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INCREMENTAL ASYMPTOTIC EXPANSION METHOD FOR CONTINUOUS PROPAGATION OF ELASTIC-PLASTIC BOUNDARIES

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1. Introduction. The response of an elastic-plastic structure to a prescribed external disturbance is characterized by the spreading and diminishing of plastic and unloaded regions and hence by the movement or propagation of the boundaries between elastic, plastic and elastically unloaded regions. A plastic region may appear from the boundary of or in an elastic or elastically unloaded region. The boundary between a growing plastic region and the corresponding diminishing elastic or unloaded region will henceforth be referred to simply as an *elastic-plastic boundary*. A region under unloading may spread from the boundary of or in a plastically deformed region. The points on the boundary between a growing elastically unloaded region and the correspondingly diminishing plastic region are under *neutral loading* and the boundary will be referred to as a *neutral loading boundary*. In the present paper, only those boundary regions which are $(n-1)$-dimensional spaces in an n-dimensional region will be considered. For the analysis of post-bifurcation paths emanating from *elastic-plastic critical points*, it seems that the propagation of elastic-plastic and/or neutral loading boundaries needs to be analyzed more accuratly and carefully than for problems of analyzing stable fundamental paths away from the critical points. An elastic-plastic and neutral loading boundary may be a surface of discontinuity in the dependent variables and their derivatives, particularly in dynamic problems. While some analytical tools for treating discontinuities have been provided by Hill in his comprehensive paper [1], no efficient numerical method of dealing with *continuous propagation* of such discontinuitied have been proposed so far to the best of the authors' knowledge.

A routine application of the finite element method to the analysis of an elastic-plastic structure furnishes element-wise elastic-plastic or neutral loading boundaries as stepwise solutions. Neither the history dependence of plastic deformation *during* an incremental step nor the effect of smooth and continuous propagation of elastic-plastic and neutral loading boundaries upon the behavior of a structure has been taken into consideration in any elaborated finite element methods. In order to remove the deficieney due to the former within the framework of constant-strain finite elements, the present authors have proposed an incremental perturbation method [2, 3] for large displacement analysis of elastic-plastic structures, which enables one to satisfy all the governing rate-equations to a desired accuracy *during* each incremental step. The purpose of this paper is to present an idea for treating the latter problem and to demonstrate, by means of two simple examples, an incremental asymptotic expansion method (or incremental perturbation method) in which the effect of continuous propagation of elastic-plastic or neutral loading boundaries has been taken into account.

The idea of a *floating joint* attached to a point on an elastic-plastic boundary and travelling with the boundary as the plastic region spreads, is first illustrated by a simple bar element under tension in Section 2. The derivation of an instantaneous stiffness equation is compared with the process of deriving it from field equations in terms of integrated quantities. In Section 3, the idea of a floating joint is applied to the problem of large displacement analysis of an idealized sandwich beam-column. A set of asymptotic expansion equations is developed for the case of a neutral loading boundary in Section 4 and for the case of an elastic-plastic boundary in Section 5.

2. A straight bar under simple tension. Let x denote the coordinate taken along the initially straight member-axis of a bar under simple tension as shown in Fig.1. The cross-section of the bar is assumed to be monotonically and smoothly increasing from the left fixed end A at $x = 0$ toward the right loaded end C at $x = l$ and denoted by $S(x)$. For the sake of simplicity, it is assumed that the bar is made of a homogeneous elastic-linear strain-hardening material which obeys the idealized uniaxial stress-strain relation shown in Fig.2.

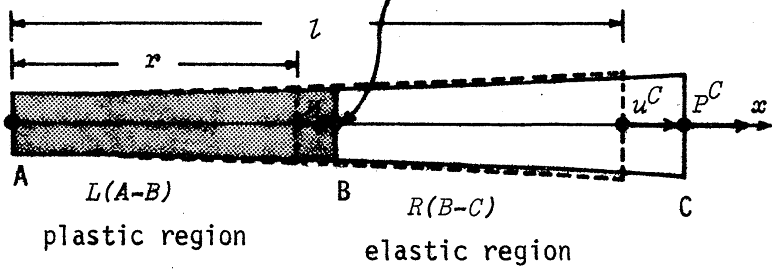

Fig.1. Straight bar under simple tension with an elastic-plastic floating node.

