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On the pluricanonical systems of algebraic manifolds.

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Any algebraic manifold is assumed to be connected, complete, non-singular and defined over the complex number field $\mathbb{C}$. Let K_M be the canonical line bundle of an algebraic manifold M. If $P_m(M) = \dim_{\mathbb{C}} H^0(M, \underline{O}(mK_M))$ is positive for a positive integer m, we can define a rational mapping

$$\begin{array}{rccc} \Phi_{mK} : & M & \longrightarrow & \mathbb{P}^N \\ & \cup & & \cup \\ & z & \longmapsto & (\varphi_0(z):\varphi_1(z):\cdots:\varphi_N(z)), \end{array}$$

where $\{\varphi_0, \varphi_1, \ldots, \varphi_N\}$ is a basis of the vector space $H^0(M, \underline{O}(mK_M))$. The rational mapping Φ_{mK} is called the m-th canonical mapping. We set $\mathbb{N}(M) = \{ m > 0 \mid P_m(M) > 0 \}$. The Kodaira dimension $\kappa(M)$ of the algebraic manifold M is defined by

$$\kappa(M) = \begin{cases} \max\limits_{m \in \mathbb{N}(M)} \dim \Phi_{mK}(M) & \text{if } \mathbb{N}(M) \neq \emptyset, \\ 0 & \text{if } \mathbb{N}(M) = \emptyset . \end{cases}$$

It is easy to show that, if two algebraic manifolds M_1 and M_2

are birationally equivalent, then $P_m(M_1) = P_m(M_2)$. Hence for an irreducible complete singular algebraic variety V, we define the Kodaira dimension $\kappa(V)$ of V by

$$\kappa(V) = \kappa(V^*)$$

where V^* is a non-singular model of the variety V. For the properties of Kodaira dimensions we refer the reader to Iitaka [2] and Ueno [6], [7].

Let S be an algebraic surface, that is, an algebraic manifold of dimension two. A complete curve C in S is called an exceptional curve of the first kind if C is a non-singular rational curve with $C^2 = -1$. If S contains an exceptional curve C of the first kind, there exist a non-singular surface $\hat{S}$ and a birational morphism $\varphi : S \longrightarrow \hat{S}$ such that $\varphi(C)$ is a point $\hat{p}$ and that φ induces an isomorphism between S - C and $\hat{S} - \hat{p}$. The following theorem is a corollary to the classification theory of algebraic surfaces.

Theorem. _Let_ S _be an algebraic surface free from exceptional curves of the first kind. Suppose that_ $\kappa(S) \geq 0$. _Then there exist a positive integers_ d _and_ m_0 _such that the complete linear system_ $|mdK_S|$ _is free from base points and fixed components if_ $m \geq m_0$.

If $\kappa(S) = 2$, then we can show that $d = 1$ and $m_0 = 4$. The proof can be found in Kodaira [4] and Bombieri [1]. If $\kappa(S) = 0$, then the number d can be taken as a divisor of 12 and

$m_0 = 1$. The proof can be found in Šafarevič et al [5] Chap. VIII. If $\kappa(S) = 1$, then the number d can be taken as a divisor of 86. This fact can be deduced from the canonical bundle formula for elliptic surfaces due to Kodaira [3].

It had not been known whether the above theorem holds for algebraic manifolds of dimension $n \geqq 3$. The main purpose of the present paper is to show that the above theorem does not necessarily hold for an algebraic manifold of dimension $n \geqq 3$. Namely, we shall prove the following :

Main Theorem. For a pair of positive integers ℓ, n with $0 \leqq \ell \leqq n$, $3 \leqq n$, there exists an algebraic manifold M of dimension n which satisfies the following conditions:

① $\kappa(M) = \ell$,

② For any birationally equivalent non-singular manifold M^* of M, if $|mK_{M^*}| \neq \emptyset$, then $|mK_{M^*}|$ has fixed components.

