数理解析研究所講究録
第 242 巻 1975 年 124-138

Operator theoretical approach for transport equations

Kiyoshi Asano

Institute of Mathemaitcs
Yoshida College
Kyoto University
Kyoto 606, Japan

§1. Introduction

The problem of neutron transport in an infinite slab leads, after an appropriate simplification, to the evolution equation

$$(1) \qquad \frac{\partial}{\partial t}u(t,x,\mu) = -\mu\frac{\partial}{\partial x}u + \frac{\kappa}{2}\int_{-1}^{1}u(t,x,\mu')d\mu' , \quad t > 0 ,$$

where $u(t,x,\mu)$ is the density of neutrons at x (going in the direction μ at time t) , and κ is a positive parameter. If the slab is extended between the planes $x = -a$, $x = a$ and the outside of the slab is a vacuum, we have the boundary conditions

$$(2) \qquad u(t,\mp a,\mu) = 0 , \quad \mu \gtrless 0 , \quad t > 0 .$$

Of course we have to add the initial condition

$$(3) \qquad u(0,x,\mu) = u_0(x,\mu) , \quad -a \le x \le a , \quad -1 \le \mu \le 1 .$$

This equation was deeply studied by J. Lehner and G. M. Wing ([2] - [4]) . In this lecture, a slight improvement will be done.

First we set the problem in an operator-theoretical framework. Put $\mathcal{H} = L^2(-a,a)$, $\mathcal{H} = L^2(-\infty,\infty)$, $M = (-1,1)$, $H = L^2(M\,;\mathcal{H})$ and $H_0 = L^2(M\,;\mathcal{H}_0)$. Define closed linear operators L in $\mathcal{H}$ and A in H (similarly L_0 in $\mathcal{H}_0$ and A_0 in H_0 with $(-a,a)$ replaced by $(-\infty,\infty)$) as follows:

$$D(L) = \{v(x) \in \mathcal{H}\ ;\ \frac{d}{dx}v(x) \in \mathcal{H}\ ,\ \ v(-a) = 0\}\ ,$$

$$(Lv)(x) = -\frac{d}{dx}v(x)$$

$$D(A) = \{u(x,\mu) \in H\ ;\ u(\cdot,\mu) \in D(L)\ \text{ for a.e.}\ \mu > 0\ ,$$
$$u(\cdot,\mu) \in D(L^*)\ \text{ for a.e.}\ \mu < 0\ ,\ \ Au \in H\}\ ,$$

$$(Au)(\cdot,\mu) = \begin{cases} \mu Lu(\cdot,\mu)\ , & \mu > 0\ , \\ -\mu L^* u(\cdot,\mu)\ , & \mu < 0\ . \end{cases}$$

Denote by J (resp. $\tilde{J}$) the projection from $\mathcal{H}_0$ to $\mathcal{H}$ (resp. from H_0 to H), and by K the "integral operator":

$$H \ni u(x,\mu) \longmapsto \frac{1}{\sqrt{2}} \int_{-1}^{1} u(x,\mu)\,d\mu \in \mathcal{H}\,.$$

If we put

$$(4)\quad B = A + \kappa K^* K\ ,\quad D(B) = D(A)\ ,$$

$$(5)\quad B_0 = A_0 + \kappa \tilde{J}^* K^* K \tilde{J}\ ,\quad D(B_0) = D(A_0)\ ,$$

then the problem (1)-(3) can be written in an evolution equiation in H :

$$\frac{d}{dt}u = Bu , \quad u(0) = u_0 .$$

Simultaneously we consider the corresponding evolution equation in H_0 :

$$\frac{d}{dt}v = B_0 v , \quad v(0) = v_0 .$$

It is easy to see that L (and hance L^*) generates a contraction semi-group e^{tL} (resp. e^{tL^*}) in $\mathcal{H}$, and L_0 generates an unitary group e^{tL_0} in $\mathcal{H}_0$. Hence A generates a contraction group e^{tA} in H , and A_0 generates an unitary group e^{tA_0} in H_0 . In addition, we obtain that

(6) $$e^{tL} = Je^{tL_0}J^* , \quad e^{tL^*} = Je^{-tL_0}J^* \quad (t \geq 0) ,$$

