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On the propagation of Error in numerical Integrations

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§0 Introduction

Even with quite simple differential equations, it can happen that their solutions are not expressible in a closed form and that a numerical approach is the most convenient way to deal with the problem.

In this case, if an approximate value y_n of the solusion $y(x)$ of a differential equation at the point x_n has been calculated by some approximate methods, the estimate on the magnitude of error

$$(0,1) \qquad e_n = y_n - y(x_n) \qquad (n=1,2,3,\cdots\cdots)$$

is of great importance.

While we posses simple and useful error estimate for the propagation of error, it seems, however, that if we concern with the problem of asymptotic behavior of the propagation of error in infinite interval, not so many resuls appeared.

The purpose of this paper is to state some results on a propagation of error of some approximate equations.

§1

First we consider the first order differencial equation;

$$(1.1)\qquad \begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

We shall try to approximate the equation (1.1) by the difference equation :

$$(1.2)\qquad y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}).$$

which is known as Euler's method.

In actual calculation, the calculated value of y_n is given by the formula;

$$(1.3)\qquad y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) - R_n.$$

(R_n; roun-off error)

On the other hand, if we denote the true value of the solution of (1.1) at the point $x = x_n$ by $y(x_n)$, we have also the relation ;

$$(1.4)\qquad y(x_n) = y(x_{n-1}) + hf(x_{n-1}, y(x_{n-1})) + \frac{1}{2}h^2 y''(\xi_{n-1})$$

where $\xi_{n-1} = x_{n-1} + \theta_{n-1}h \quad (0 \leqq \theta_{n-1} \leqq 1)$.

If we substract (1.3) from (1.4) and write

$$(1.5)\qquad E_n = R_n + \frac{1}{2}h^2 y(\xi_{n-1})$$

$$e_n = y(x_n) - y_n \quad ; \quad e_0 = 0,$$

we find the difference equation ;

$$(1.6) \qquad e_n = e_{n-1} + h\left(f(x_{n-1}, y(x_{n-1})) - f(x_{n-1}, y_{n-1})\right) + E_n.$$

We notice first that we may write

$$f(x_{n-1}, y(x_{n-1})) - f(x_{n-1}, y_{n-1}) = f_y(x_{n-1}, \eta_{n-1})(y(x_{n-1}) - y_{n-1})$$

if f_y exists, where η_n is a number between y_n and $y(x_n)$, so that (1.6) may be written in the form;

$$(1.7) \qquad e_n = e_{n-1} + h e_{n-1} f_y(x_{n-1}, \eta_{n-1}) + E_n,$$

or

$$(1.8) \qquad \nabla e_n = A_n e_{n-1} + (h f_y(x_{n-1}, \eta_{n-1}) - A_n) e_{n-1} + E_n.$$

Here we discuss respectively the asymptotic behavior of the solution of (1.7) and (1.8).

Theorem [1].

Consider the difference equation (1.7) under the assumptions;

(1) $\left|\frac{d}{dx} f(x, y(x))\right| \leq \Psi(x) \qquad (-\infty < y < +\infty)$

$$\sum_{\nu=0}^{\infty} \{h^2 \Psi(x_0 + (\nu + \theta_\nu)h) + |R_\nu|\} \leq L \quad \text{for } 0 < h < h_0,$$

(2) $|f_y(x, y)| \leq \Phi(x) \qquad (-\infty < y < +\infty)$

where $\exists M$, $\exists h_1, > 0$

$$\sum_{\nu=0}^{\infty} h\, \Phi(x_0 + \nu h) \leq \frac{C-L}{C} \quad \text{for } 0 < h < h_1,$$

(3) $e_0 = 0$,

then we have

$$|e_n| \leq C \quad (L < C) \quad \text{for } 0 < h \leq \min\{h_0, h_1\}.$$

Proof.

The proof is derived by mathematical inducti[on].

Let us assume

$$|e_\nu| \leq C \qquad (\nu = 0, 1, 2, \cdots, n-1)$$

and we shall show

$$|e_n| \leq C .$$

From (1.7) and the hypothesis of the induction, we have

$$|e_n| \leq \sum_{\nu=0}^{n-1} |h f_y(x_0 + \nu h, \eta_\nu) e_\nu| + \sum_{\nu=0}^{n-1} |E(x_0 + (\nu+1)h)|$$
$$\leq hc \sum_{\nu=0}^{n-1} |f_y(x_0 + \nu h, \eta_\nu)| + L,$$

and taking h so small that conditions (1) and (2) are satisfied,

$$|e_n| \leq C \quad . \qquad \text{Q. E. D.}$$

Next we shall show that, under certain conditions, the solution of difference equation (1.8) monotonically decreases as $n \to \infty$. Before stating the Theorem, we shall given a Lemma.

