数理解析研究所講究録
第241巻 1975年 47-71

Finite Element Method of Incompressible Viscous Fluid Flow by Means of Perturbation Method

Mutsuto Kawahara*

1. Introduction

As is generally known, laminar flow analysis of incompressible viscous fluid is to analize initial and boundary value problem given by the Navier-Stokes equation. The purpose of this paper is to discuss finite element methods concerned with steady and unsteady fluid flow problems and to present related illustrative examples.

There have already been presented some contributions to the solution procedures for the problem given by the Navier-Stokes equations ([1] -[7]).
In the present paper, steady flow and unsteady flow are analized by using perturbation method. In case of steady flow, solutions obtained by the Newton-Raphson method and perturbation method are compared numerically. In case of unsteady flow, assuming that the basic flow is known, the unsteady flow is calculated by the linearized equation increasing the boundary values by small amounts.

2. Basic Equation

Basic equations, namely, equation of motion, constitutive equation, equation of continuity and boundary conditions employed in the present paper are described in this section. Throughout the paper, equations are expressed

* Department of Civil Engineering, Chuo University, Kasuga, Bunkyo-ku, Tokyo

by using spatial description with rectangular coordinate system x_i ($i = 1, 2$ and 3) and time t. Indecial notation is used and the usual summation convention is employed for repeated indices, which run from 1 to 3, unless otherwise noted.

Denoting velocity as u_i, the equation of motion of a fluid is expressed as in the following form.

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j u_{i,j}\right) - \tau_{ij,j} = \rho \hat{f}_i \qquad (2.1)$$

where ρ and $\hat{f}_i$ are density and body force, respectively. In case of incompressible nonlinear viscous fluid, stress tensor τ_{ij} is described as follows using Kronecker's delta function δ_{ij}.

$$\tau_{ij} = -p\delta_{ij} + 2\mu(II_d, III_d)\, d_{ij} \qquad (2.2)$$

where p denotes pressure and μ is nonlinear viscosity function and is assumed as the function of the second and third order invariants of deformation rate tensor d_{ij}, i. e.,

$$II_d = -d_{ij}d_{ij} \qquad (2.3)$$

$$III_d = \det(d_{ij}) \qquad (2.4)$$

where deformation rate tensor d_{ij} is related to velocity u_i as:

$$d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (2.5)$$

In case of Newtonian fluid, μ is the constant and equation (2.2) and (2.5) yield:

$$\tau_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \qquad (2.6)$$

Moreover, the well known Navier-Stokes equation can be obtained by introducing equation (2.6) into equation (2.1) and using equation of continuity,

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j u_{i,j}\right) + P_{,i} - \mu u_{i,jj} = \rho \hat{f}_i \tag{2.7}$$

Equation of continuity of incompressible fluid is as follows.

$$u_{i,i} = 0 \tag{2.8}$$

As the boundary condition, the following two conditions are employed for simplicity. Boundary condition for velocity is expressed as:

$$u_i = \hat{u}_i \qquad \text{on } S_1 \tag{2.9}$$

where superscripted ^ means the given value on the boundary. The surface force is prescribed on the boundary S_2, i. e.,

$$S_i = T_{ij} n_j = \hat{S}_i \qquad \text{on } S_2 \tag{2.10}$$

where n_j is the components of the unit normals to the boundary surface S_2. Let it be assumed that

$$S_1 \cup S_2 = S \tag{2.11}$$

$$S_1 \cap S_2 = \phi \tag{2.12}$$

where S is the surface of the body V and ϕ means null set.

In order to apply finite element method, variational equation corresponding to equations (2.1), (2.2), (2.9) and (2.10) is required and it can be obtained by following the conventional procedure of the Galerkin method. Let the weighting function u_i^* be the function, the value of which is arbitray except on the boundary S_1, where it takes the value zero. Multiplying both sides of equation (2.1) by u_i^*, integrating over the whole volume V and using Green's theorem, it follows that

$$\int_V (\rho u_i^* \frac{\partial u_i}{\partial t}) dV + \int_V (\rho u_i^* u_j u_{i,j}) dV + \int_V (u_{i,j}^* \tau_{ij}) dV$$

$$= \int_V (\rho u_i^* \hat{f}_i) dV + \int_{S_2} (u_i^* \hat{S}_i) dS \qquad (2.13)$$

Introducing equation (2.2) into equation (2.13) and rearranging it, the following variational equation is obtained.

$$\int_V (\rho u_i^* \frac{\partial u_i}{\partial t}) dV + \int_V (\rho u_i^* u_j u_{i,j}) dV - \int_V (u_{i,i}^* p) dV$$

$$+ \int_V \mu (u_{i,j}^* u_{i,j}) dV + \int_V \mu (u_{i,j}^* u_{j,i}) dV$$

$$= \int_V \rho (u_i^* \hat{f}_i) dV + \int_{S_2} (u_i^* \hat{S}_i) dS \qquad (2.14)$$

As to the equation of continuity (2.8), the variational equation:

$$\int_V (p^* u_{i,i}) dV = 0 \qquad (2.15)$$

is followed, in which p^* is the arbitrary weighting function.

3. Finite Element Method

It is assumed that the flow field to be analized is divided into small regions called finite elements. Let the interpolating equation for velocity and pressure in each finite element be expressed by the following forms.

$$u_i = \Phi_\alpha U_{\alpha i} \qquad (3.1)$$

$$p = \Psi_\lambda P_\lambda \qquad (3.2)$$

where Φ_α and Ψ_λ mean interpolation function for velocity and pressure, $U_{\alpha i}$ denote the velocity at α th node of finite element in the i th direction, and P_λ is the pressure at the λ th node, respectirely. The selection of interpolating function Φ_α and Ψ_λ is one of the most important procedures in the finite element analysis. Taylor and Hood, [5] , [6] Kawahara et. al. [1] - [4] and others have been used the ralation that Φ_α is selected so as to be higher order polynomials than Ψ_λ is. On the contrary, Oden and Wellford [7] used the relation that Φ_α is the same order polynomial as Ψ_λ is. In case of the selection by Oden and Wellford, the aditional boundary conditions should be imposed in the analysis. Oden and Wellford employed the boundary condition of the pressure gradient normal to the wall, i. e.,

$$\frac{\partial p}{\partial n} = \hat{r} \qquad \text{on } S_3 \qquad (3.3)$$

where n is the normal coordinate to the boundary, and $\hat{r}$ means prescribed boundary value of the pressure gradient.

In the numerical examples of the present paper, two dimensional fluid flow analysis is obtained, and, in that case, perfect polynomial series to second order terms is used for velocity and linear polynomial series for pressure. Triangular finite element with six nodes is employed for velocity and with three nodes for pressure.

According as the conventional Galerkin method, the equations:

$$u_i^* = \Phi_\alpha U_{\alpha i}^* \quad (3.4)$$

$$p^* = \Psi_\lambda P_\lambda^* \quad (3.5)$$

are used for weighting functions u_i^* and P_λ^* . Introducing equations (3.1), (3.2), (3.4) and (3.5) into equations (2.14) and (2.15), and making use of the arbitrariness of the quantities $U_{\alpha i}^*$ and P_λ^* , the following finite element equations are obtained.

$$M_{\alpha i\beta j}\dot{U}_{\beta j} + K_{\alpha\beta\gamma j}U_{\beta j}U_{\gamma i} + H_{\alpha i\lambda}P_\lambda + S_{\alpha i\beta j}U_{\beta j} = \hat{\Omega}_{\alpha i} \quad (3.6)$$

$$H_{\alpha i\lambda}U_{\alpha i} = 0 \quad (3.7)$$

where superposed dot denotes the partial differentiation with respect to time t , and

$$M_{\alpha i\beta j} = \int_V (\Phi_\alpha\Phi_\beta)\delta_{ij}\,dV$$

$$K_{\alpha\beta\gamma j} = \int_V \rho(\Phi_\alpha\Phi_\beta\Phi_{\gamma,j})\,dV$$

$$H_{\alpha i\lambda} = \int_V (\Phi_\alpha\Psi_{\lambda,i})\,dV$$

$$S_{\alpha i\beta j} = \int_V \mu(\Phi_{\alpha,k}\Phi_{\beta,k})\delta_{ij}\,dV + \int_V \mu(\Phi_{\alpha,i}\Phi_{\beta,j})\,dV$$

$$\hat{\Omega}_{\alpha i} = \int_V \rho(\hat{f}_i\Phi_\alpha)\,dV + \int_{S_2}(\Phi_\alpha\hat{S}_i)\,dS$$