(7) $$e^{tA} = \tilde{J}e^{tA_0}J^* , \quad e^{tA^*} = \tilde{J}e^{-tA_0}\tilde{J}^* \quad (t \geq 0) .$$

Since $C = K^*K$ (resp. $C_0 \equiv \tilde{J}^*K^*K\tilde{J}$) is a bounded linear operator in H (resp. H_0) , B (resp. B_0) generates a semi-group e^{tB} in H (resp. a group e^{tB_0} in H_0) . Furthermore we have

(8) $$e^{tB} = \tilde{J}e^{tB_0}\tilde{J}^* , \quad t \geq 0 .$$

Following Lehner and Wing, we are concerned with spectral properties of B and B_0, and asymptotic properties of e^{tB} and e^{tB_0}. However the relation (8) implies that there are no essential differences between e^{tB} and e^{tB_0} in the physical meaning. Thus we treat only B_0 and e^{tB_0} in this lecture.

Our main result is as follows:

The continuous spectrum of B_0, which is the whole imaginary axis, is similar to the spectrum of A_0 except for the discrete values of κ.

§2. The spectrum of B_0

Put $\tilde{K} = K\tilde{J}$. Then the second resolvent equation for A_0 and B_0 :

(9) $(\lambda-B_0)^{-1} = (\lambda-A_0)^{-1} + \kappa(\lambda-A_0)^{-1}\tilde{K}^*\tilde{K}(\lambda-B_0)^{-1}$

gives the following

(10) $(\lambda-B_0)^{-1} = (\lambda-A_0)^{-1} + \kappa(\lambda-A_0)^{-1}\tilde{K}^*(1-\kappa G(\lambda))^{-1}\tilde{K}(\lambda-A_0)^{-1}$,

where

$$G(\lambda) = \tilde{K}(\lambda-A_0)^{-1}\tilde{K}^* = K\tilde{J}(\lambda-A_0)^{-1}\tilde{J}^*K^* .$$

Thus the study of $G(\lambda)$ is essential for our purpose. Denoting by $\mathbb{B}(\mathcal{H})$ (resp. $\mathbb{C}_\infty(\mathcal{H})$) the set of all bounded (resp. compact) linear operators in $\mathcal{H}$, and by $\|T\|$ the operator norm of $T \in \mathbb{B}(\mathcal{H})$, we summarize some properties of $G(\lambda)$.

Lemma 2.1. (i) $G(\lambda)$ is a $\mathbb{C}_\infty(\mathcal{H})$-valued analytic function in $\mathbb{C}_\pm = \{\lambda \,;\, \mathrm{Re}\lambda \gtrless 0\}$ and satisfies

$$G(\bar{\lambda}) = G(\lambda)^* , \quad G(-\bar{\lambda}) = -G(\lambda)^* .$$

(ii) Let $\lambda \in \mathbb{C}_\pm$. λ belongs to the resolvent set $\rho(B_0)$ of B_0 (i.e., there exists $(\lambda-B_0)^{-1} \in \mathbb{B}(H_0)$) if and only if there exists $(1-\kappa G(\lambda))^{-1} \in \mathbb{B}(\mathcal{H})$.

(iii) For $\lambda \in \mathbb{C}_+$, $G(\lambda)$ satisfies

$$0 < \mathrm{Re}G(\lambda) = \frac{1}{2}\{G(\lambda) + G(\lambda)^*\} \leq \frac{1}{\mathrm{Re}\lambda} ,$$

$$\mathrm{Im}G(\lambda) = \frac{1}{2i}\{G(\lambda) - G(\lambda)^*\} \lesseqgtr 0 \quad (\mathrm{Im}\lambda \gtreqless 0) .$$

(iv) For $0 < \beta < \beta'$, $G(\beta) > G(\beta') > G(+\infty) = 0$.