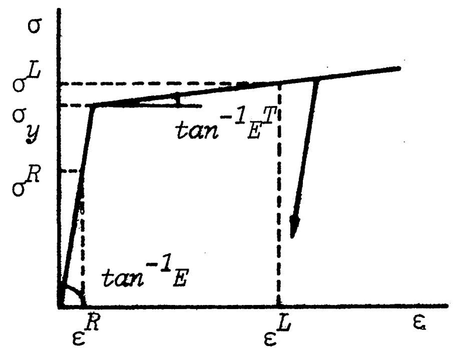

Fig.2. Bilinear uniaxial stress-strain diagram

If the tensile force P^C applied at C is increased from zero, it is at A with the smallest cross-sectional area that the first yield occurs and that an elastic-plastic boundary appears. As P^C is increased further, the elastic-plastic boundary moves toward C. Let t denote a path parameter such that at $t = 0$ the elastic-plastic boundary is at $x = 0$. Let $\sigma(x, t)$ denote the normal stress acting on a deformed cross-section and measured per unit area of the corresponding undeformed cross-section located at x. The equilibrium equation may be written as:

$$\left(S(x)\sigma(x, t)\right)' = 0 \ , \qquad (1)$$

where a prime denotes differentiation with respect to x.

A rate-type formulation of this problem is first shown in the following. The partial derivative of a field quantity with respect to t is denoted by a dot. Then the rate equation of equilibrium may be written as:

$$\left(S(x)\dot{\sigma}(x, t)\right)' = 0 \ . \qquad (2)$$

The boundary conditions in the rate form are given by

$$\dot{u}(x, t) = 0 \qquad \text{at } x = 0 \ , \qquad (3)$$

$$S(x)\dot{\sigma}(x, t) = \dot{P}^C(t) \quad \text{at } x = l \ . \qquad (4)$$

The relation between the infinitesimal longitudinal strain $\varepsilon(x,t)$ and the longitudinal displacement $u(x, t)$ may be written as:

$$\varepsilon(x, t) = u'(x, t) \ , \qquad (5)$$

$$\dot{\varepsilon}(x, t) = \dot{u}'(x, t) \ . \qquad (6)$$

The stress rate-strain rate relations are given by

$$\dot{\sigma} = E\dot{\varepsilon} \quad \text{if} \quad \sigma < \sigma_y \quad \text{or if} \quad \sigma = \sigma_y \quad \text{and} \quad \dot{\sigma} < 0 \ , \qquad (7)$$

$$\dot{\sigma} = E^T\dot{\varepsilon} \quad \text{if} \quad \sigma = \sigma_y \quad \text{and} \quad \dot{\sigma} \geq 0 \ , \qquad (8)$$

where σ_y denotes the initial yield stress in tension.

Let $x = r(t)$ represent the position of the elastic-plastic boundary B at t. Use of Eq.(4) in the general solution of Eq.(2) leads to the expression:

$$\dot{\sigma}(x, t) = \dot{P}^C(t)/S(x) \ . \qquad (9)$$

Since the region AB of $0 \leq x \leq r(t)$ is in the plastic range and since the region BC of $r(t) \leq x \leq l$ still remains in the elastic range, the stress-strain relations (7) and (8) may be applied to BC and AB, respectively. Let the subscripts L and R denote those quantities which belong to the regions or elements AB and BC, respectively. Then

$$\dot{\sigma}_L = E^T \dot{\varepsilon}_L \qquad \text{for} \qquad 0 \leq x \leq r(t) \ , \tag{10}$$

$$\dot{\sigma}_R = E \dot{\varepsilon}_R \qquad \text{for} \quad r(t) \leq x \leq l \ . \tag{11}$$

Integration of Eq.(6) with respect to x subject to the boundary condition (3) and substitution of Eqs.(10) and (11)and then (9) into the resulting equation provide the following relation between $\dot{P}^C$ and the displacement rate $\dot{\Delta}^C$ of the loaded end C:

$$\begin{aligned}\dot{\Delta}^C &= \int_0^{r(t)} \dot{\varepsilon}_L(x,t)\,dx + \int_{r(t)}^{l} \dot{\varepsilon}_R(x,t)\,dx \\ &= \left\{ \frac{1}{E^T} \int_0^r \frac{1}{S(x)}\,dx + \frac{1}{E} \int_r^l \frac{1}{S(x)}\,dx \right\} \dot{P}^C \\ &\equiv \left\{ \frac{\phi(r)}{E^T} - \frac{\psi(r)}{E} \right\} \dot{P}^C \ ,\end{aligned} \tag{12}$$

where

$$\phi\left[r(t)\right] \equiv \int_0^{r(t)} \frac{1}{S(x)}\,dx \quad \text{and} \quad \psi\left[r(t)\right] \equiv \int_{r(t)}^{l} \frac{1}{S(x)}\,dx \ . \tag{13}$$