To prove the theorem we shall construct algebraic manifolds which satisfy the above conditions ①, ② using the canonical resolutions of cyclic quotient singularities. For simplicity, in this paper, we shall only consider the quotient singularity by a cyclic group of order 2. It is not dificult to generalize our construction to the case of arbitrary quotient singularities.

§1. Let M be an algebraic manifold and let $\underline{S}^k(\Omega^\ell_M)$ be the k-th symmetric tensor product of the sheaf Ω^ℓ_M of germs of holomorphic ℓ-forms on M. The following lemma is well-known. A proof is found in Ueno [6].

Lemma 1.1 <u>Let</u> M <u>and</u> M^* <u>be algebraic manifolds. Suppose that there exists a surjective rational mapping</u> $f : M \longrightarrow M^*$. <u>Then for any positive integer</u> k , f <u>induces an injective linear mapping</u>

$$f^* : H^0(M^*, \underline{S}^k(\Omega^{\ell}_{M^*})) \longrightarrow H^0(M, \underline{S}^k(\Omega^{\ell}_M)).$$

<u>Moreover if</u> f <u>is birational</u>, f^* <u>is an isomorphism</u>.

Now we shall consider resolutions of quotient singularities. Let U be an open set in $\mathbb{C}^n$ defined by inequalities :

$$|z_i| < (\varepsilon)^{1/2}, \quad i = 1, 2, \ldots, n.$$

We let G be a group of order 2 of analytic automorphisms of U generated by the automorphism

$$g : (z_1, z_2, \ldots, z_n) \longmapsto (-z_1, -z_2, \ldots, -z_n).$$

The quotient space $\hat{U} = U/G$ has a singular point p which corresponds to the origin of $\mathbb{C}^n$. A resolution of the singularity of $\hat{U}$ can be given as follows. Let W_i, i = 1, 2, ... , n be open set of $\mathbb{C}^n$ defined by the inequalities :

$$|(w_i^k)^2 w_i^i| < \varepsilon, \quad k \neq i, \quad |w_i^i| < \varepsilon.$$

We shall construct a complex manifold $W = \bigcup_{i=1}^{n} W_i$ by identifying W_{i-1} and W_i through the following relations :

$$\begin{cases} w_i^k = w_{i-1}^k / w_{i-1}^i , & k \neq i-1, i, \\ w_i^{i-1} = 1/w_{i-1}^i \end{cases}$$

$$\left\{ w_i^i = (w_{i-1}^i)^2 w_{i-1}^{i-1} \right. .$$

Let us consider a meromorphic mapping

$$(1.2) \qquad \begin{array}{ccc} T_i : U & \longrightarrow & W_i \\ \cup & & \cup \\ (z_1, z_2, \ldots, z_n) & \longmapsto & (\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, (z_i)^2, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}). \end{array}$$

The meromrophic mappings T_i, $i = 1, \ldots, n$ induce a meromorphic mapping $T : \hat{U} \longrightarrow W$. Let E be a submanifold of W defined by the equations :

$$w_i^i = 0 \quad \text{in} \quad W_i, \quad i = 1, 2, \ldots, n .$$

E is analytically isomorphic to an $(n-1)$-dimensional complex projective space $\mathbb{P}^{n-1}$. The meromorphic mapping T incudes an isomorphism between $\hat{U} - p$ and $W - E$. Hence we infer that W is a non-singular model of the quotient space $\hat{U} = U/G$. The procedure of resolving the singularity is called the <u>canonical resolution</u>.

Let us consider the G-invariant subspace $H^0(U, \underline{S}^k(\Omega_U^\ell))^G$ of $H^0(U, \underline{S}^k(\Omega_U^\ell))$. Any element φ of $H^0(U, \underline{S}^k(\Omega_U^\ell))^G$ gives an element $\hat{\varphi}'$ of $H^0(\hat{U}-p, \underline{S}^k(\Omega_{\hat{U}-p}^\ell))^G$.
By 1.2 we can easily show that $\hat{\varphi}'$ can be uniquely extended to a meromorphic section $\hat{\varphi}$ of the locally free sheaf $\underline{S}^k(\Omega_W^\ell)$.
By explicit calculations we can prove the following:

Lemma 1.3. ① <u>If</u> $n \geqq 2$, <u>there is a canonical isomorphism</u>

$$\begin{array}{ccc} H^0(U, \underline{O}(mK_U))^G & \xrightarrow{\sim} & H^0(W, \underline{O}(mK_W)) \\ \cup & & \cup \\ \varphi & \longmapsto & \hat{\varphi} . \end{array}$$

Moreover, if $n \geqq 3$, *any element* φ *of* $H^0(U, \underline{O}(mK_U))^G$ *has a zero of order at least* $[\frac{m}{2}]$ *on* E *where* $[\ ]$ *is the Gauss symbol*.

② *The form* $(dz_1)^2 \in H^0(U, \underline{S}^2(\Omega_U^1))^G$ *induces a meromorphic section* ψ *of the sheaf* $\underline{S}^2(\Omega_W^1)$ *which has a pole of order* 1 *on* E.

Remark 1.4. If $n = 2$, $(dz_1 \wedge dz_2)^m$ is an element of $H^0(U, \underline{O}(mK_U))^G$ and induces a nowhere vanishing element of $H^0(W, \underline{O}(mK_W))^G$. This is one of the main differences between dimention two and dimension $n \geqq 3$.

§ 2. Main Theorem is a corollary of the following theorem.

Theorem 2.1. *Let* V *be an algebraic manifold of dimension* $n \geqq 3$. *Suppose that* V *has an analytic involution* g. *Suppose, moreover,*

① *the involution* g *has at least one fixed point and any fixed manifold of* g *is an isolated point* ;

② *there exists a holomorphic 1-form* ω *on* V *such that* ω *does not vanish at a fixed point* p_1 *of the involution* g *and that* $g^*\omega = -\omega$.

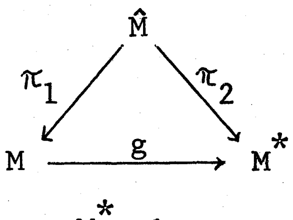
Let M *be any non-singular model of the quotient variety* V/G *where* G *is a cyclic group generated by* g. *Then, if* $|mK_M| \neq \phi$, $|mK_M|$ *has a fixed component*.

Proof. Let $p_1, p_2, \ldots, p_k$ be fixed points of the involution g. The quotient space V/G has singular points $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_\ell$ which correspond to the fixed points. Each singular point has a neighbourhood which is analytically isomorphic to $\hat{U}$ in § 1. Let M be a non-singular model of V/G obtained by the canonical resolution of its singularities. First we shall show that if $|mK_M| \neq \emptyset$, then $|mK_M|$ has fixed components. Let $E_1, \ldots, E_k$ be subvarieties of M appearing in the canonical resolution. From Lemma 1.2, ① we infer that there is an isomorphism

$$H^0(V, \underline{O}(mK_V))^G \xrightarrow{\sim} H^0(M, \underline{O}(mK_M))$$

and any element of $H^0(M, \underline{O}(mK_M))$ has zero of order at least $[\frac{m}{2}]$ on E_i. Hence the divisor $[\frac{m}{2}](E_1 + \cdots + E_k)$ is a fixed component of $|mK_M|$.

Next let us consider a birationally equivalent non-singular model M^* of M. Let $g : M \longrightarrow M^*$ be a birational morphism. By elimination of the points of indeterminacy of a rational mapping due to Hironaka, there exist an algebraic manifold $\hat{M}$ and a



morphism $\pi_1 : \hat{M} \longrightarrow M$ obtained by a finite succession of monoidal transformations with non-singular centers such that $\pi_2 = g \circ \pi_1 : \hat{M} \longrightarrow M^*$ is a morphism. Let $\mathcal{E}$ be the exceptional divisors appearing in the monoidal transformations. Then for any element $\varphi \longrightarrow H^0(M, \underline{O}(mK_M))$, $\pi_1^*(\varphi)$ has zeros on $\mathcal{E}$. Hence if $|mK_{\hat{M}}|$

$\neq \emptyset$, $|mK_{\hat{M}}|$ has a fixed component. We let $\hat{E}_i$, $i = 1,\dots,k$ be the strict transform of E_i to $\hat{M}$.