(v) $G(\lambda)$ is continuous in $\mathbb{C}_+ - \{0\} = \{\lambda ; \mathrm{Re}\lambda \geq 0 , \lambda \neq 0\}$ with respect to the norm of $\mathbb{B}(\mathcal{H})$ and satisfies

$$0 < \mathrm{Re}G(\beta+i\gamma) \leq \frac{1}{|\gamma|}(1+\pi) ,$$

$$\mathrm{Im}G(\beta+i\gamma) \gtreqless 0 \quad \text{for} \quad \gamma \lesseqgtr 0 \quad \text{and} \quad \beta \geq 0 .$$

(vi) For $\lambda \in \mathbb{C}_+ - [0,\infty)$, there exists $(1-\kappa G(\lambda))^{-1} \in \mathbb{B}(\mathcal{H})$. For any $\delta > 0$, there exists a constant $c_{\kappa,\delta} > 0$ such that

$$\|(1-\kappa G(\lambda))^{-1}\| \leq C_{\kappa,\delta} \quad (\mathrm{Re}\lambda \geq 0 , \ |\mathrm{Im}\lambda| \geq \delta) .$$

For $\lambda \in \mathbb{C}_- - \{0\}$, there holds

$$\|(1-\kappa G(\lambda))^{-1}\| \leq 1 .$$

For $\beta > 0$, there exists $(1-\kappa G(\beta))^{-1} \in \mathbb{B}(\mathcal{H})$ except for the fimite set of β which depends on κ .

Carrying out simple calculations we obtain

$$G(\lambda) = \int_0^\infty \frac{1}{2}(e^{tL} + e^{tL^*})dt \int_0^1 \frac{1}{\mu} e^{-\frac{\lambda t}{\mu}} d\mu .$$

Using the equality

$$\int_0^1 \frac{1}{\mu} e^{-\frac{z}{\mu}} d\mu = \int_1^\infty \frac{1}{\mu} e^{-\mu z} d\mu$$

$$= -\log z - b + E_0(z) ,$$

where b is Euler number and $E_0(z)$ is an entire analytic function of z which satisfies $|E_0(z)| \leq |z|$ for $z \in \overline{\mathbb{C}}_+$, we have

(11) $G(\lambda) = \int_0^\infty \mathrm{Re}\, e^{tL}\{-\log \lambda t - b - E_0(\lambda t)\}dt$.

We put

$$K(\lambda) = -\int_0^\infty \mathrm{Re}\, e^{tL} dt(\log\lambda + b) + \int_0^\infty \mathrm{Re}\, e^{tL}(-\log t)dt ,$$

$$G_0(\lambda) = \int_0^\infty \mathrm{Re}\, e^{tL} E_0(\lambda t)dt .$$

Since $\int_0^\infty \mathrm{Re}\, e^{tL} dt = \mathrm{Re}\, L^{-1}$ reduces to the 1-dimensional operator:

$$\mathcal{H} \in u(x) \longmapsto \frac{1}{2} \int_{-a}^{a} u(x)dx = a\frac{1}{2a}(u,1)1 \in \mathcal{H} ,$$

we have

(12) $K(\lambda) = -aN\log \lambda - baN + K_0$

where N is the orthogonal projection $\frac{1}{2a}(\ \ ,1)1$ in $\mathcal{H}$ and

$$K_0 = \int_0^\infty \mathrm{Re}\, e^{tL}(-\log t)dt \in \mathbf{C}_\infty(\mathcal{H}) .$$

The inequality $|E_0(z)| \le |z|$ $(z \in \overline{\mathbb{C}}_+)$ implies that

$$\| G_0(\lambda) \| \le \int_0^a |\lambda t| dt = \frac{a^2}{2} |\lambda| .$$

This implies that the spectrum $\sigma(G(\beta))$ of $G(\beta)$ converges to the spectrum $\sigma(K(\beta))$ of $K(\beta)$ as $\beta \downarrow 0$. Thus we have the following

Lemma 2.2. Let $\{\rho_n(\beta)\}$ be the set of (positive) eigen values of $G(\beta)$ (counted as many times as multiplicities). We can arrange $\{\rho_n(\beta)\}$ in the following way;

$\rho_n(\beta)$ is monotone decreasing in $\beta \in (0,\infty)$,

$\rho_n(\beta) \downarrow 0$ as $\beta \uparrow \infty$,

$\rho_n(\beta) \uparrow \rho_n^*$ as $\beta \downarrow 0$,

$\rho_n(\beta)$ is real analytic in $\beta \in (0,\infty)$.

Here $\rho_1^* = \infty$ and $\rho_2^* \ge \rho_3^* \ge \cdots$ are the eigen values of $N'K_0N'$ arranged in the decreasing order. (In above we have put $N' = 1 - N$. Note that $N'K_0N' > 0$ on the range $R(N')$ of N'.)

