数理解析研究所講究録
第 240 巻 1975 年 57-63

A remark on deformations of Calabi-Eckmann manifolds

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In this note we give explicit constructions of effectively parametrized and complete families of deformations of Calabi-Eckmann manifolds. In [1], Calabi and Eckmann constructed a class of complex manifolds homeomorphic to $S^{2p+1}\times S^{2q+1}$ $(p,q\geq 1)$. These are holomorphic fibre bundles over $\mathbb{P}^p\times\mathbb{P}^q$ with elliptic curves as fibres. Moreover these are so-called non-Kähler C-manifolds ,i.e., simply connected homogeneous spaces. P. Griffiths proved in [2] that, for any compact complex non-Kähler homogeneous space M, there exists an effectively parametrized family of deformations corresponding to each abelian Lie subalgebra in $H^1(M,\Theta_M)$, where Θ_M denotes the sheaf of germes of holomorphic vector fields on M. His construction is, however, not explicit one, using similar method as in the proof of the existence theorem of Kuranishi families. Our construction is quite elementary, but in the same manner we may construct effectively parametrized and complete families of deformations of any non-Kähler C-manifolds, using Tits' constructions of homogeneous spaces [6].

Remark. For Kähler homogeneous manifolds, it is quite easy to construct complete and effectively parametrized families of their deformations.

Let $W = (\mathbb{C}^{p+1} - (0)) \times (\mathbb{C}^{q+1} - (0))$. For each pair $(A, B) \in GL(p+1, \mathbb{C}) \times GL(q+1, \mathbb{C})$, and for any complex number t, we define $g_t(A, B) \in \mathrm{Aut}(W)$ as follows.

$$\text{For } (z, w) \in W, \quad G\, g_t(A, B)(z, w) = (e^{tA}z, e^{tB}w).$$

Lemma. Let λ be a complex number with $\mathrm{Im}\,\lambda \neq 0$. Then for each pair (A, B) sufficiently near to $(I_{p+1}, \lambda I_{q+1})$[1] the group $G(A, B) = \{g_t(A, B)\}_{t \in \mathbb{C}}$ operates on W locally properly, and the quotient space $W/G(A, B)$ is a compact complex manifold diffeomorphic to $S^{2p+1} \times S^{2q+1}$.

Proof. A simple calculation.

Note that for $(A, B) = (I_{p+1}, \lambda I_{q+1})$, $W/G(A, B)$ is nothing but a usual Calabi-Eckmann manifold. Every Calabi-Eckmann manifold is obtained by the above construction.

Let $M = W/G(I_{p+1}, \lambda I_{q+1})$ be a Calabi-Eckmann manifold. Let U be a sufficiently small neighbourhood of $(I_{p+1}, \lambda I_{q+1})$ in $SL(p+1, \mathbb{C}) \times GL(q+1, \mathbb{C})$, such that, for every $(A, B) \in U$, $G(A, B)$ operates locally properly on W. We define a 1-parameter Lie group $G = \{g_t\}_{t \in \mathbb{C}}$ of automorphisms of $W \times U$ as follows.

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1) For $n \in \mathbb{Z}$, I_n denotes the unit matrix in $M_n(\mathbb{C})$.

For $t \in \mathbf{C}$ and $(z, w, (A, B)) \in W \times U$,

$$g_t(z, w, (A, B)) = (g_t(A, B)(z, w), (A, B)).$$

Then the quotient space $\mathcal{W} = W \times U/G$ is a complex manifold and the natural projection ϖ to the 2^{nd} component makes $\mathcal{W}$ a complex fibre space over U with compact fibres, which is differentiably a fibre bundle. Moreover the fibre on $(I_{p+1}, \lambda I_{q+1})$ is isomorphic to M.

Theorem. The above constructed family $\mathcal{W}$ of deformations of M is effectively parametrized and complete at $(I_{p+1}, \lambda I_{q+1})$.