Then the instantaneous stiffness equation of the bar AC at $t \geq 0$ may be written as

$$\frac{E^T E}{E\phi\left[r(t)\right] + E^T \psi\left[r(t)\right]} \dot{\Delta}^C(t) = \dot{P}^C(t) \ . \tag{14}$$

The stiffness coefficient in Eq.(14) contains $r(t)$. The formulation is completed by imposing the following additional equation which characterizes $r(t)$ as the elastic-plastic boundary:

$$\sigma_R \ r(t),\ t \ = \sigma_L \ r(t),\ t \ = \sigma_y \ . \tag{15}$$

Since the stress field in the element BC may be written as

$$\begin{aligned}\sigma_R(x,t) &= \sigma_R(x,0) + \int_0^t \dot{\sigma}_R(x,\tau)\,d\tau \\ &= \overset{(0)}{\sigma}(x) + \frac{1}{S(x)} \int_0^t \dot{P}^C(\tau)\,d\tau = \overset{(0)}{\sigma}(x) + \frac{P^C(t) - P^C(0)}{S(x)} \ ,\end{aligned}$$

Eq.(15) may be reduced to

$$\overset{(0)}{\sigma}(r) + \frac{P^C(t)-P^C(0)}{S(r)} = \sigma_y \quad \text{or} \quad P_C(t) = \sigma_y S(r) \ . \tag{16}$$

Although the second equation in Eq.(16) can be written directly for such a simple statically determinate model, it has been the purpose of the lengthy derivation of Eq.(16) to demonstrate the general procedure. It should be noted that, in view of the properties

$$\phi[r(t)] \to 0 \text{ as } r \to 0 \text{ and } \psi[r(t)] \to 0 \text{ as } r \to l \ , \tag{17}$$

the stiffness coefficient in Eq.(14) takes on an intermediate value between the following two extreme values:

$$E\Big/\int_0^l \frac{1}{S(x)}\,dx \quad \text{when the whole element is elastic and}$$

$$E^T\Big/\int_0^l \frac{1}{S(x)}\,dx \quad \text{when the whole element is plastic.}$$

For this particularly simple problem, an alternative formulation with respect to the integrated quantities or total quantities is possible. The solution of Eq.(1) subject to the mechanical boundary condition given by the integrated form of Eq.(4) may readily be obtained as

$$\sigma(x,\ t) = P^C(t)/S(x) \ . \tag{18}$$

The (total)stress-(total)strain relations for the plastic and elastic regions may be written, respectively, as

$$\varepsilon_L(x,t) = \frac{\sigma_y}{E} + \frac{\sigma_L(x,t)-\sigma_y}{E^T} \quad \text{for} \quad 0 \leq x \leq r(t) \ , \tag{19}$$

$$\varepsilon_R(x,t) = \frac{\sigma_R(x,t)}{E} \qquad\qquad r(t) \leq x \leq l \ . \tag{20}$$

The (total) displacement $\Delta^C(t)$ of the end C may readily be expressed as follows:

$$\Delta^C(t) = \int_0^r \varepsilon_L(x,t)\,dx + \int_r^l \varepsilon_R(x,t)\,dx$$

$$= \left(\frac{1}{E} - \frac{1}{E^T}\right)\sigma_y r(t) + \left\{\frac{\phi(r)}{E^T} - \frac{\psi(r)}{E}\right\}P^C(t) \ . \tag{21}$$

The rate expression derived by differentiating (21) with respect to t:

$$\dot{\Delta}^C(t) = \left(\frac{1}{E} - \frac{1}{E^T}\right)\sigma_y \dot{r}(t) + \left\{\frac{1}{E^T}\frac{d\phi}{dr} + \frac{1}{E}\frac{d\psi}{dr}\right\}P^C(t)\dot{r}(t)$$
$$+ \left\{\frac{\phi(r)}{E^T} + \frac{\psi(r)}{E}\right\}\dot{P}^C(t)$$

may be shown to coincide with Eq.(12) if the definitions (13) and Eq.(16) are utilized. The repeated derivation of Eq.(12) has been carried out here not only for the purpose of demonstrating the equivalence between the two formulations, but also for pointing out that the formulation in terms of the integrated quantities may be possible only if the constitutive equations are written in terms of the integrated quantities. Since the constitutive equations in the flow theory are not given in terms of the integrated quantities and are history-dependent, the rate-type formulation will in general be inevitable and should be distinguished from its linearized version in terms of the linear finite increments as will be discussed later.