Following the conventional finite element procedure, the equation system for the whole flow field can be obtained using superposition.

$$M_{\alpha\beta}\dot{v}_{\beta} + K_{\alpha\beta\gamma}v_{\beta}v_{\gamma} + H_{\alpha\lambda}g_{\lambda} + S_{\alpha\beta}v_{\beta} - \hat{\Omega}_{\alpha} = 0 \qquad (3.8)$$

$$H_{\lambda\alpha}v_{\alpha} = 0 \qquad (3.9)$$

where v_{α} and g_{λ} denote the velocity and pressure of the nodes in the whole flow field and whole direction, and coefficients can be constructed by superposing the coefficients of equation (3.6) and (3.7) applied to the whole flow field. In equations (3.8) and (3.9), indices α, β and γ take the value from 1 to $2N$ and λ from 1 to Q ,where N is the total number of nodal points of the flow field and Q is the total number of the nodes at which pressure is taken to be the unknowns.

In case of steady flow analysis, equations (3.8) and (3.9) are rewritten as in the following form.

$$K_{\alpha\beta\gamma}v_{\beta}v_{\gamma} + H_{\alpha\lambda}g_{\lambda} + S_{\alpha\beta}v_{\beta} - \hat{\Omega}_{\alpha} = 0 \qquad (3.10)$$

$$H_{\lambda\alpha}v_{\alpha} = 0 \qquad (3.11)$$

Equations (3.10) and (3.11) are the finite dimensional nonlinear simultaneous equation system.

4. Steady Flow Analysis

In order to solve nonlinear equation system, incremental solution method is one of the most commonly used procedures in the finite element analysis. It seems to be more convenient to proceed the calculation considering the effects of the terms of the higher order.

In this purpose, the perturbation method is suitable. Assume that the boundary term Ω_α can be expanded into the Taylor series in small perturbation parameter ε as:

$$\hat{\Omega}_\alpha = \hat{\Omega}_\alpha^{(0)} + \varepsilon \hat{\Omega}_\alpha^{(1)} + \varepsilon^2 \hat{\Omega}_\alpha^{(2)} + \cdots \tag{4.1}$$

The velocity v_β and pressure p_λ are also assumed to be expanded into the form that

$$v_\beta = v_\beta^{(0)} + \varepsilon v_\beta^{(1)} + \varepsilon^2 v_\beta^{(2)} + \cdots \tag{4.2}$$

$$p_\lambda = p_\lambda^{(0)} + \varepsilon p_\lambda^{(1)} + \varepsilon^2 p_\lambda^{(2)} + \cdots \tag{4.3}$$

Assumming that viscocity coefficient μ is constant for simplicity, introducing equation (4.1) to (4.3) into equations (3.10) and (3.11), and equating the coefficients of the same order terms in ε, the following linear simultaneous equation system can be obtained: for the 0th order term:

$$K_{\alpha\beta\gamma} v_\beta^{(0)} v_\gamma^{(0)} + H_{\alpha\lambda} p_\lambda^{(0)} + S_{\alpha\beta} v_\beta^{(0)} - \hat{\Omega}_\alpha^{(0)} = 0 \tag{4.4}$$

$$H_{\alpha\lambda} v_\alpha^{(0)} = 0 \tag{4.5}$$

for the 1st order term:

$$K_{\alpha\beta\gamma} (v_\beta^{(0)} v_\gamma^{(1)} + v_\beta^{(1)} v_\gamma^{(0)}) + H_{\alpha\lambda} p_\lambda^{(1)} + S_{\alpha\beta} v_\beta^{(1)} - \hat{\Omega}_\alpha^{(1)} = 0 \tag{4.6}$$

$$H_{\alpha\lambda} v_\alpha^{(1)} = 0 \tag{4.7}$$

for the nth order term:

$$K_{\alpha\beta\gamma}(v_\beta^{(0)} v_\gamma^{(n)} + v_\beta^{(n)} v_\gamma^{(0)}) + H_{\alpha\lambda} g_\lambda^{(n)} + S_{\alpha\beta} v_\beta^{(n)} = \hat{\Omega}_\alpha^{(n)} - \sum_{r=1}^{n-1} K_{\alpha\beta\gamma} v_\beta^{(r)} v_\gamma^{(n-r)} \quad (4.8)$$

$$H_{\alpha\lambda} v_\alpha^{(n)} = 0 \quad (4.9)$$

Postulating that the intial values $v_\beta^{(0)}$ and $g_\lambda^{(0)}$ are previously known, equations (4.6) - (4.9) give the increments $v_\beta^{(k)}$ and $g_\lambda^{(k)}$ $(k=1,2,\cdots n)$ calculating recursively up to the required order n. Substituting the increments $v_\beta^{(k)}$ and $g_\lambda^{(k)}$ into equations (4.2) and (4.3), velocity v_β and pressure g_λ at each nodal point of a finite element can be calculated.

Equations (4.8) and (4.9) are rewritten as in the following form to calculate the n th order increment.

$$\begin{bmatrix} K_{\alpha\beta\gamma} v_\gamma^{(0)} + K_{\alpha\gamma\beta} v_\gamma^{(0)} + S_{\alpha\beta} & H_{\alpha\lambda} \\ H_{\alpha\lambda} & 0 \end{bmatrix} \cdot \begin{bmatrix} v_\beta^{(n)} \\ g_\lambda^{(n)} \end{bmatrix} = \begin{bmatrix} \hat{\Omega}_\alpha^{(n)} - \sum_{r=1}^{n-1} K_{\alpha\beta\gamma} v_\beta^{(r)} v_\gamma^{(n-r)} \\ 0 \end{bmatrix} \quad (4.10)$$

On the contrary, iteration procedure of Newton-Raphson method can be obtained as follows.

$$\begin{bmatrix} K_{\alpha\beta\gamma} v_\gamma^{(n-1)} + K_{\alpha\gamma\beta} v_\gamma^{(n-1)} + S_{\alpha\beta} & H_{\alpha\lambda} \\ H_{\alpha\lambda} & 0 \end{bmatrix} \left(\begin{bmatrix} v_\beta^{(n)} \\ g_\lambda^{(n)} \end{bmatrix} - \begin{bmatrix} v_\beta^{(n-1)} \\ g_\lambda^{(n-1)} \end{bmatrix} \right) = \begin{bmatrix} F_\alpha^{(n-1)} \\ G_\lambda^{(n-1)} \end{bmatrix} \quad (4.11)$$

where

$$\begin{bmatrix} F_\alpha^{(n-1)} \\ \\ G_\lambda^{(n-1)} \end{bmatrix} = \begin{bmatrix} K_{\alpha\beta\gamma} v_\beta^{(n-1)} v_\gamma^{(n-1)} + H_{\alpha\lambda} g_\lambda^{(n-1)} + S_{\alpha\beta} v_\beta^{(n-1)} - \hat{\Omega}_\alpha^{(n-1)} \\ \\ H_{\alpha\lambda} v_\alpha^{(n-1)} \end{bmatrix} \quad (4.12)$$

Equation (4.11) can be derived by a little modification for equation (4.10). Perturbation method, i. e., solution procedure by equation (4.10) is advantageous to use because the coefficient matrix needs not reformulate in each iteration cycle. However, given boundary condition term $\hat{\Omega}_\alpha$ should be able to be expanded into Taylor series as in equation (4.1). On the contrary, Newton-Raphson method, i. e., solution procedure by equations (4.11) and (4.12) proceeds the iterative calculation by reformulating the coefficient matrix by use of previously calculated values $v_\beta^{(n-1)}$ and $g_\lambda^{(n-1)}$.

