p}$ is SMC $\forall p$, or, equivalently, if the stabilizer of Λ in $G_{\mathbb{Q}}$ is SAG. (Here we have freely used the theorem of strong approximation.)

We take as our basic problem the classification of <u>outer</u> isomorphism classes of SAG's in G, i.e., the determination of the orbits in the set of SAG's of $\mathrm{Aut}(G)_{\mathbb{Q}}$. In many, but not all, cases it turns out there are only finitely many such classes, in some instances only one. For an example of the latter type, let $G = SL_2$, $H = GL_2$, V = two dimensional vector space defined over $\mathbb{Q}$, and then identify H with GL(V). Direct calculation shows that for each p there are two conjugacy classes of special maximal compact subgroups in $SL_2(\mathbb{Q}_p)$ and that these are interchanged by $\mathrm{Ad}\,\alpha_p$, where $\alpha_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. Since $\mathbb{Q}$ has class number one, it follows that any two SAG's are conjugate with respect to $\mathrm{Ad}(H_{\mathbb{Q}})$, which may be viewed as a subgroup of $\mathrm{Aut}(G)_{\mathbb{Q}}$. If we replace $\mathbb{Q}$ by a number field k and p by a prime $\mathfrak{p}$ of k, $\alpha_{\mathfrak{p}}$ will be a <u>local</u> element, but not global if $\mathfrak{p}$ does not belong to the principal ideal class; in this case, one may show that the number of outer isomorphism classes with respect to $Ad(H_{\mathbb{Q}})$ is $[\mathcal{L} : \mathcal{L}^2] = 2^{c_2}$, where $\mathcal{L}$ is the class group of k and c_2 = number of 2^{power}-cyclic summands of $\mathcal{L}$.

More generally, let G be a connected, simply connected, semi-simple linear algebraic group defined over $\mathbb{Q}$. We <u>assume</u> $G \subset H \subset GL(V)$, where H is also connected, defined over $\mathbb{Q}$, and reductive, such that for a central torus T of H defined over $\mathbb{Q}$ we have $H = T.G$ ($T \cap G$ must then be finite) and such that $Ad(H_{\mathbb{Q}_p}) = Ad(G)_{\mathbb{Q}_p}$. Then we investigate the orbits of $Ad(H_{\mathbb{Q}})$ among the SAG's. For simplicity, we assume G to be almost absolutely simple; the general case may be treated in the same way using the ground-field reduction functor (A. Weil: Adeles and Algebraic Groups, IAS notes, 1961).

Let Γ be a special arithmetic group in $G_{\mathbb{Q}}$ stabilizing a special lattice Λ in V. For each p, let Γ_p be the closure of Γ in $G_{\mathbb{Q}_p}$, which is the same as $Stab_{G_{\mathbb{Q}_p}}(\Lambda_p)$, and let $\Gamma''_p = Stab_{H_{\mathbb{Q}_p}}(\Lambda_p)$. Let G_A, H_A be the adele groups of G and H (i.e., the infinite direct products $\prod' G_{\mathbb{Q}_p}$, $\prod' H_{\mathbb{Q}_p}$, restricted with respect to the families $\{\Gamma_p\}$, $\{\Gamma''_p\}$ of open compact groups of the local factors). Let

$$U'' = H_\infty \times \prod_{p<\infty} \Gamma''_p ,$$

$$U = G_\infty \times \prod_{p<\infty} \Gamma_p ;$$

then U'' (resp. U) is open in H_A (resp. in G_A) and U'' normalizes

U. We have

$$H_A = \bigcup_{\alpha \in \mathcal{E}} H_{\mathbb{Q}} \alpha U'', \quad \mathcal{E} \text{ a finite set } \text{ (A. Borel).}$$

If Γ' is another SAG, we say Γ and Γ' are in the same genus if for each p there exists $\alpha_p \in H_{\mathbb{Q}_p}$ such that $\mathrm{Ad}\,\alpha_p(\Gamma_p) = \Gamma_p'$; since $\Gamma_p = \Gamma_p'$ $\forall' p$ anyway, we may choose α_p for each prime p such that $(\alpha_p)_p \in H_A$. Hence, there are not more than card($\mathcal{E}$) ($< \infty$) classes of SAG's in the genus of Γ. Moreover, the number of genera will be finite if there exists a finite set S of primes such that for $p \notin S$, and for any two SMC's K_p, K_p' of $G_{\mathbb{Q}_p}$, there exists $\alpha_p \in H_{\mathbb{Q}_p}$ such that $\mathrm{Ad}\,\alpha_p(K_p) = K_p'$. (It should be noted, by the way, that the number of classes of SAG's in a genus may actually be smaller than the number of classes of lattices in a genus of lattices; this is illustrated by our example with $G = SL_2(k)$, $H = GL_2(k)$, when the class number of k is even but not a power of 2.)

Calculations indicate that for many such G we may find an H for which the number of genera is finite. An exception appears to be the case of the unitary group of a Hermitian form in an odd number of variable over a quadratic extension of a number field k. In this case, G is of type ${}^2A_{2n}$ and for <u>infinitely many</u> primes* $\mathfrak{p}$, the extended Dynkin diagram of its restricted, $k_{\mathfrak{p}}$-relative root system

* I.e., for those primes $\mathfrak{p}$ of k which remain prime ideals in the quadratic extension.

is unsymmetrical with respect to its two special points. (Cf. F. Bruhat and J. Tits: Groupes réductifs sur un corps local, Publ'ns I.H.E.S. v.41. Esp. pp.225-226 and p.30 in regard to a laddering (= échelonnage) of type $C\text{-}BC_n^{IV}$).