First we show that there exist at least one $\hat{E}_i$ or an irreducible component $\mathcal{E}_1$ of $\mathcal{E}$ such that $\pi_2(\hat{E}_i)$ or $\pi_2(\mathcal{E}_1)$ is a divisor on M^*. Assume the contrary. Then $\pi_2(E_i)$ and $\pi_2(\mathcal{E})$ are of codimension at least two in M^*. Let us consider the holomorphic 1-form ω on M. Since M is algebraic, ω is a closed form. Hence we can choose a coordinate neighbourhood U of the fixed point p_1 in M with local coordinates $z_1, z_2, \dots, z_n$ with center p_1 such that ω has a form dz_1 and that the involution is expressed in the form

$$(z_1, z_2, \dots, z_n) \longrightarrow (-z_1, -z_2, \dots, -z_n).$$

The form $(\omega)^2 \in H^0(V, \underline{S}^2(\Omega^1_V))^G$ induces a meromorphic section ψ of $\underline{S}^2(\Omega^1_M)$ which is holomorphic on $M - \bigcup_{i=1}^{k} E_i$. Therefore the pull-back $\pi_1^*(\psi)$ is holomorphic on $\hat{M} - \bigcup_{i=1}^{k} \pi_1^{-1}(E_i)$. On the other hand if S is the smallest analytic subset of $\hat{M}$ such that π_2 is an isomorphism on $\hat{M} - S$, then $\pi_2(S)$ is of codimension at least two by Zariski's Main Theorem. Hence $\pi_1^*(\psi)$ induces a holomorphic form on $M^* - \{\pi_2(\bigcup_{i=1}^{k} \pi_1^{-1}(E_i) \cup S)\}$.
By our assumption $\pi_2(\pi_1^{-1}(E_i))$ is of codimension at least two. Since $\pi_2^*(g^*(\psi)) = \pi^*(\psi)$, $g^*(\psi)$ is holomorphic on M^*. Then by Lemma 1.1, ψ must be holomorphic on M. But by Lemma 1.3,②, ψ has a pole on E_1. This is a contradiction. Hence

$\pi_2(E_i)$ or $\pi_2(\mathcal{E}_1)$ is a divisor. For simplicity we assume that $\pi_2(E)$ is a divisor. By Zariski's Main Theorem, there exists a nowhere dense algebraic subset S such that $S \neq E_1$ and that at any point of $S-E_1$, π_2 is an isomorphism. Hence for any element $\varphi \in H^0(M, \underline{O}(mK_M))$ $g^*(\varphi)$ has a zero on $\pi_2(E_1)$. Since $\pi_2^*(g^*(\varphi)) = \pi_1^*(\varphi)$. By Lemma 1.1, if $|mK_{M^*}| \neq \emptyset$, then $|mK_{M^*}|$ has a fixed component $\pi_2(E_1)$. Q.E.D.

Remark 2.2. ① The above theorem holds for a compact complex manifold V if we assume, furthermore, that a holomorphic 1-form ω in the above condition ② is d-closed.

② In the above theorem, if we assume that any fixed manifold of the involution g is of codimension at least three and that there exists a holomorphic 1-form ω on V such that ω has no zeros on a fixed manifold F, and that $\omega|F = 0$ and $g^*(\omega) = -\omega$, then the same conclusion holds.

§3. Now we prove Main Theorem. For simplicity we shall prove the theorem when $n = 3$.