In the remaining part of this section, it is shown that certain singularities appearing in the disconnected element stiffness equations for the elastic and plastic regions and in the superposed stiffness equation for the whole bar turn out to be removed after elimination of the nodal quantities belonging to the elastic-plastic boundary. The well-known direct stiffness method is formally applied first to the system consisting of the two elements AB and BC. In other words, a tentative *floating node* B is considered to lie on the elastic-plastic boundary. The location of the node B depends upon t and has been denoted by $x = r(t)$. The element stiffness equations for the elements AB and BC with *varying lengths* may be written formally as

$$\frac{E^T}{\phi(r)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \dot{\Delta}^A \\ \dot{\Delta}^B \end{Bmatrix} = \begin{Bmatrix} \dot{P}^A \\ \dot{P}^B \end{Bmatrix}, \tag{22a}$$

$$\frac{E}{\psi(r)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \dot{\Delta}^B \\ \dot{\Delta}^C \end{Bmatrix} = \begin{Bmatrix} \dot{P}^B \\ \dot{P}^C \end{Bmatrix}, \tag{22b}$$

respectively, where the superscripts A and B denote those nodal quantities belonging to the nodes A and B, respectively. It should be noted in view of the properties of $\phi(r)$ and $\psi(r)$ given by Eq.(17), that the stiffness matrices in Eqs.(22a and b) are divergent as $r \to 0$ and $r \to l$, respectively. This means that the element stiffnesses would be increased infinitely if their lengths are reduced simply to zero. The system stiffness equation derived by simple superposition:

$$\begin{bmatrix} E^T/\phi & -E^T/\phi & 0 \\ -E^T/\phi & E^T/\phi+E/\psi & -E/\psi \\ 0 & -E/\psi & E/\psi \end{bmatrix} \begin{Bmatrix} \dot{\Delta}^A \\ \dot{\Delta}^B \\ \dot{\Delta}^C \end{Bmatrix} = \begin{Bmatrix} \dot{P}^A \\ \dot{P}^B \\ \dot{P}^C \end{Bmatrix} \qquad (23)$$

retains still the same singularities. Since $\dot{P}^B$ can be assumed to be zero without losing generality, $\dot{\Delta}^B$ may readily be eliminated from Eq.(23). The resulting equation:

$$\frac{EE^T}{E\phi+E^T\psi} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\Delta}^A \\ \dot{\Delta}^C \end{Bmatrix} = \begin{Bmatrix} \dot{P}^A \\ \dot{P}^C \end{Bmatrix} \qquad (24)$$

coincides essentially with Eq.(14) and no longer contains the afore-mentioned singularities. The nature of the floating node B may be observed also from the remaining equation:

$$\dot{\Delta}_B = \frac{E^T\psi\dot{\Delta}^A+E\phi\dot{\Delta}^C}{E\phi+E^T\psi} . \qquad (25)$$

The solution procedure for Eq.(24) by the application of the perturbation method or asymptotic expansion method will be discussed through the more practical example in the following sections.

3. The tangential stiffness equation for a cantilever beam-column.

Fig.3 shows a cantilever beam-column element with an idealized sandwich cross-section. The tangential stiffness relation in terms of nodal force-rates and nodal displacement-rates is derived by the same procedure as in Section 2 for such an element when containing an elastic-plastic boundary or a neutral loading boundary. For this purpose, a tentative floating joint B is again considered to lie on the boundary. The joint B moves from the fixed end A toward the free end C as the boundary moves. Let x denote the coordinate taken along the initially-straight member-axis. For the sake of simplicity