For $\kappa > 0$, denote by $N(\kappa)$ the number of ρ_n^* such that $\kappa\rho_n^* > 1$. Let $\beta_n = \beta_n(\kappa)$ be the root of $\kappa\rho_n(\beta) = 1$ for $n = 1, \cdots, N(\kappa)$. Then $(1-\kappa G(\lambda))^{-1} \in \mathbb{B}(\mathcal{H})$ exists for $\lambda \in \overline{\mathbb{C}}_- \cup \overline{\mathbb{C}}_+ - \{0, \beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\}$. The $\beta_n(\kappa)$'s are simple roots of $(1-\kappa G(\lambda))^{-1}$. Hence $(\lambda - B_0)^{-1} \in \mathbb{B}(\mathcal{H})$ exists for $\lambda \in \mathbb{C}_- \cup C_+ - \{\beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\}$ and has simple poles at $\{\beta_1(\kappa), \cdots \beta_{N(\kappa)}(\kappa)\}$. A simple argument connected with Lemma 2.1 shows that the there is not the point spectrum $\sigma_p(B_0)$ of B_0 on the imaginary axis $i\mathbb{R}$. Hence $\sigma_p(B_0)$ coincides with the discrete spectrum $\sigma_d(B_0)$ of B_0, i.e. $\sigma_p(B_0) = \sigma_d(B_0) = \{\beta_n(\kappa)\}$. Similarly $\sigma_p(B_0^*) = \sigma_d(B_0^*) = \{\beta_n(\kappa)\}$. Furthermore the inequality (proved by Ukai)

$$\begin{aligned} \mathrm{Re}(\tilde{K}^* u, (\lambda - A_0)^{-1}\tilde{K}^* u) &\geq \mathrm{Re}((\lambda - A_0)(\lambda - A_0)^{-1}\tilde{K}^* u, (\lambda - A_0)^{-1}\tilde{K}^* u) \\ &= \mathrm{Re}\lambda \| (\lambda - A_0)^{-1}\tilde{K}^* u \|^2 \end{aligned}$$

shows that for $\lambda \in \mathbb{C}_+$

$$\begin{aligned} \| (\lambda - A_0)^{-1}\tilde{K}^* u \|^2 &\leq \frac{1}{\mathrm{Re}\lambda} \mathrm{Re}(u, G(\lambda)u) \\ &\leq \frac{1}{\mathrm{Re}\lambda} \|u\| \|G(\lambda)u\| . \end{aligned}$$

Thus the compactness of $G(\lambda)$ implies that of $(\lambda - A_0)^{-1}\tilde{K}^*$. This implies that the essential spectrum of B_0 coincides with that of A_0, which is the whole imaginary axis. All these arguments show that the continuous spectrum $\sigma_0(B_0)$ of B_0 is the imaginary axis $i\mathbb{R}$, and the residual spectrum $\sigma_r(B_0)$ of B_0 is empty. Thus we have the following theorem due to Lehner.

Theorem 1. Let $\kappa > 0$ and B_0 be defined by (5). Then

$$\rho(B_0) = \mathbb{C}_- \cup \mathbb{C}_+ - \{\beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\}$$

$$\sigma_p(B_0) = \sigma_d(B_0) = \{\beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\}$$

$$\sigma_c(B_0) = i\mathbb{R} , \quad \sigma_r(B_0) = \phi$$

$(\lambda - B_0)^{-1}$ has simple poles at $\{\beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\}$.

§3. The similarity of the continuous spectra of A_0 and B_0

Denote by $P_j = P_j(\kappa)$ the residue of $(\lambda-B_0)^{-1}$ at $\lambda = \beta_j(\kappa)$, that is the eigen projection of B_0 belonging to $\beta_j(\kappa)$, $j = 1,\cdots,N(\kappa)$. Put $Q_1 = \Sigma P_j$, $Q_2 = 1 - Q_1$, $B_1 = B_0Q_1$ and $B_2 = B_0Q_2$. Then $(\lambda-B_0)^{-1}Q_2 = (\lambda-B_2)^{-1}Q_2$ is analytic in $\mathbb{C}_\pm$ and there hold