Lemma [1.1].

The solution of the equation;

$$\nabla z(x_0 + nh) = A_n z(x_0 + (n-1)h) + B(x_0 + (n-1)h) z(x_0 + (n-1)h)$$
$$+ w(x_0 + nh)$$

is given by

$$z(x_0+nh) = z(x_0)\,y(x_0+nh) + \sum_{\nu=0}^{n-1} y(x_0+nh)\,y^{-1}(x_0+(\nu+1)h)\,B(x_0+\nu h)\,z(x_0+\nu h)$$
$$+ \sum_{\nu=0}^{n-1} y(x_0+nh)\,y^{-1}(x_0+(\nu+1)h)\,w(x_0+(\nu+1)h),$$

where $y(x)$ is a solution of the following equation;

$$\begin{cases} \nabla y(x_0+nh) = A_n\, y(x_0+(n-1)h) \\ y(x_0) = 1 \end{cases}$$

Proof.

The proof proceeds by the well known method, namely the variation of parameters.

Let
$$z(x_0+nh) = y(x_0+nh)\,u(x_0+nh),$$
then
$$\nabla z(x_0+nh) = \nabla y(x_0+nh)\,u(x_0+nh)$$
$$= u(x_0+(n-1)h)\,\nabla y(x_0+nh) + y(x_0+nh)\,\nabla u(x_0+nh),$$
and
$$\nabla y(x_0+nh) = A_n\, y(x_0+(n-1)h).$$
Thus
$$A_n\, y(x_0+(n-1)h)\,u(x_0+(n-1)h) + y(x_0+nh)\,\nabla u(x_0+nh)$$
$$= A_n\, y(x_0+(n-1)h)\,u(x_0+(n-1)h) + B(x_0+(n-1)h)\,y(x_0+(n-1)h)\,u(x_0+(n-1)h)$$
$$+ w(x_0+nh),$$
and hense
$$\nabla u(x_0+nh) = y^{-1}(x_0+nh)\,B(x_0+(n-1)h)\,z(x_0+(n-1)h)$$
$$+ y^{-1}(x_0+nh)\,w(x_0+nh).$$

From the above equation, we have
$$u(x_0+nh) = u(x_0) + \sum_{\nu=0}^{n-1} y^{-1}(x_0+(\nu+1)h)\,B(x_0+\nu h)\,z(x_0+\nu h)$$
$$+ \sum_{\nu=0}^{n-1} y^{-1}(x_0+(\nu+1)h)\,w(x_0+(\nu+1)h),$$

where

$$Z(x_0) = Y(x_0)U(x_0) = U(x_0).$$

Thus, we have the solution;

$$\begin{aligned} Z(x_0+nh) &= Y(x_0+nh)U(x_0+nh) \\ &= Z(x_0)Y(x_0+nh) + Y(x_0+nh)\sum_{\nu=0}^{n-1} Y^{-1}(x_0+(\nu+1)h)B(x_0+\nu h)Z(x_0+\nu h) \\ &\quad + Y(x_0+nh)\sum_{\nu=0}^{n-1} Y^{-1}(x_0+(\nu+1)h)\omega(x_0+(\nu+1)h) \end{aligned}$$

Q.E.D.

Theorem [2].

Suppose that there exist constants

$\exists M>0$, $\exists h_0>0$, $\exists h_1>0$, $\exists\{\lambda_\nu\}$, $\exists\{a_\nu\}$, $\exists L_1>0$, $\exists L_2>0$,

$\exists L_3>0$, $(L_1+L_2<1)$, which satisfy the following conditions

(1). $0<|1+A_{\nu+1}|\le e^{\lambda_\nu}$ $\quad(\nu=1,2,\cdots)$

$$\sum_{\nu=1}^{n-1}|A_{\nu+1}|\le L_1 e^{\lambda_n},$$

(2) $|f_y(x,y)|\le \Phi(x)$ $\quad(-\infty<y<\infty)$

$$\sum_{\nu=0}^{n-1} h\Phi(x_0+\nu h)\le(1-L_1-L_2)e^{\lambda_n} \qquad \text{for } 0<h\le h_0,$$