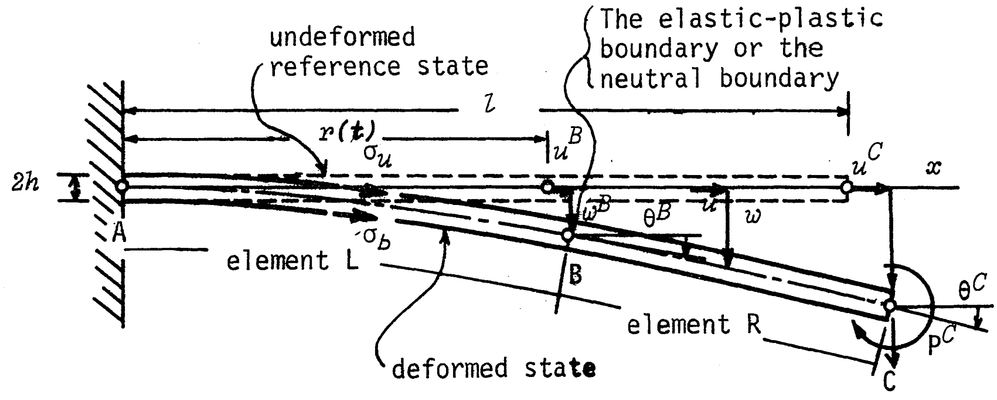

Fig.3. Sandwich beam-column with an elastic-plastic floating node.

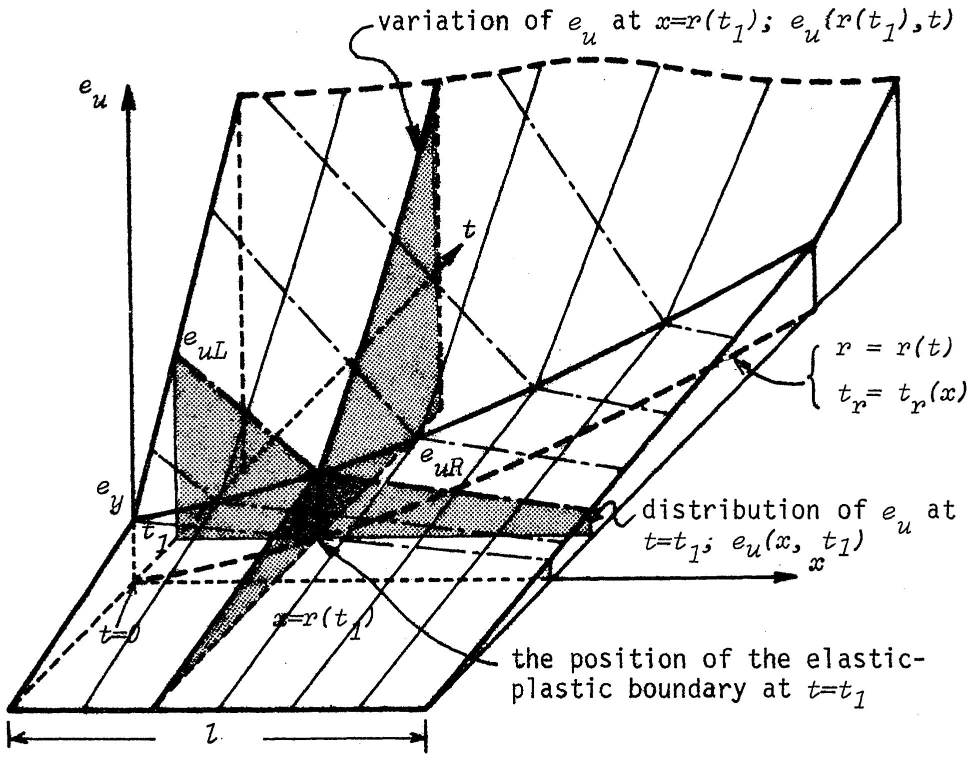

Fig.4. Timewise variation of the strain distribution and of the elastic-plastic floating node.

without losing generality, the elastic-plastic or neutral loading boundary is assumed to be at A at $t = 0$. Let $\overset{(0)}{u}(x)$, $\overset{(0)}{w}(x)$ and $\overset{(0)}{\sigma}_u(x)$ and $\overset{(0)}{\sigma}_b(x)$ denote those initial tangential and normal displacements and the initial stresses in the upper and bottom flanges, respectively, which the beam-column has experienced by the time $t = 0$. The subscripts L and R denote those quantities belonging to the elements $\overline{AB}$ and $\overline{BC}$, respectively. The subscripts u and b denote those quantities belonging to the upper and lower flanges, respectively. The elastic-plastic boundary in a flange at time $t > 0$ is denoted by $x = r(t)$.