$$(\lambda-B_0)^{-1} = (\lambda-B_0)^{-1}Q_2 + \sum_{j=1}^{N(\kappa)} \frac{1}{\lambda-\beta_j} P_j ,$$

$$e^{tB_0} = e^{tB_0}Q_2 + \Sigma e^{t\beta_j}P_j .$$

In order to study the spectral property of B_2, we use the method of A_0-smooth perturbation developed by Kato [1]. In what follows, we put for a fixed $\alpha \in (0, 1)$

$$\alpha_1(s) = \begin{cases} 2^\alpha - \log|s| , & |s| \leq 1 , \\ (1+|s|)^\alpha , & |s| \geq 1 , \end{cases}$$

$$\alpha_2(s) = (1+|s|)^\alpha ,$$

and for later conveniens $N_1 = N$ and $N_2 = N'$. From Lemma 2.1, (11) and (12), we obtain for some constant a_0

$$\| \mathrm{Re}\, N_jG(\pm\sigma+i\gamma)N_j\| \leq \frac{1}{2} a_0\alpha_j(\gamma)^{-1} , \quad j = 1,2 .$$

Let $\{E_0(s)\}$ be the spectral resolution of $-iA_0$ and put $R(\lambda)$

$= (\lambda - A_0)^{-1} = \int (\lambda - is)^{-1} dE_0(s)$. Following Kato [1] , we have

$$\| N_j \tilde{K}(\lambda - A_0)^{-1}u - N_j \tilde{K}(-\bar{\lambda} - A_0)^{-1}u \|^2$$

$$\leq 2 \|\mathrm{Re}\ N_j G(\lambda) N_j\| (\{(\lambda - A_0)^{-1} - (-\bar{\lambda} - A_0)^{-1}\}u, u)$$

$$\leq a_0 \alpha_j(\gamma)^{-1} \int_{-\infty}^{\infty} \frac{2\sigma}{\sigma^2 + (\gamma - s)^2} d\|E_0(s)\|^2 , \quad \lambda = \sigma + i\gamma .$$

This implies

$$\int_{-\infty}^{\infty} \alpha_j(\gamma) \| N_j \tilde{K}R(\sigma + i\gamma)u - N_j \tilde{K}R(-\sigma + i\gamma)u \|^2 d\gamma$$

$$\leq 2\pi a_0 \| u \|^2 , \quad j = 1,2 .$$

Using estimates for Hilbert transforms with weighted norms, we have

$$\int_{-\infty}^{\infty} \alpha_j(\gamma) \| N_j \tilde{K}R(\sigma + i\gamma)u \|^2 d\gamma$$

$$\leq C_0 \int_{-\infty}^{\infty} \alpha_j(\gamma) \| N_j \tilde{K}R(\sigma + i\gamma)u - N_j \tilde{K}R(-\sigma + i\gamma)u \|^2 d\gamma$$

$$\leq 2\pi a_0 C_0 \| u \|^2 ,$$

Hence $N_j \tilde{K}R(\sigma + i\gamma)u$ is an element of a $\mathcal{H}$-valued Hardy class with a weighted norm, and is a continuous function of $\sigma \geq 0$ and $\sigma \leq 0$ with values in $L^2(\mathbb{R}, \alpha_j(\gamma)^{\frac{1}{2}} d\gamma ; \mathcal{H})$.
Putting $R_1(\lambda) = (\lambda - B_0)^{-1}$ and recalling that

$$\tilde{K}(\lambda - B_0)^{-1} = (1 - \kappa G(\lambda))^{-1} \tilde{K}(\lambda - A_0)^{-1} ,$$

we define so called wave operators $W_{\pm}$ and $Z_{\pm}$ as follows:

$$(W_{\pm}u,v) = (u,v) \pm \frac{\kappa}{2\pi i}\int_{-\infty}^{\infty}(\tilde{K}R(\pm 0+i\gamma)u,\ \tilde{K}R_1(\mp i0+i\gamma)^* v)d\gamma ,$$

$$(Z_{\pm}u,v) = (Q_2u,v) \mp \frac{\kappa}{2\pi i}\int_{-\infty}^{\infty}(\tilde{K}R_1(\pm 0+i\gamma)Q_2u,\ \tilde{K}R(\mp 0+i\gamma)^* v)d\gamma .$$