Before proving this theorem, we must calculate $H^{\nu}(M, \Theta)$

Proposition 1. ([4]). Let M be as above. Then

$$\dim H^1(M, \Theta) = \dim H^0(M, \Theta)$$
$$= (p+1)^2 + (q+1)^2 - 1,$$
$$\dim H^{\nu}(M, \Theta) = 0 \quad \text{for } \nu \geqq 2.$$

Moreover there exists a natural isomorphism

$$H^0(M, \Theta) \otimes H^1(M, \mathcal{O}) \xrightarrow{\sim} H^1(M, \Theta).$$

Note that, in particular, M has no obstruction.

Next, we need a result on the deformations of homogeneous Hopf manifolds. Let $V = \mathbf{C}^{p+1} - (0)$, α a complex number with $0 < |\alpha| < 1$, and $N = V/\{\alpha^n I_{p+1}\}_{n \in \mathbb{Z}}$ a Hopf manifold. Let U be a sufficiently small neighbourhood of 0 in $M_{p+1}(\mathbf{C})$. We define an automorphism g of V U as follows,

For $(z, u) \in V \times U$, $g(z, u) = ((\alpha I_{p+1} + u)z, u)$.

Then $G = \{g^n\}_{n \in \mathbb{Z}}$ operates on $V \times U$ properly discontinuously and $\mathcal{V} + V \times U/G$ is a family of deformations of N over U.

Proposition 2 ([5]). $\mathcal{V}$ is effectively parametrized and complete at $0 \in U$.

Proof of Theorem. As mentioned above, M is a complex analytic fibre bundle over $\mathbb{P}^p \times \mathbb{P}^q$. Let $\pi : M \longrightarrow \mathbb{P}^p \times \mathbb{P}^q$ be the bundle projection, p_i $(i = 1,2)$ the projection from $\mathbb{P}^p \times \mathbb{P}^q$ to the i-th component, and $\pi_i = p_i \circ \pi$. Consider the fibre bundle $\pi_1 : M \longrightarrow \mathbb{P}^p$. The fibre is a homogeneous Hopf manifold. Since π_1 is smooth, there exists the following exact sequence of sheaves

$$0 \longrightarrow \Theta_{M/\mathbb{P}^p} \longrightarrow \Theta_M \longrightarrow \pi_1^* \Theta_{\mathbb{P}^p} \longrightarrow 0.$$

From this we get the exact sequence

$$\cdots \to H^0(M, \Theta_M) \to H^0(M, \pi_1^* \Theta_{\mathbb{P}^p}) \to H^1(M, \Theta_{M/\mathbb{P}^p}) \to H^1(M, \Theta_M) \to H^1(M, \pi_1^* \Theta_{\mathbb{P}^p}) \to H^2(M, \Theta_{M/\mathbb{P}^p})$$

Since M is homogeneous, the first map is surjective. Also we can prove $H^2(M, \Theta_{M/\mathbb{P}^p}) = 0$. Hence we get the exact sequence

$$0 \longrightarrow H^1(M, \Theta_{M/\mathbb{P}^p}) \xrightarrow{\ \psi_1\ } H^1(M, \Theta_M) \longrightarrow H^1(M, \pi_1^* \Theta_{\mathbb{P}^p}) \longrightarrow 0.$$

Let $\mathcal{W}_1$ be a submanifold of $\mathcal{W}$, defined by $A = I_{p+1}$, and $U_1 = \{(A, B) \in U \mid A = I_{p+1}\}$. Then $\mathcal{W}_1 \xrightarrow{\varpi|\mathcal{W}_1} U_1$ gives a subfamily of $\mathcal{W} \xrightarrow{\varpi} U$, where the fibre structure by π_1 is preserved. Moreover for every $u \in U_1$, $\varpi^{-1}(u)$ is a complex analytic fibre bundle over $\mathbb{P}^p$ with Hopf fibre.

Hence we get the following diagramme,

$$\begin{array}{ccc} \mathcal{W}_1 & \xrightarrow{\varpi|_{\mathcal{W}_1}} & \\ \downarrow & & \\ \mathbb{P}^p \times U_1 & \xrightarrow[\varphi_1]{} & U_1 \end{array}$$

where φ_1 is the projection to the 2^{nd} component.