The following displacement-rate functions are employed:

$$\left.\begin{aligned} \dot{w}_L(x,t) &= \dot{a}_{2L}(t)x^2 + \dot{a}_{3L}(t)x^3, \\ \dot{u}_L(x,t) &= \dot{b}_{1L}(t)x, \end{aligned}\right\} \text{for the element } L(\overline{AB}), \tag{26L}$$

$$\left.\begin{aligned} \dot{w}_R(x,t) &= \dot{a}_{0R} + \dot{a}_{1R}x + \dot{a}_{2R}x^2 + \dot{a}_{3R}x^3, \\ \dot{u}_R(x,t) &= \dot{b}_{0R} + \dot{b}_{1R}x \ . \end{aligned}\right\} \text{for the element } R(\overline{BC}). \tag{26R}$$

The corresponding nodal displacement rate-generalized displacement rate relations may be written as

$$\dot{\mathrm{a}}_L(t) = \mathrm{A}_L(r)\dot{\Delta}_L(t) \tag{27L}$$

where

$$\dot{\mathrm{a}}_L = \begin{Bmatrix} \dot{a}_{2L} \\ \dot{a}_{3L} \\ \dot{b}_{1L} \end{Bmatrix}, \quad \dot{\Delta}_L = \dot{\Delta}^B = \begin{Bmatrix} \dot{w}^B \\ \dot{\theta}^B \\ \dot{u}^B \end{Bmatrix} \text{ and } \mathrm{A}_L = \begin{bmatrix} 3/r^2 & -1/r & 0 \\ -2/r^3 & 1/r^2 & 0 \\ 0 & 0 & 1/r \end{bmatrix}.$$

Similar equations for the element R may be written readily but are omitted. Similar omissions will be made hereafter unless otherwise stated for particular distinctions due to the difference in forms. The strain rate-displacement rate relations are given by

$$\dot{\mathrm{e}}_L(x,t) = \begin{Bmatrix} \dot{e}_{uL} \\ \dot{e}_{bL} \end{Bmatrix} = \begin{Bmatrix} \dot{u}_L' + w_L'\dot{w}_L' + h\dot{w}_L'' \\ \dot{u}_L' + w_L'\dot{w}_L' - h\dot{w}_L'' \end{Bmatrix}$$

$$= \mathrm{B}_L(x,t)\dot{\mathrm{a}}_L(t) = \mathrm{B}_L(x,t)\mathrm{A}_L(r)\dot{\Delta}_L(t) \ , \tag{28L}$$

where

$$B_L = \begin{bmatrix} 2w_L'x+2h & 3w_L'x^2+6hx & 1 \\ 2w_L'x-2h & 3w_L'x^2-6hx & 1 \end{bmatrix},$$

and where h denotes the distance between a sandwich flange and the member axis. The current displacement gradients w_L' and w_R' in B_L and B_R may be expressed as follows:

$$w_L'(x,t) = \overset{(0)}{w}{}'(x) + \int_0^{t_r(x)} \dot{w}_R'(x,s)ds + \int_{t_r(x)}^{t} \dot{w}_L'(x,s)ds\ , \tag{29L}$$

$$w_R'(x,t) = \overset{(0)}{w}{}'(x) + \int_0^{t} \dot{w}_R'(x,s)ds\ , \tag{29R}$$

where $t_r(x)$ represents that time at which the elastic-plastic boundary or neutral loading boundary passes through x and is the inverse of $r(t)$ as shown in Fig.4. The stress rate-strain rate relations for the flanges may be written as:

$$\dot{\boldsymbol{\sigma}}_L(x,t) = \begin{Bmatrix} \dot{\sigma}_{uL} \\ \dot{\sigma}_{bL} \end{Bmatrix} = \begin{bmatrix} E_{uL} & 0 \\ 0 & E_{bL} \end{bmatrix} \begin{Bmatrix} \dot{e}_{uL} \\ \dot{e}_{bL} \end{Bmatrix} = E_L\dot{e}_L\ . \tag{30L}$$