(3) $\left|\frac{d}{dx}f(x,y(x))\right|\le\psi(x)$ $\quad(-\infty<y<\infty)$

where

$$\frac{1}{2}h^2\psi(x_0+(\nu+\theta_\nu)h)+R_{\nu+1}\le a_\nu L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_\nu}$$

$$\sum_{\nu=0}^{n-1}a_\nu\le L_2e^{\lambda_n}, \qquad \text{for } 0<h\le h_1$$

(4) $e_0=0$,

then we have

$$|e_n|\le L_3e^{\lambda_1+\lambda_2+\cdots+\lambda_n} \qquad (n=1,2,\cdots)$$

for $0<h\le\min\{h_0,h_1\}$.

Proof.

The proof is carried out by mathematical induction. For the case $n=1$, the proposition is clearly true, and for the case $n=2$, we have

$$|e_2| \leq h\,|e_1 f_y(x_1, \eta_1)| + |E_1| + |E_2|$$
$$\leq h\,|e_1|\,\Phi(x_1) + (a_0 + a_1)\,e^{\lambda_1}$$
$$\leq L_3 e^{\lambda_2}.$$

Let us assume

$$|e_m| \leq L_3\, e^{\lambda_1+\lambda_2+\cdots+\lambda_m} \qquad (m = 1, 2, \cdots, n-1),$$

and we shall show that the above inequality holds for $m=n$ as well.

From Lemma [1.1], we have

$$|e_n| \leq |y(x_0+nh)\sum_{\nu=0}^{n-1} y^{-1}(x_0+(\nu+1)h)(hf_y(x_0+\nu h, \eta_\nu)-A_{\nu+1})e_\nu|$$
$$+\,|y(x_0+nh)\sum_{\nu=0}^{n-1} y^{-1}(x_0+(\nu+1)h)E_{\nu+1}|.$$

Thus from the conditions (1), (2), and the hypothesis of induction

$$|y(x_0+nh)\sum_{\nu=0}^{n-1} y^{-1}(x_0+(\nu+1)h)(hf_y(x_0+\nu h, \eta_\nu)-A_{\nu+1})e_\nu|$$
$$\leq h\,|y(x_0+nh)\sum_{\nu=0}^{n-1} y^{-1}(x_0+(\nu+1)h)e_\nu\,\Phi(x_0+\nu h)|$$
$$+\,|y(x_0+nh)\sum_{\nu=0}^{n-1} y^{-1}(x_0+(\nu+1)h)|\,|A_{\nu+1}|\,|e_\nu|$$
$$\leq L_3(1-L_2)\,e^{\lambda_1+\lambda_2+\cdots+\lambda_n},$$

and by making (2) and (3),

$$|y(x_0+nh)\sum_{\nu=0}^{n-1} y^{-1}(x_0+(\nu+1)h)E_{\nu+1}|$$
$$\leq |E_1\prod_{\nu=2}^{n}(1+A_\nu)| + |E_2\prod_{\nu=3}^{n}(1+A_\nu)| + \cdots + |E_n|$$

$$\leq L_2 L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_m}.$$

Hence we have the inequality

$$|e_m| \leq L_3^{\lambda_1+\lambda_2+\cdots+\lambda_m}$$

Q. E. D.

Remark.

The following example satisfies the conditions of Theorem [2],

$$y' = \frac{e^{-y^2}}{(x^2+1)}$$

In Theorem [2], if we take the constants A_m as

$$-1 < A_m < A_{m+1} < 0$$

and

$$|1+A_m| = e^{-\lambda_{m-1}},$$

then

$\lambda_{m-1} > \lambda_m > 0$, and we have the following result.

Corollary.