Two dimensional steady fluid flow between two parallel rigid wall is treated as the first numerical example. The computed results obtained by the perturbation method and Newton-Raphson method are compared in figures (1) -(4). Figures (1) and (2) illustrate the finite element idealizations named model A and model B, respectively and computed velocity profiles. On the boundaries indicated by hatched lines, it is assumed that two components of velocities are zero. On the boundaries numbered from 31 to 56 of the model A and from 38 to 63 of the model B, velocities are assumed to be given, of which profiles are shown parabola. The Reynolds number calculated by using inlet flow velocity and inlet width is 150. The boundary numbered from 6 to 67 of model A and from 8 to 74 of model B, is regarded as the boundary S_2 in equation (2.10), i. e., surface force are assumed to be zero. Numerals in the figures are computed pressure. In figures (3) and

(4), computed results by perturbation method and Newton-Raphson method are shown. Velocities at certain representative nodal points are plotted according as the Reynolds number calculated using inlet length and inlet maximum velocities. The computation results by both methods shows well in agreement all through Reynolds number in the figures.

5. Unsteady Flow Analysis

In order to solve unsteady flow in equations (3.8) and (3.9), perturbation method and Newton-Raphson method are used, both of which are similar procedures employed in the steady flow analysis. It is assumed that the term $\hat{\Omega}_\alpha$ can be expanded into the following form in small parameter ε .

$$\hat{\Omega}_\alpha = \hat{\Omega}_\alpha^{(0)} + \varepsilon \hat{\Omega}_\alpha^{(1)} + \varepsilon^2 \hat{\Omega}_\alpha^{(2)} + \cdots \qquad (5.1)$$

Also, expanding the velocity v_β and pressure p_λ as:

$$v_\beta = v_\beta^{(0)} + \varepsilon v_\beta^{(1)} + \varepsilon^2 v_\beta^{(2)} + \cdots \qquad (5.2)$$

$$p_\lambda = p_\lambda^{(0)} + \varepsilon p_\lambda^{(1)} + \varepsilon^2 p_\lambda^{(2)} + \cdots \qquad (5.3)$$

introducing equations (5.1) - (5.3) into equations (3.8) and (3.9) and rearranging them lead to the following simultaneous equation system equating coefficients of the same order terms in ε .

$$M_{\alpha\beta} \dot{v}_\beta^{(0)} + K_{\alpha\beta\gamma} v_\beta^{(0)} v_\gamma^{(0)} + H_{\alpha\lambda} g_\lambda^{(0)} + S_{\alpha\beta} v_\beta^{(0)} = \hat{\Omega}_\alpha^{(0)} \qquad (5.4)$$