(3.1) Let C be a non-singular complete curve of genus g. Suppose that C has an involution $\imath$ which has at least one fixed point. We set $\hat{C} = C/\langle\imath\rangle$. Assume that the genus of $\hat{C}$ is strictly greater than one. Let S be a surface in $\mathbb{P}^3$ defined by the homogeneous equation

$$z_0^{10} + z_1^{10} + z_2^{10} + z_3^{10} = 0.$$

S has an involution h defined by

$$h : (z_0:z_1:z_2:z_3) \longrightarrow (z_0:-z_1:-z_2:z_3).$$

The involution h has twenty fixed points on S. Let $\tilde{S}$ be a non-singular model of the quotient variety $S/\langle h\rangle$. Since there exists a surjective rational mapping of $\tilde{S}$ onto the surface F in $\mathbb{P}^3$ defined by the homogeneous equation

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 = 0,$$

we have $2 \geqq \kappa(\tilde{S}) \geqq \kappa(F) = 2$.

Let g be an involution of $V = C \times S$ defined by

$$g : C \times S \longrightarrow C \times S$$

$$(z, w) \longmapsto (\iota(z), h(w)).$$

Since the canonical series $|K_C|$ of the curve C has no base points, there exists a holomorphic 1-form ω on C which does not vanish at a fixed point p of C. We can consider ω as a holomorphic 1-form on V. Then the conditions (1) and (2) in Theorem 2.1 are satisfied. We let M be the non-singular model of the quotient variety $V/\langle g\rangle$. By Theorem 2.1 M satisfies the condition 2) in Main Theorem. Since there exists a surjective rational mapping of M onto $\hat{C} \times \tilde{S}$, we have

$$3 \geqq \kappa(M) = \kappa(\hat{C} \times \tilde{S}) = \kappa(\hat{C}) + \kappa(S) = 3.$$

(3.2) Let E be an elliptic curve. We set $V = E \times S$ where S is the same as above. V has an involution

$$g : E \times S \longrightarrow E \times S$$

$$(z, w) \longmapsto (-z, h(w))$$

where h is the same involution as above. It is easy to show that V, g and a holomorphic 1-form ω on E satisfy the conditions in Theorem 2.1. Let M be a non-singular model of $V/\langle g\rangle$ obtained by the canonical resolution of its singularity. Then M satisfies the condition 2) in Main Theorem. There exists a surjective rational mapping $f : M \longrightarrow \tilde{S}$ whose general fibre is the elliptic curve C. Hence $f : M \longrightarrow \tilde{S}$ is birationally equivalent to an elliptic threefold. From the canonical bundle formula for elliptic threefolds (see Ueno [6], Theorem 6.1), we infer that $\kappa(M) = 2$.

(3.3) Let C, ι and ω be the same as those in 3.1. We let T be an abelian surface. We set $V = C \times T$. V has an involution g defined by

$$g : C \times T \longrightarrow C \times T$$
$$(z, w) \longmapsto (\iota(z), -w).$$

It is easy to show that V and g satisfy the conditions in Theorem 2.1. Let M be a non-singular model of the quotient variety $V/\langle g\rangle$ obtained by the canonical resolution of its singularities. There exists a surjective rational mapping $f : M \longrightarrow \hat{C}$ whose general fibre is the abelian surface S. It is easy to calculate the canonical bundle formula of such a fibre space (see Ueno [8]) and we obtain

$$\kappa(M) = 1.$$

(3.4) Let V be an abelian variety of dimension 3. V has a natural involution

$$g : V \longrightarrow V$$
$$z \longmapsto -z .$$

A non-singular model M of the quotient manifold $V/\langle g\rangle$ obtained by the canonical resolution of its singularities is usually called a Kummer manifold. $\kappa(M) = 0$ and M satisfies the conditions of Main Theorem. Such a manifold has been studied in Ueno [7], §16.

Remark 3.5. Let M be an algebraic threefold defined in 3.1. It is easy to show that $|mK_M| \neq \emptyset$ for any positive integer m. The m-th canonical mapping

$$\Phi_{mK} : M \longrightarrow \mathbb{P}^N$$

associated with the complete linear system $|mK|$ is a morphism. If m is sufficiently large, the image $\Phi_{mK}(M)$ is analytically isomorphic to the quotient variety $V/\langle g\rangle$. Hence the image variety $\Phi_{mK}(M)$ is normal.

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