Under the same assumptions as in Theorem [2] on the constants $^\exists M$, $^\exists h_0$, $^\exists\{a_\nu\}_{\nu=0}^{\infty}$, $^\exists L_1$, $^\exists L_2$, $^\exists L_3$, if the following conditions are satisfied

(1) $|1+A_{\nu+1}| = e^{-\lambda_\nu}$, $\sum_{\nu=1}^{n-1} |A_{\nu+1}| \leq L_1^{-\lambda_m}$, $-1 < A_\nu < 0$,

(2) $|f_y(x, y)| \leq \Phi(x)$ $(-\infty < y < \infty)$

$\sum_{\nu=1}^{n-1} h\Phi(x_0+\nu h) \leq (1-L_1-L_2)e^{-\lambda_m}$ for $0<h\leq h_0$,

(3) $\left|\frac{d}{dx} f_y(x, y)\right| \leq \Psi(x)$ $(-\infty<y<\infty)$

where

$$\frac{1}{2}h^2\psi(x_0+(\nu+\theta_\nu)h)+R_{\nu+1}\leq a_\nu L_3 e^{-\lambda_1-\lambda_2-\cdots-\lambda_\nu}$$
for $0<h\leq h_1$

$$\sum_{\nu=1}^{n-1} a_\nu \leq L_2 e^{-\lambda_n} \qquad (\lambda_\nu>0.\ \nu=1,2,\cdots),$$

then
$$|e_n|\leq L_3 e^{-\lambda_1-\lambda_2-\cdots-\lambda_n} \qquad \text{for } 0<h\leq \min\{h_0,h_1\}.$$

In the above corollary, for instance, if we take the constants A_n as

$$A_m=\frac{-1}{(\rho+m-1)^\alpha} \qquad (m=1,2,3,\cdots)\ (\rho>1:\text{constant}).$$

and
$$\alpha>\frac{e^{\lambda_1}}{L_1}\rho+1$$

then the condition (1) is satisfied.

§2

In §1, we consider the propagation error of the open type formula, ie Adams-Bash formula type

$$(2.1) \qquad y(x_{n+1})=y(x_n)+hf(x_{n+1},y(x_{n+1}))+T_{n+1}$$
$$(T_{n+1}=\tfrac{1}{2}h^2y''(x_n+\theta_n h),\ 0\leq\theta_n\leq 1)$$
$$Y_{n+1}^{(i+1)}=Y_n+hf(x_{n+1},Y_{n+1}^{(i)}), \qquad (n=0,1,2,\cdots)$$

where we denote the i-th approximation to $y(x_n)$ by $Y_n^{(i)}$ and the truncation error of (i+1)th iteration of n step by $T_n^{(i+1)}$.

And the calculated value of y_n based on (2.1) will be given by the formula;

$$(2.2) \qquad Y_{n+1}^{(i+1)} = Y_n + h f(x_{n+1}, Y_{n+1}^{(i)}) - R_{n+1}^{(i+1)}$$

where we denote the round-off error corresponding to (i+1)th iteration of n step by $R_n^{(i+1)}$.

If the difference

$$L_n^{(i+1)} = Y_n^{(i+1)} - Y_n^{(i)}$$

is smaller than the constant L:

$$|L_n^{(i+1)}| \leq L$$

then we set

$$Y_{n+1} = Y_{n+1}^{(i+1)}.$$

From the equation (2.1), (2.2), we obtain the relation

$$y(x_{n+1}) - Y_{n+1}^{(i+1)} = y(x_n) - Y_n + h f(x_{n+1}, y(x_{n+1}))$$
$$- h f(x_{n+1}, Y_{n+1}^{(i)}) + T_{n+1} + R_{n+1}^{(i+1)}.$$

If $\frac{\partial f}{\partial y}(x, y)$ exists, we have the relations,

$$f(x_{n+1}, \eta(x_{n+1})) - f(x_{n+1}, Y_{n+1}^{(i)}) = \frac{\partial f}{\partial y}(x_{n+1}, \eta_{n+1})(y(x_{n+1}) - Y_{n+1}^{(i)}),$$

for some η_{n+1} which lies between $Y_{n+1}^{(i)}$ and $y(x_{n+1})$.

Setting

$$e_n = y(x_n) - Y_n,$$

$$(2.3) \qquad W_{n+1}^{(i)} = T_{n+1} + R_{n+1}^{(i)} + h\frac{\partial f}{\partial y}(x_{n+1}, \eta_{n+1})e_{n+1} + W_{n+1}^{(i)}$$

And by using the backward difference operator, the above equation may be written in the form

$$(2.4) \qquad \nabla e_{n+1} = A_{n+1} e_{n+1} + \left(h\frac{\partial f}{\partial y}(x_{n+1}, \eta_{n+1}) - A_{n+1}\right) e_{n+1} + W_{n+1}^{(1)}$$

where A_n is some constant.

Here we discuss the asymptotic behavior of the difference equations (2.3), (2.4).

Theorem 3.