$$H_{\alpha\lambda} v_\alpha^{(0)} = 0 \qquad (5.5)$$

Fig.(1) Computed Velocity and Pressure of Model A at $R_e = 150$

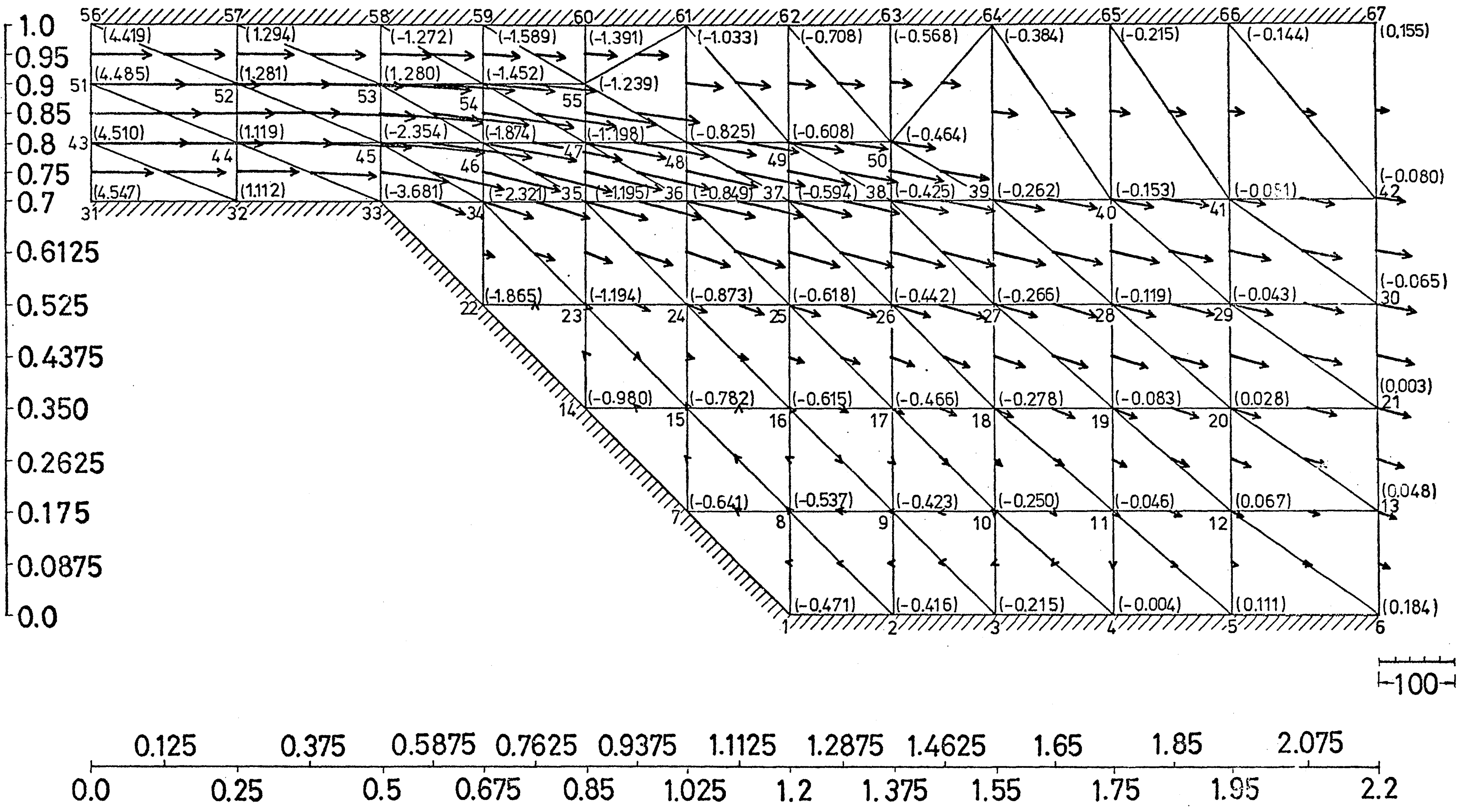


Fig.(2) Computed Velocity and Pressure of Model B at $R_e = 150$

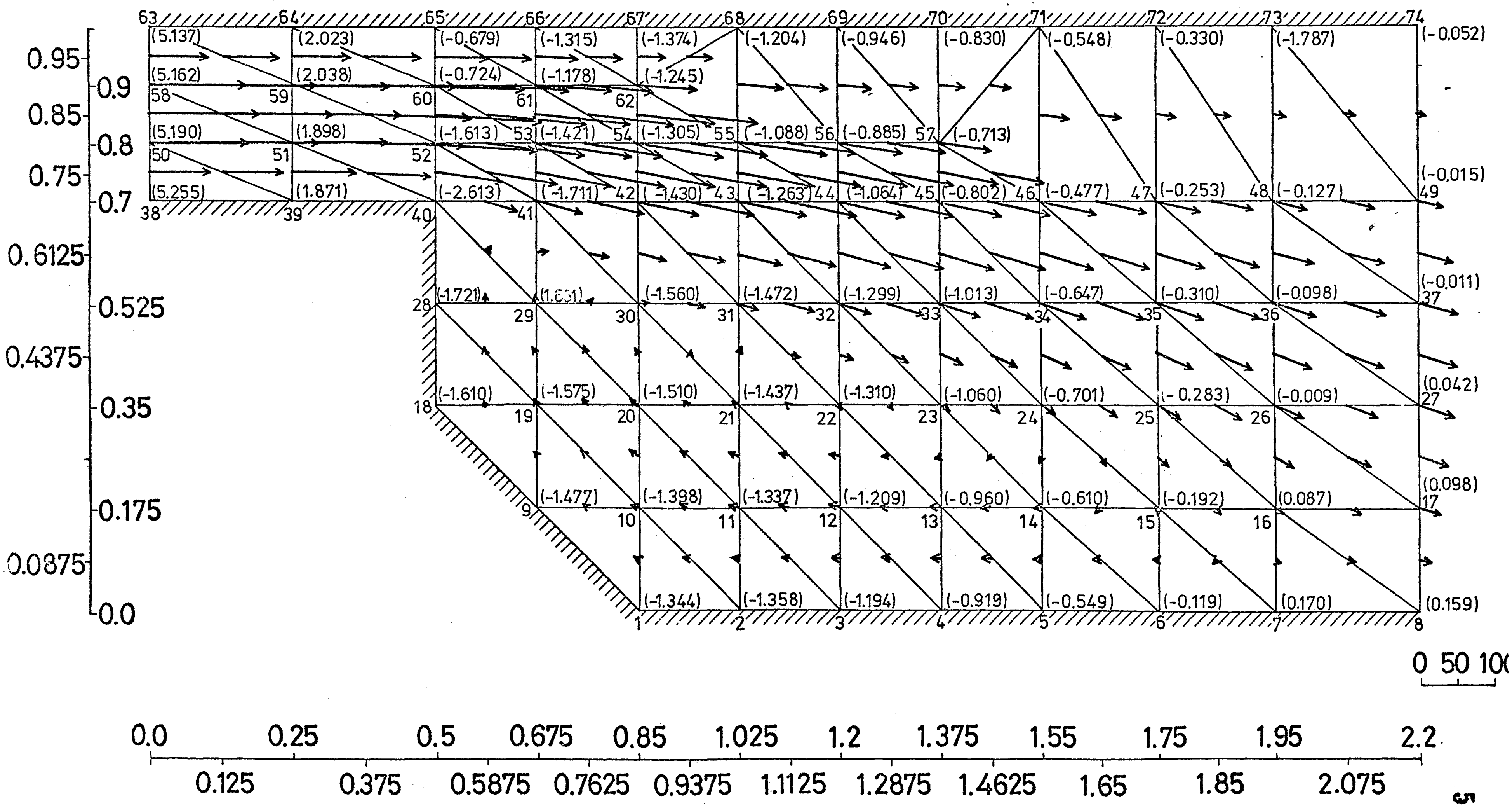


Fig.(3) Comparison of Computed Velocity of Model A

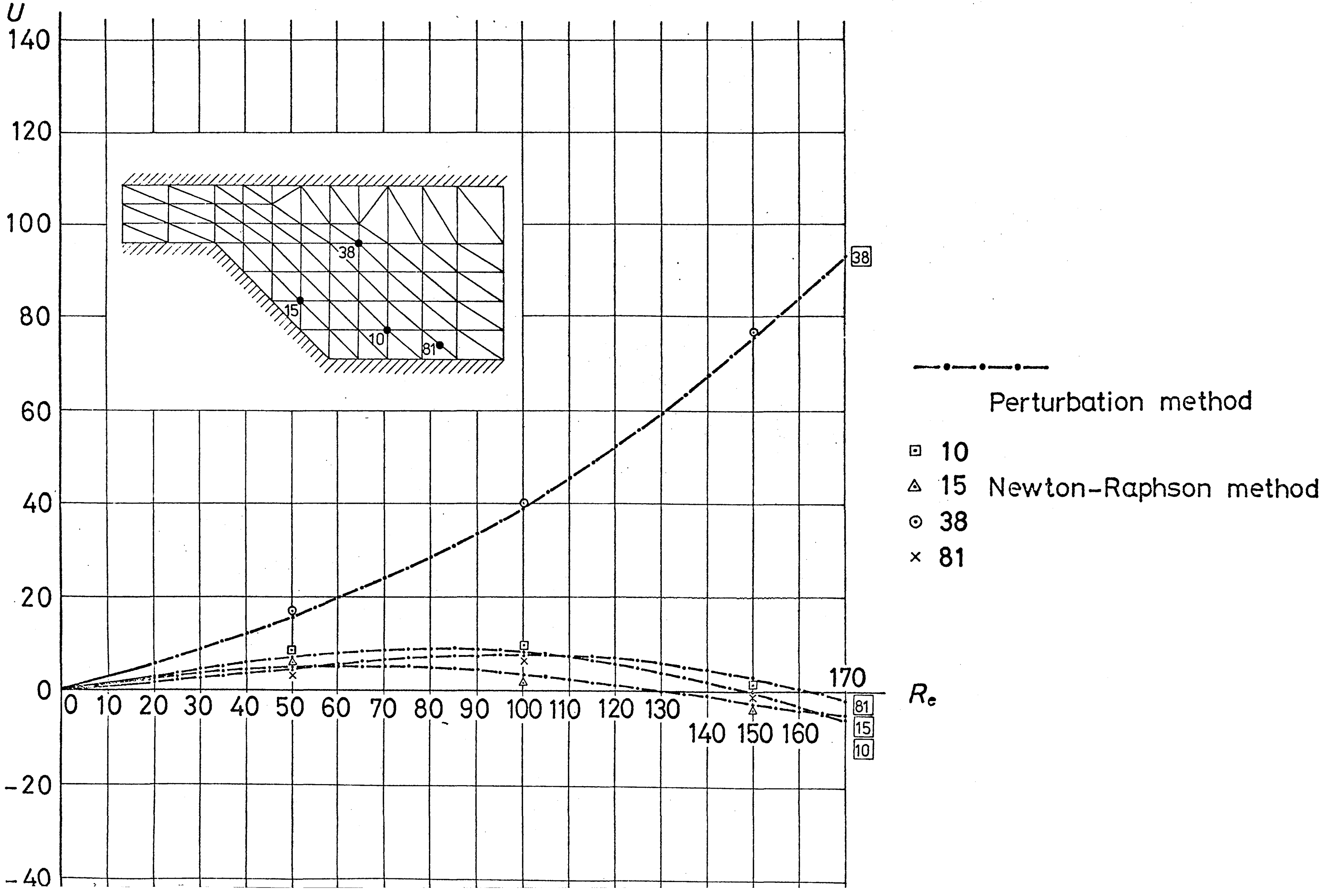
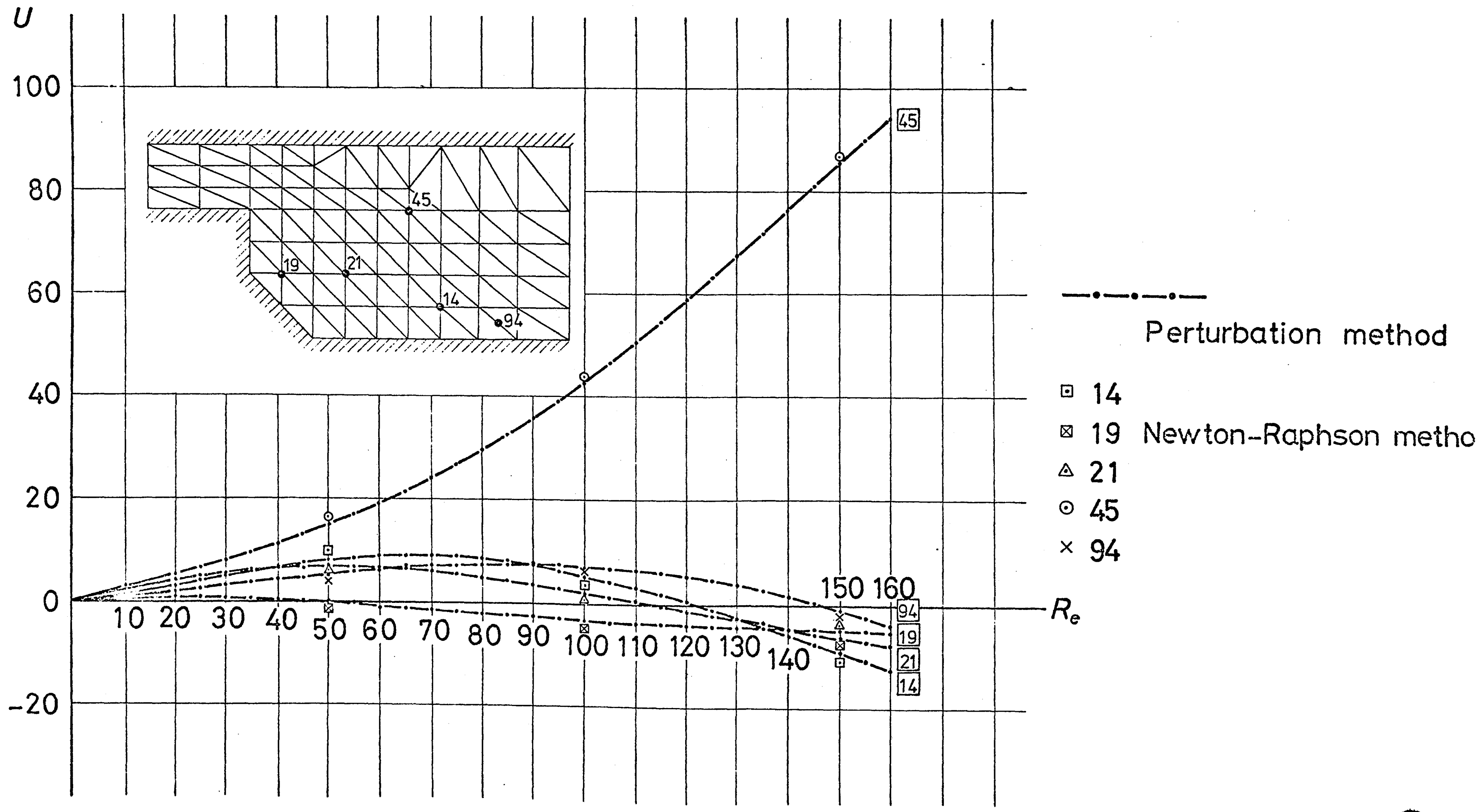