The variational principles for the rate quantities may be written as:

$$\int_0^{r(t)} \left[\delta\dot{e}_L{}^T\dot{\boldsymbol{\sigma}}_L + (\sigma_{uL}+\sigma_{bL})\delta\dot{w}'\dot{w}_L' \right] dx = \delta\dot{\Delta}_L{}^T\dot{P}_L\ , \tag{31L}$$

$$\int_{r(t)}^{l} \left[\delta\dot{e}_R{}^T\dot{\boldsymbol{\sigma}}_R + (\sigma_{uR}+\sigma_{bR})\delta\dot{w}_R'\dot{w}_R' \right] dx = \delta\dot{\Delta}_R{}^T\dot{P}_R\ . \tag{31R}$$

Substitution of Equations (27), (28) and (30) into Equations (31) provides the tangential stiffness equations for the two elements L and R,,which may be written as:

$$K_L(r,t)\dot{\Delta}_L(t) = \dot{P}_L(t) \tag{32L}$$

where

$$K_L = A_L^T \int_0^r \left[B_L{}^T E_L B_L + S_L \right] dx\ A_L\ . \tag{33L}$$

S_L and S_R represent, respectively, the terms involving the current stresses $\boldsymbol{\sigma}_L(x,\ t)$ and $\boldsymbol{\sigma}_R(x,\ t)$ given by

$$\boldsymbol{\sigma}_L(x,t) = \overset{(0)}{\boldsymbol{\sigma}}(x) + \int_0^{t_r(x)} E_R\dot{e}_R(x,s)\ ds + \int_{t_r(x)}^{t} E_L\dot{e}_L(x,s)\ ds\ , \tag{34L}$$

$$\sigma_R(x,t) = \overset{(0)}{\sigma}(x) + \int_0^t E_R \dot{e}_R(x,s)\,ds \quad . \tag{34R}$$

Since the elements of K_L involve terms of negative powers of r, $K_L \rightarrow [\infty]$ as $r \rightarrow 0$. On the other hand, $K_R \rightarrow [\infty]$ as $r \rightarrow l$. These apparent singularities due to direct use of K_L and K_R may readily be removed if $\dot{\Delta}^B$ is eliminated from the superposed stiffness equation for the whole cantilever $\overline{ABC}$:

$$\begin{bmatrix} K_L^{BB}+K_R^{BB} & K_R^{BC} \\ K_R^{CB} & K_R^{CC} \end{bmatrix} \begin{Bmatrix} \dot{\Delta}^B \\ \dot{\Delta}^C \end{Bmatrix} \equiv \begin{bmatrix} K^{BB} & K^{BC} \\ K^{CB} & K^{CC} \end{bmatrix} \begin{Bmatrix} \dot{\Delta}^B \\ \dot{\Delta}^C \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dot{P}^C \end{Bmatrix} \tag{35}$$

After elimination, the stiffness equation with respect to $\{\Delta^C\}$ and $\{\dot{P}^C\}$ may be written as

$$K(r,\ t)\dot{\Delta}^C(t) = \dot{P}^C(t) \tag{36}$$

where

$$K = K^{CC} - K^{CB}K^{BB^{-1}}K^{BC} \quad . \tag{37}$$

Neither for $r \rightarrow 0$ nor for $r \rightarrow l$, the matrix K given by Eq.(37) is divergent any longer. The continuous variation of the elastic-plastic stiffness of the whole beam-column has now been expressed with r as a parameter.

4. Neutral loading boundary condition and asymptotic expansion.

If propagation of a neutral boundary is to occur in the upper flange, the neutral loading boundary condition may be written as

$$\dot{\sigma}_{uR}(r,t) = E_{uR}B_{uR}(r,t)A_R(r)\begin{Bmatrix} \Delta^B \\ \Delta^C \end{Bmatrix} \equiv F(r,t)\Delta^C(t) = 0 \quad . \tag{38}$$