Under the following assumptions,

(1) $\quad |f_y(x, y)| \le \Phi(x) \qquad (-\infty < y < +\infty)$

where $\Phi(x)$ is a continuous function satisfying the following condition

$$\exists K > 0, \ \exists h > 0 \quad \sum_{\nu=1}^{\infty} h\Phi(x_0 + \nu h) \le \frac{C-E}{C}$$

for $0 < h \le h_0$, and $\Phi(x) \le M$,

(2) $\quad \sum_{\nu=1}^{\infty} |W_\nu^{(1)}| \le E \qquad$ for $\ 0 < h \le h_1$,

we have

$$|e_n| \le C(<E) \qquad \text{for} \ \ 0 < h \le \min\{h_0, h_1, \tfrac{1}{M}\}.$$

Proof.

The proof is derived by mathematical induction.

Let us assume

$$|e_\nu| \le C \qquad (\nu = 1, 2, 3, \cdots n-1),$$

and we shall show

$$|e_n| \le C.$$

From (4), we have the inequality

$$|(1 - hf_y(x_0 + nh, \eta_n))e_n| < h\sum_{\nu=1}^{n-1} |f_y(x_0 + \nu h, \eta_\nu)e_\nu| + \sum_{\nu=1}^{n} |W_\nu^{(1)}|,$$

and taking the constant h small, we have

$$|\dot{e}_n| \le C$$

Q. E. D.

Next we shall investigate the propagation of error more explicity.

Lemma [2.1]

The solution of the equation

$$\nabla z(x_0+nh) = A_n z(x_0+nh) + B(x_0+nh)z(x_0+nh) + W(x_0+nh)$$
$$(n=1,2,3,\cdots).$$
$$z(x_0) = z_0, \quad (A_n \neq 1 : n=1,2,\cdots),$$

is given by

$$z(x_0+nh) = z_0 Y(x_0+nh) + Y(x_0+nh)\sum_{\nu=0}^{n-1}\{B(x_0+(\nu+1)h)Y^{-1}(x_0+\nu h)z(x_0+(\nu+1)h)\}$$
$$+ Y(x_0+nh)\sum_{\nu=0}^{n-1} Y^{-1}(x_0+\nu h)\, W(x_0+(\nu+1)h),$$

where $Y(x)$ is the solution of the following difference equation

$$\begin{cases} \nabla Y(x_0+\nu h) = A_\nu Y(x_0+\nu h) & (\nu=1,2,\cdots), \\ Y(x_0)=1. \end{cases}$$

Proof.

we can derive the proof in a simillar fashion of Lemma [1.1].

Theorem [4].

Suppose that there exist constants $^\exists K>0$, $^\exists h_0>0$, $^\exists h_1>0$, $^\exists\{\lambda_\nu\}$, $^\exists\{a_\nu\}$, $^\exists L_1>0$, $^\exists L_2>0$, $^\exists L_3$ $(L_1+L_2<1)$ which satisfy the following conditions

(1) $\quad 0 < |1+A_\nu|^{-1} \le e^{\lambda_{\nu+1}}$

$$\sum_{\nu=1}^{\infty} |A_\nu| \le L_2$$

(2) $\quad |f_y(x, y)| \le \Phi(x) \qquad (-\infty < y < +\infty)$

$$\sum_{\nu=0}^{\infty} h\,\Phi(x_0+\nu h) \le 1 - L_1 - L_2 \qquad \text{for} \quad 0 < h \le h_0$$

(3) $\quad \ldots))| \le \Psi(x) \qquad (-\infty < y < +\infty)$

where

$$\frac{1}{2}h^2\Psi(x_0+(\nu+\theta_\nu)h) + R_{\nu+1} \le a_\nu L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_\nu} \qquad \text{for} \quad 0 < h \le h_1$$

$$\sum_{\nu=1}^{\infty} a_\nu \le L_1$$

(4) $\quad e_0 = 0$

then we have

$$|e_n| \le L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_n}$$

for $\quad 0 < h \le \min\{h_0, h_1\}$.

Proof.