Fig.(4) Comparison of Computed Velocity of Model B



$$M_{\alpha\beta}\dot{v}_\beta^{(1)} + K_{\alpha\beta\gamma}(v_\beta^{(1)}v_\gamma^{(0)} + v_\beta^{(0)}v_\gamma^{(1)}) + H_{\alpha\lambda}g_\lambda^{(1)} + S_{\alpha\beta}v_\beta^{(1)} = \hat{\Omega}_\alpha^{(1)} \tag{5.6}$$

$$H_{\alpha\lambda}v_\alpha^{(1)} = 0 \tag{5.7}$$

$$M_{\alpha\beta}\dot{v}_\beta^{(n)} + K_{\alpha\beta\gamma}(v_\beta^{(0)}v_\gamma^{(n)} + v_\beta^{(n)}v_\gamma^{(0)}) + H_{\alpha\lambda}g_\lambda^{(n)} + S_{\alpha\beta}v_\beta^{(n)} = \hat{\Omega}_\alpha^{(n)} - \sum_{r=1}^{n-1} K_{\alpha\beta\gamma}v_\beta^{(r)}v_\gamma^{(n-r)} \tag{5.8}$$

$$H_{\alpha\lambda}v_\alpha^{(n)} = 0 \tag{5.9}$$

Equation (5.8) can be rewritten as in the form:

$$M_{\alpha\beta}\dot{v}_\beta^{(n)} + A_{\alpha\beta}v_\beta^{(n)} + H_{\alpha\lambda}g_\lambda^{(n)} = \hat{B}_\alpha^{(n)} \tag{5.10}$$

where

$$A_{\alpha\beta} = (K_{\alpha\beta\gamma}v_\gamma^{(0)} + K_{\alpha\gamma\beta}v_\gamma^{(0)}) + S_{\alpha\beta}$$

$$\hat{B}_\alpha^{(n)} = \hat{\Omega}_\alpha^{(n)} - \sum_{r=1}^{n-r} K_{\alpha\beta\gamma}v_\beta^{(r)}v_\gamma^{(n-r)}$$

Replacing the differetiation with respet to time with the difference as:

$$\dot{v}_\beta^{(n)} = \frac{v_\beta^{(n)} - v_\beta^{(n)}(0)}{\Delta t} \tag{5.11}$$

where $v_\beta^{(n)}(0)$ denotes the initial value of the velocity increment $v_\beta^{(n)}$ for a short time increment Δt. Introducing equation (5.11) into equation (5.10) and combining with equation (5.9), it follows that

$$\begin{bmatrix} A_{\alpha\beta} + \frac{1}{\Delta t} M_{\alpha\beta} & H_{\alpha\lambda} \\ H_{\alpha\lambda} & 0 \end{bmatrix} \begin{bmatrix} v_{\beta}^{(n)} \\ g_{\lambda}^{(n)} \end{bmatrix} = \begin{bmatrix} \hat{B}_{\alpha}^{(n)} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{\Delta t} M_{\alpha\beta} \, v_{\beta}^{(n)}(0) \\ 0 \end{bmatrix} \tag{5.12}$$

As is seen in equation (5.12), coefficient matrices are the same through the all increments, and the equivalent load terms should be modified as in right side of the above equation.

First of all, the basic flow, which satisfies equations (5.4) and (5.5), is assumed to be known. And then, the flow at the boundary S_1 or the surface force at the boundary S_2 is increased by a little quantity. The responce to the flow or surface force can be calculated by using equation (5.12). This method of solution is called perturbation method. Figures (5) to (11) show the computed results of unsteady flow analysis obtained by the perturbation method. In figures (5), (7) and (9), finite element idealization, the computed velocities and the pressures are illustrated. The Reynolds number in the figures is calculated using inlet maximum velocity and inlet length. As the basic flow, the steady flow obtained by the method mensioned in the previous section is employed. Changes of unsteady velocity ratio to the corresponding steady flow velocity versus time are plotted in figures (6), (8) and (10) computed using various basic flows of different Reynolds number. Figure (11) shows the changes of the velocity versus time at node No. 9 according as the velocity of basic flow.