Usual application of the asymptotic expansion procedure to Eq. (38) with appropriate precaution of treating r as a function of t, furnishes the following ordered set of equations:

$$F|_{t=0}\,\overset{(1)}{\Delta}{}^C = 0 \quad ,$$

$$F|_{t=0}\,\overset{(i)}{\Delta}{}^C + \frac{\partial F}{\partial r}\Big|_{t=0}\,\overset{(1)}{\Delta}{}^C\,\frac{\overset{(i-1)}{r}}{i} = \overset{(i)}{q}{}^*(\overset{(1)}{\Delta},\ldots\overset{(i-1)}{\Delta},\ \overset{(1)}{r},\ldots\overset{(i-2)}{r}) \ , \tag{39}$$

where $\overset{(i)}{\Delta}$ and $\overset{(i)}{r}$ denote, respectively, the coefficients in the expansions of $\Delta(t)$ and $r(t)$ in powers of t:

$$\Delta(t) = \overset{(0)}{\Delta} + \overset{(1)}{\Delta}t + \overset{(2)}{\Delta}t^2 + \cdots\cdots,$$
$$r(t) = \overset{(1)}{r}\,t + \overset{(2)}{r}\,t^2 + \cdots\cdots. \tag{40}$$

A similar ordered set of equations may be derived also for Eq. (11):

$$K|_{t=0}\,\overset{(1)}{\Delta}{}^{C} = \overset{(1)}{P}{}^{C},$$
$$K|_{t=0}\,\overset{(i)}{\Delta}{}^{C} + \frac{\partial K}{\partial r}\Big|_{t=0}\overset{(1)}{\Delta}{}^{C}\frac{\overset{(i-1)}{r}}{i} = \overset{(i)}{P}{}^{C} + \overset{(i)}{Q}{}^{*}(\overset{(1)}{\Delta},\ldots,\overset{(i-1)}{\Delta},\overset{(1)}{r},\ldots\overset{(i-2)}{r}). \tag{41}$$

The system of simultaneous equations (39) and (41) for $\overset{(i)}{\Delta}$ and $\overset{(i-1)}{r}$ must be solved successively.

5. Elastic-plastic boundary condition and asymptotic expansion.

If propagation of an elastic-plastic boundary is to occur in the upper flange which has not experienced any plastic deformation, the elastic-plastic boundary condition may be written as:

$$\sigma_{uR}(r.t) = \sigma_y, \tag{42}$$

where σ_y denotes the constant yield stress. The ordered set of expanded equations may be reduced to the following form:

$$\overset{(0)}{\sigma}_u|_{t=0} = \sigma_y,$$
$$\overset{(0)}{G}\,\overset{(i)}{\Delta} + \frac{\partial\overset{(0)}{\sigma}_{uR}}{\partial r}\Big|_{t=0}\overset{(i)}{r} = \overset{(i)}{g}{}^{*}(\overset{(1)}{\Delta},\ldots,\overset{(i-1)}{\Delta},\overset{(1)}{r},\ldots,\overset{(i-1)}{r}), \tag{43}$$

where

$$G = E_{uR}B_{uR}(0,0)A_R(0). \tag{44}$$

Since the i-th order stiffness equation does not contain $\overset{(i)}{r}$, Eq.(41) may be solved independently of Eq.(43) for $\overset{(i)}{\Delta}$. The solution $\overset{(i)}{\Delta}$ may then be substituted into Eq.(43) to find $\overset{(i)}{r}$. The first order equation of (39) and the 0-th order equation of (43) define, respectively, that the initial positions of the boundaries are at $x = 0$. It has been tacitly assumed in the foregoing development that an appropriate choice of the parameter t is to be made so that $r'(0) \neq 0$.

In practical problems of a beam-column or a framed structure subjected to repeated loads, the subsequent yield stress of the material in a plastically deformed portion is dependent up on the previous history of the stress-strain path. In such a problem, σ_y in Eq.(43) must be treated as a function of x which are known from the analysis in the preceding step.

6. Concluding remarks. In view of the results of Cicala [4] and of Hutchinson [5], a careful treatment of the variation of an unload region and hence of the propagation of a neutral boundary with a high accuracy seems to be crucial in the elastic-plastic bifurcation and post-bifurcation analysis of a structure. With such problems in mind as a field of potential application, a numerical method has been devised of dealing with continuous propagation of an elastic-plastic boundary and a neutral loading boundary across which some discontinuities in the dependent variables and/or their derivatives may occur. The proposed idea has been illustrated by the two examples to considerable details. Although certain difficulties may arise in two- and three-dimensional finite elements, the proposed procedure furnishes, at least for beam-columns and framed structures, a means of taking into account, not only the continuous stiffness variation of an element containing a possibly discontinuous moving boundary, but also the termination condition when the boundary surface passes through the terminal point or surface.

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