We denote by $T_0(U)$ and $T_0(U_1)$ the tangent spaces to U and U_1 at $(I_{p+1}, \lambda I_{q+1})$ respectively, and identify $T_0(U_1)$ with a subspace of $T_0(U)$. Let ρ be the Kodaira-Spencer map corresponding to $\mathcal{W}$:

$$\rho \ : \ T_0(U) \longrightarrow H^1(M, \Theta_M).$$

Then, since the fibre structure by π_1 is preserved over U_1, ρ maps $T_0(U_1)$ into $\psi_1(H^1(M, \Theta_{M/\mathbb{P}^p}))$. We shall show that this map is bijective, or equivalently, the map $\tilde{\rho}$ from $T_0(U_1)$ to $H^1(M, \Theta_{M/\mathbb{P}^p})$, which factorizes ,
is surjective.

Lemma. There exists a natural isomorphism, induced by the restriction,

$$H^1(M, \Theta_{M/\mathbb{P}^p}) \xrightarrow{\sim} H^0(F, \Theta_F),$$

where F is a fibre of π_1.

In fact, by Leray's spectral sequence,

$$H^1(M, \Theta_{M/\mathbb{P}^p}) \xrightarrow{\sim} H^0(\, P^p, \ R^1\pi_{1*}\Theta_{M/\mathbb{P}^p}),$$

and $R^1\pi_{1*}\Theta_{M/\mathbb{P}^p} \simeq \mathcal{O}(H^0(F, \Theta_F))$.

Let φ_2 be the projection from $P^p \times U_1$ to th 1^{st} component, and γ a curve on $P^p \times U_1$, defined by $\varphi_2(\gamma) = z_0 =$ const. Then γ is tangent to $\varphi_2^{-1}(z_0)$ at $\gamma \cap \varphi_1^{-1}(I_{p+1}, \lambda I_{q+1})$, and $\varpi^{-1}(\gamma)$ is a family of Hopf manifolds on $\gamma \rightarrow U_1$, isomorphic to that of Proposition 2 if U_1 is sufficiently small. Therefore by Proposition 2 the map $\rho|_{T_c(U_1)}$ is surjective.

Similarly we can prove the corresponding statement for π_2. Since it is obvious that $\dim(\psi_1(H^1(M, \Theta_{H/P})) \cap \psi_2(H^1(M, \Theta_{H/P})))$ $= 1$, we see that ρ is surjective. Hence $\mathcal{W}$ is effectively parametrized and complete at $(I_{p+1}, \lambda I_{q+1})$, because $\dim T_0(U) = \dim H^1(M, \Theta_H)$.

Corollary. For $u = (A, B) \in U$ sufficiently near to $(I_{p+1}, \lambda I_{q+1})$,

$$\dim H^1(\varpi^{-1}(u), \Theta) = \dim H^0(\varpi^{-1}(u), \Theta)$$

$$= \dim \left\{(P, Q) \in M_{p+1}(\mathbb{C}) \times M_{q+1}(\mathbb{C}) \;\middle|\; PA = AP,\ QB = BQ\right\} - 1,$$

$$\dim H^{\nu}(\varpi^{-1}(u), \Theta) = 0 \quad \text{for } \nu \geq 2.$$

References.

[1] E. Calabi and B. Eckmann, A class of compact, complex manifolds which are not algebraic, Ann. of Math., Vol. 58 (1953), 494 - 500.

[2] P. Griffiths, Some geometric and analytic properties of homogeneous complex manifolds, Part II, Acta Math., Vol. 110 (1963), 157 - 208.

[3] H. Holmann, Quotientenräume komplexer Mannigfaltigkeiten nach komplexen Lieschen Automorphismengruppen,

Math. Ann., 139 (1960), 383 - 402.

[4] M. Ise, Some Properties of Complex Analytic Vector Bundles over Compact Complex Homogeneous Spaces, Osaka Math, J., Vol. 12 (1960), 217 - 252.

[5] K. Kodaira and D.C. Spencer, On deformations of complex analytic structures II, Ann. of Math., Vol. 67 (1958), 403 - 466.

[6] J. Tits, Espaces homogènes complexes compacts, Comm. Math. Helv., Vol. 37 (1962 - 1963), 111 - 120.