To see the convergence of these integrals, we have to investigate the behavior of $(1-\kappa G(\lambda))^{-1}$ near $\lambda = \pm 0 \in \mathbb{C}_{\pm}$. We put $N_iG_{ij}(\lambda)N_j = G_{ij}(\lambda)$, $i = 1,2$. Then $G_{ij}(\lambda)$'s have the following forms:

$$G_{11}(\lambda) = \{-a\log\lambda - ab - g_1(\lambda)\}N_1 ,$$

$$G_{12}(\lambda) = G_{21}(\bar{\lambda})^* = N_1K_0N_2 + N_1G_0(\lambda)N_2 ,$$

$$G_{22}(\lambda) = N_2K_0N_2 + N_2G_0(\lambda)N_2 ,$$

$$|g_1(\lambda)| \leq \frac{1}{2}a^2|\lambda| , \quad \|N_iG_0(\lambda)N_j\| \leq \frac{1}{2}a^2|\lambda| .$$

Let us assume that $\kappa > 0$ and $\kappa^{-1} \notin \sigma(N_2K_0N_2)$. Then for sufficiently small $\lambda \in \mathbb{C}_+$, there exists $(1 - \kappa G_{22}(\lambda))^{-1} \in \mathbb{B}(\mathcal{H})$ with uniformly bounded norm. Hence we have

$$\|(1 - \kappa G(\lambda))^{-1}u\| \leq \frac{c_1}{2-\log|\lambda|}\|N_1u\| + c_2\|N_2u\|$$

for sufficiently small $\lambda \in \mathbb{C}_+$ (and hence for small $\lambda \in \mathbb{C}_-$) . This implies

$$\|\tilde{K}R_1(\lambda)u\| \leq \frac{c_1}{2-\log|\lambda|}\|N_1\tilde{K}R(\lambda)u\| + c_2\|N_2\tilde{K}R(\lambda)u\|$$

for sufficiently small $\lambda \in \mathbb{C}_\pm$. Thus the above integrals converge absolutely, and $W_\pm$, $Z_\pm \in B(H_0)$. Following Kato's argument, we can easily see that

(13) $Z_\pm W_\pm = 1$, $W_\pm Z_\pm = Q_2$

$$(\lambda - B_2)W_\pm = W_\pm(\lambda - A_0)^{-1} \quad \text{i.e.} \quad B_2 = W_\pm A_0 Z_\pm .$$

(14) $e^{tB_2} = W_\pm e^{tA_0} Z_\pm$.

Thus we have

Theorem 2. Let $\kappa > 0$ and $\kappa^{-1} \notin \sigma(N_2K_0N_2)$. Then A_0 and $B_2 = B_0Q_2$ are similar to each other. That is, $W_\pm$ and $Z_\pm \in \mathbb{B}(H_0)$ exist and satisfy (13) and (14). Furthermore we have

$$W_\pm = s - \lim_{t\to\pm\infty} Q_2 e^{tB_0} e^{-tA_0} ,$$

$$Z_\pm = s - \lim_{t\to\pm\infty} e^{tA_0} e^{-tB_0} Q_2 .$$

If we put $F(\Delta) = W_\pm(\Delta)E_0(\Delta)Z_\pm(\Delta)$, $\Delta = (a,b)$, then $F(\Delta)$ is the "spectral resolution" of B_2, i.e.,

$$B_0 = i \int_{-\infty}^{\infty} \lambda dF(\lambda) + \Sigma \beta_j P_j .$$

References

[1] Kato, T., Wave operators and similarity for some non-self-adjoint operators, Math. Ann., 162(1966), 258-279.

[2] Lehner, J. and Wing, G. M., On the spectrum of an unsymmetric operator arising in the transport theory of neutrons, C.P.A.M., 8(1955), 217-234.

[3] ——————, Solution of the linearized Boltzmann transport equation for the slab geometry, Duke Math. J., 23(1956), 125-142.

[4] Lehner, J., The spectrum of the neutron transport operator for the infinite slab, J. Math. Mech. 11(1962), 173-181.

[5] Asano, K., Sizuta, Y. and Ukai, S., Operator theoretical approach for transport equations, RIMS-Kōkyūroku. 164, 56-106.