Fig. (6) Computed Unsteady Velocity from $R_e = 50$ to $R_e = 60$

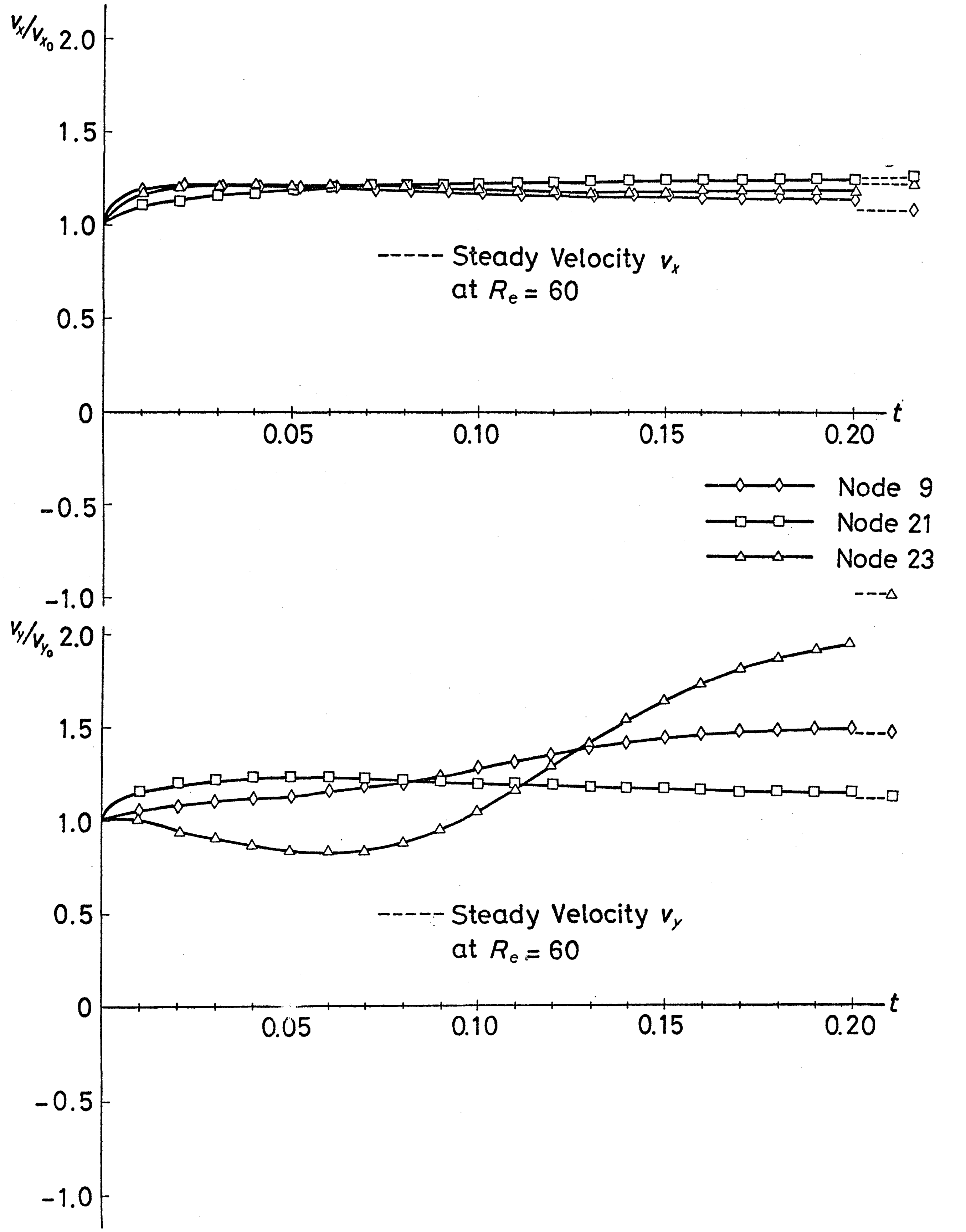


Fig. (6) Computed Steady Velocity at $R_e = 60$

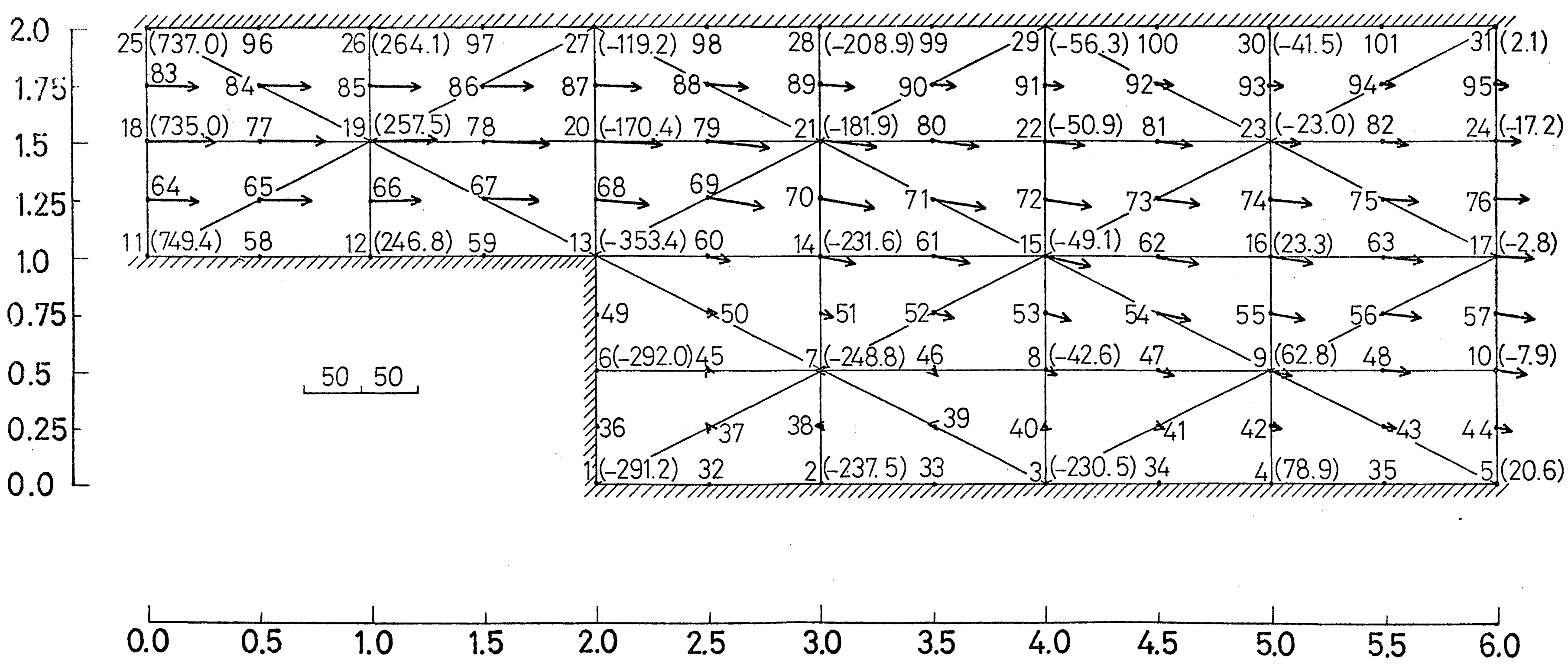


Fig. (7) Computed Unsteady Velocity from R_e=100 to R_e=12(

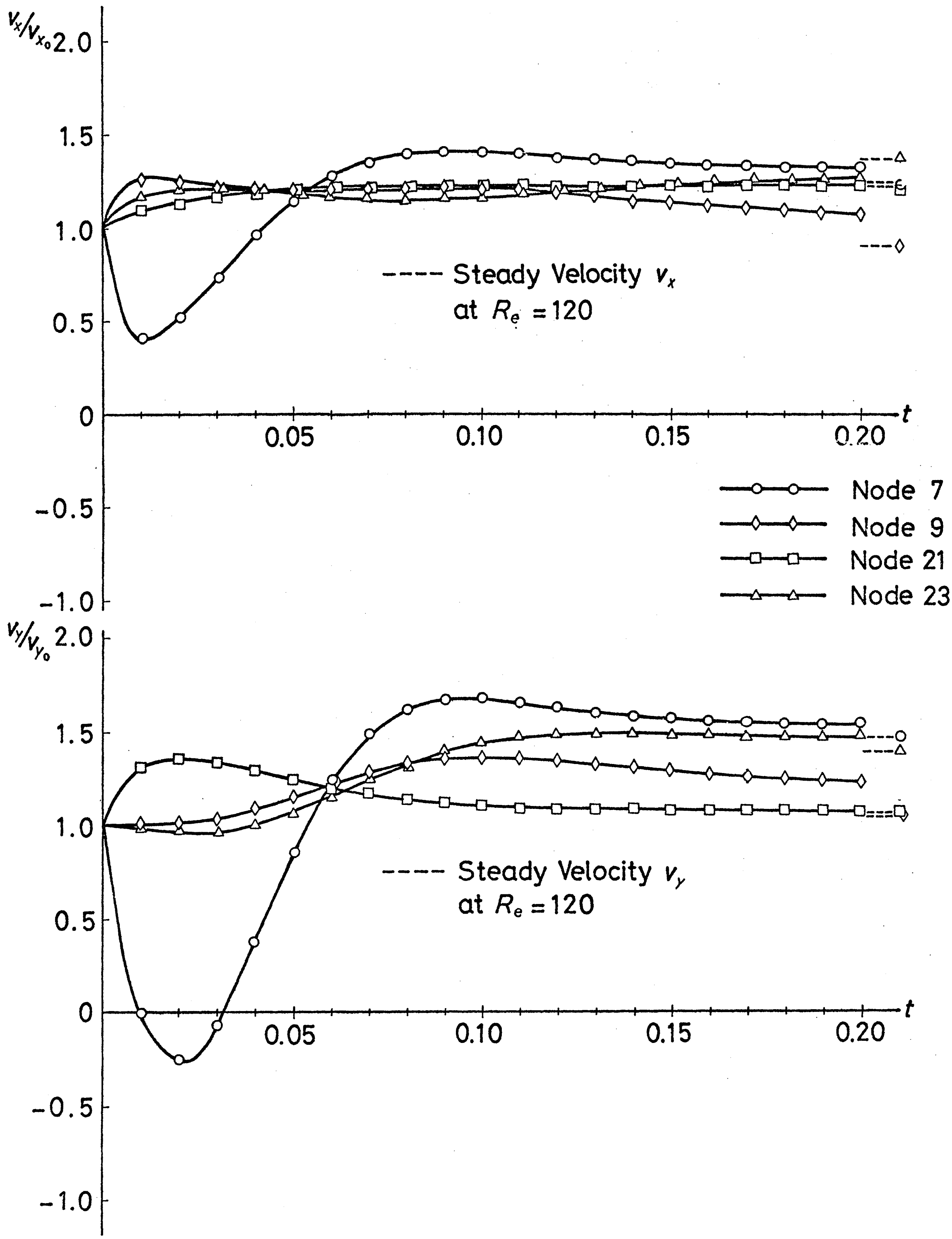


Fig. (8) Computed Steady Velocity at $R_e = 100$

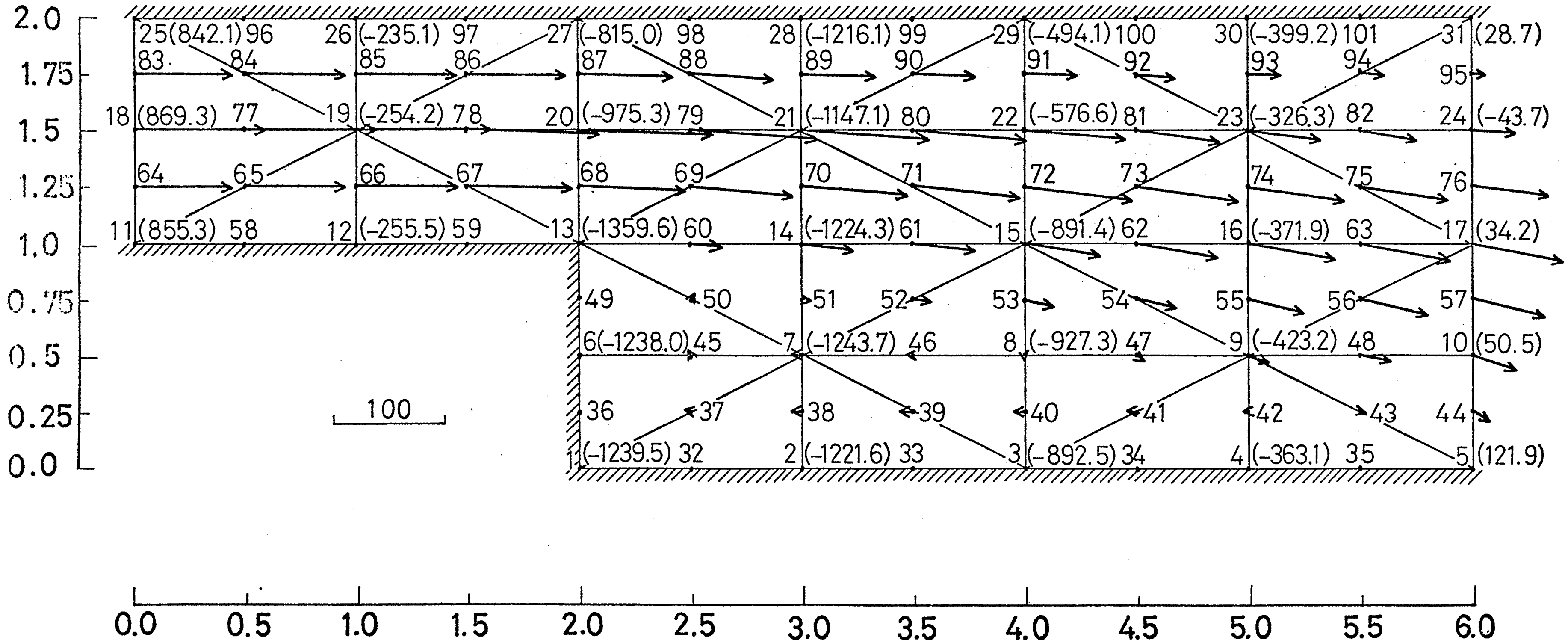


Fig. (9) Computed Unsteady Velocity from $R_e=200$ to $R_e=240$

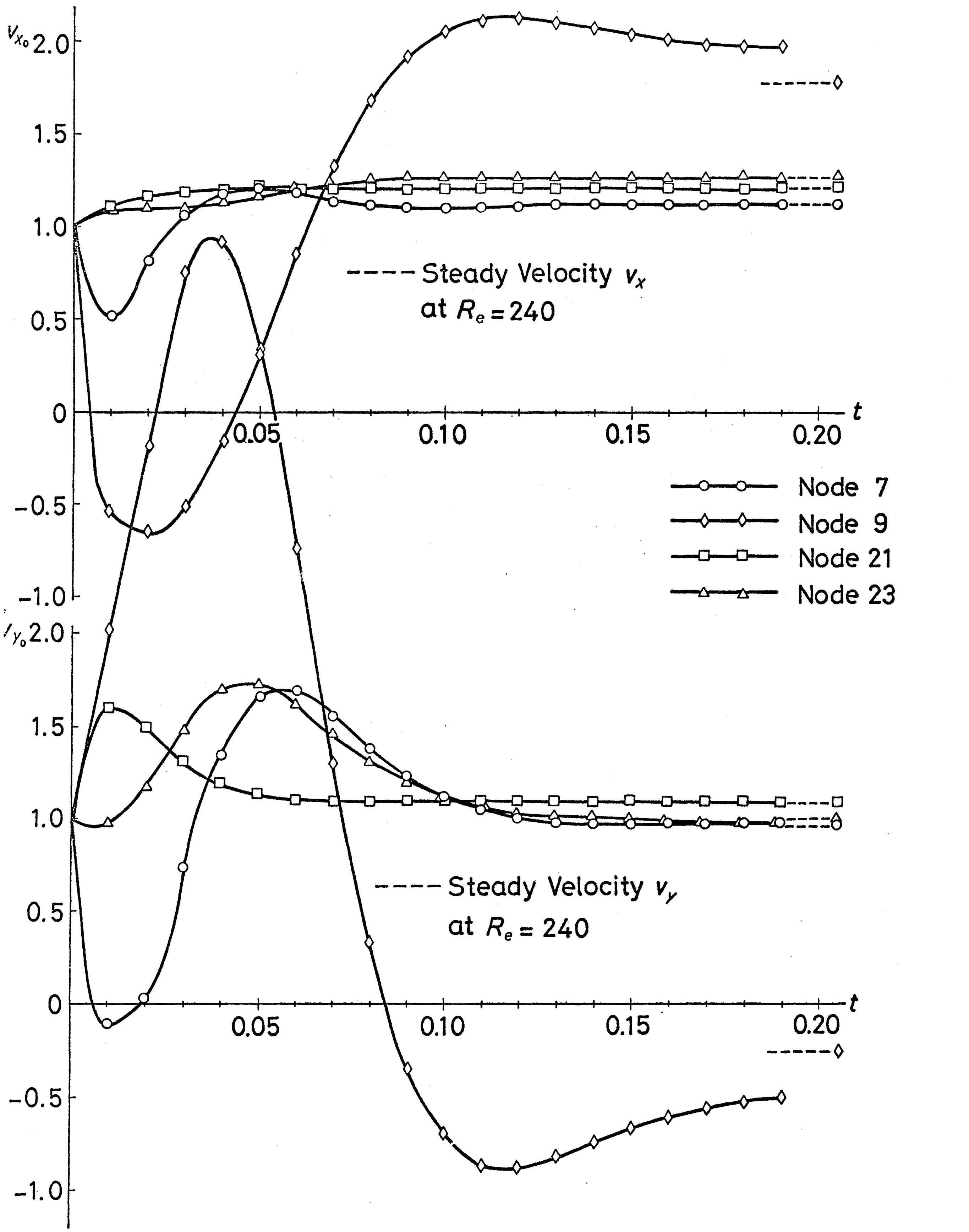


Fig. (10) Computed Steady Velocity at $R_e = 240$

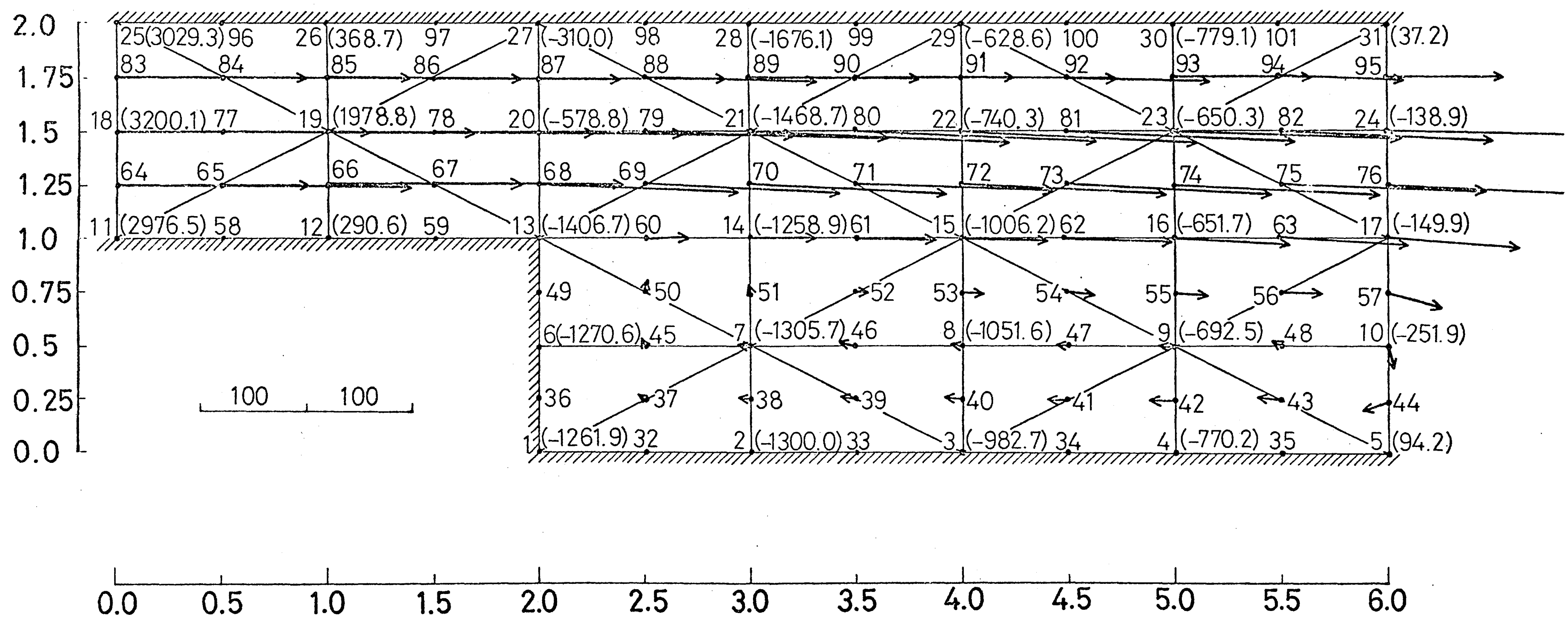
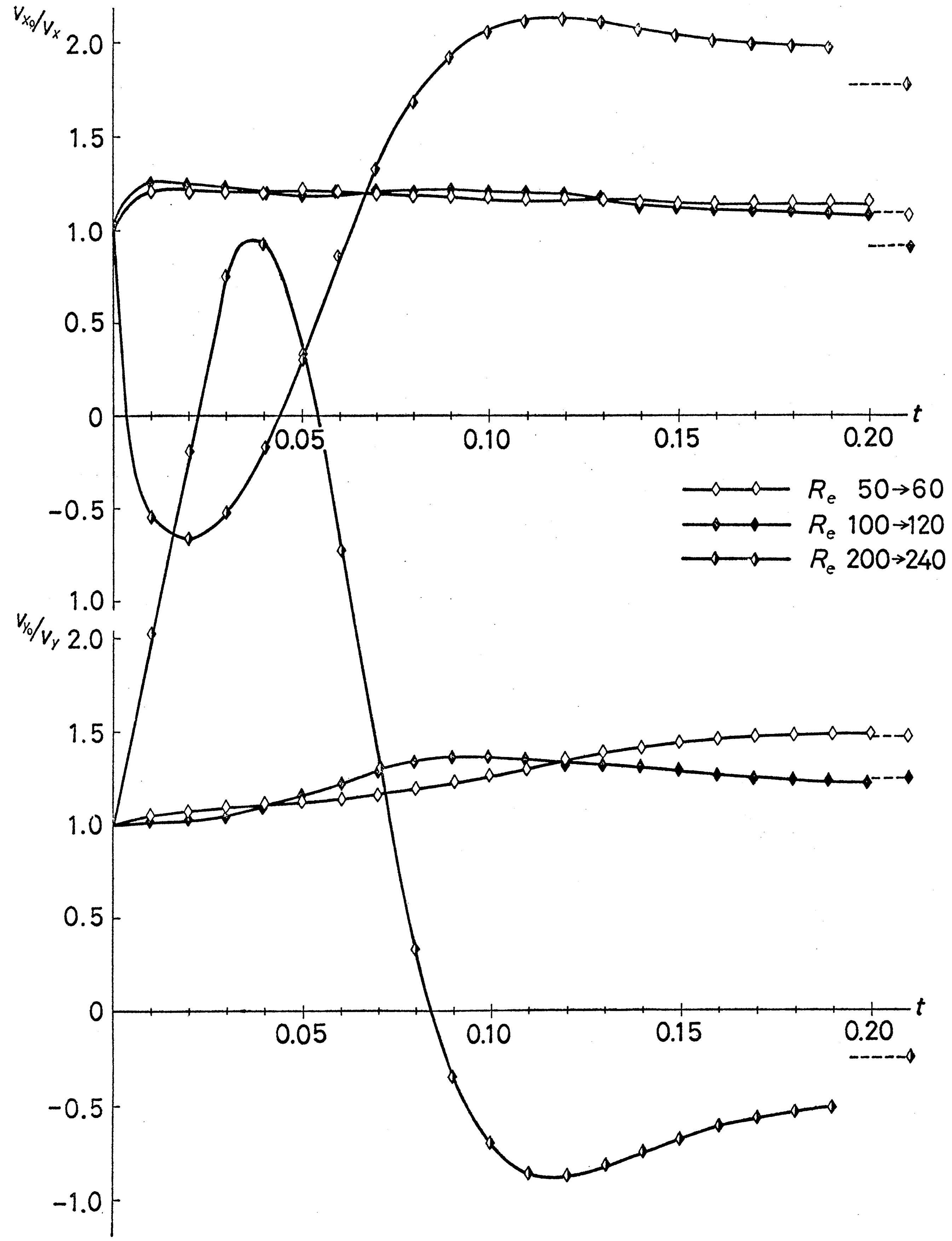


Fig. (11) Computed Unsteady Velocity at Node No. 9



References

1. M. Kawahara: "On Finite Element Method in Fluid Flow," 22nd Japan National Conference on Theoretical and Applied Mechanics, 1972

2. M. Kawahara, N. Yoshimura, K. Nakagawa and H. Ohsaka "Steady Flow Analysis of Incompressible Viscous Fluid by the Finite Element Method," Theory and Practice in Finite Element Structural Analysis Edited by Y. Yamada and R. H. Gallagher, University of Tokyo press, 1973

3. M. Kawahara, N. Yoshimura and K. Nakagawa: "Analysis of Steady Incompressible Viscous Flow," Finite Element Methods in Flow Problems edited by J. T. Oden, O. C. Zienkiewiez, R. H. Gallagher and C. Taylor, U. A. H. press, 1974

4. M. Kawahara, N. Yoshimura, K. Nakagawa and H. Ohsaka: "Steady and Unsteady Finite Element Analysis of Incompressible Viscous Fluid," (to be published)

5. C. Taylor and P. Hood: "A Numerical Solution of the Navier Stokes Equations Using the Finite Element Technique," Computers and Fluids, Vol. 1, 1973

6. P. Hood and C. Taylor: "Navier Stokes Equations Usinf Mixed Interpolation," Finite Element Method in Flow Problems edited by J. T. Oden, O. C. Zienkiewicz, R. H. Gallagher and C. Taylor, U. A. H. press, 1974

7. J. T. Oden and L. C. Wellford, Jr. , : "Analysis of Flow of Viscous Fluids by the Finite Element Method," A. I. A. A. J. Vol. 10, No, 12, 1972