The proof is carried out by mathematical induction. For the case $n=1$ the proposition is clearly true, and for the case $n=2$, we have

$$|e_2| \le \frac{1}{|1-hf_y(x_2,\eta_2)|}\left\{\frac{|hf_y(x_1,\eta_1)-A_1|}{|1-A_1|}|e_1| + \frac{|W_1|}{|1-A_1|} + |W_2|\right\}$$

$$\le \frac{1}{|1-hf_y(x_2,\eta_2)|}\left\{|hf_y(x_1,\eta_1)-A_1|L_3e^{\lambda_1+\lambda_2} + (a_1+a_2)L_3e^{\lambda_1+\lambda_2}\right\}$$

$$\le L_3 e^{\lambda_1+\lambda_2}.$$

Let us assume

$$|e_m| \le L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_m} \qquad (m=1, 2, \cdots, n-1),$$

and we shall show that the above inequality holds for $m=n$ as well.

From Lemma [2.1], we have

$$\{1-y(x_0+nh)y^{-1}(x_0+(n-1)h)(hf_y(x_n,\eta_n)\}e_n$$
$$= e_0 y(x_0+nh) + y(x_0+nh)\sum_{\nu=0}^{n-2}(hf_y(x_{\nu+1},\eta_{\nu+1})-A_{\nu+1})y^{-1}(x_0+\nu h)e_{\nu+1}$$
$$+ y(x_0+nh)\sum_{\nu=0}^{n-1} y^{-1}(x_0+\nu h)W_{\nu+1}$$

Thus from the condition (1) (3) and the hyphothesis of induction

$$|y(x_0+nh)\sum_{\nu=0}^{n-2}(hf_y(x_{\nu+1},\eta_{\nu+1})-A_{\nu+1})y^{-1}(x_0+\nu h)e_{\nu+1}|$$
$$+|y(x_0+nh)\sum_{\nu=0}^{n-1}y^{-1}(x_0+\nu h)W_{\nu+1}|$$
$$\leq L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_n}|1+A_n|^{-1}\{\sum_{\nu=0}^{n-2}|hf_y(x_{\nu+1},\eta_{\nu+1})-A_{\nu+1}|+\sum_{\nu=0}^{n-1}a_\nu\}.$$

And by making (1), (2), (3), we have the inequality,

$$|e_n| \leq L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_n}$$

Q.E.D.

In the above theorem, if we take the constant

$$-1<A_\nu<0, \quad \text{and}, \quad |1+A_\nu| = e^{-\lambda_{\nu+1}},$$

then $\lambda_\nu<0$, and we have the following result.

Corollary

Under the same assumptions as in Theorem [4] on the constants $^\exists k>0$, $^\exists h_1>0$, $^\exists h_2>0$, $^\exists\{\lambda_\nu\}$, $^\exists\{a_\nu\}$, $^\exists L_1>0$, $^\exists L_2>0$, $^\exists L_3>0$, $(L_1+L_2<1)$ if the following conditions are satisfied

(1) $|1+A_\nu| = e^{-\lambda_{\nu+1}}$, $-1<A_\nu<0$

$$\sum_{\nu=1}^{\infty} |A_\nu| \le L_2 ,$$

(2) $\quad |f_y(x, y)| \le \Phi(x) \qquad (-\infty < y < +\infty),$

$$\sum_{\nu=0}^{\infty} h\Phi(x_0+\nu h) \le 1-L_1-L_2 \qquad \text{for } 0<h\le h_1,$$

(3) $\quad |\frac{d}{dx} f(x, y(x))| \le \Psi(x) \qquad (-\infty < y < +\infty)$

where

$$\frac{1}{2} h^2 \Psi(x_0+(\nu+\theta_\nu)h) + R_{\nu+1} \le a_\nu L_3 e^{-\lambda_1-\lambda_2-\cdots-\lambda_\nu},$$

$$\sum_{\nu=1}^{\infty} a_\nu \le L_1 ,$$

(4) $\quad e_0 = 0,$

then we have

$$|e_n| \le L_3 e^{-\lambda_1-\lambda_2-\cdots-\lambda_n} \qquad (n=1, 2, 3, \cdots)$$

for $0<h \le \min\{h_1, h_2\}$.

§3

In §1, §2, we consider the propagation of error which based on a special approximation method, and it will be investigate in this section the propagation of error which is caused by the general one step method. General one step method may be regarded as a generaligation of EULER's method, but, indeed it is more effective than EULER's method.

The general one step formula for the solution of differential equation, if we use the same notation of §1, §2, can be written in the form;

$$(3.1)\qquad y_{n+1} = y_n + h\Phi(x_n, y_n; h)$$

where $\Phi(x_n, y_n; h)$ is a function of the arguments x_n, y_n, h.

We now define the function Δ by $(x, y; h)$ by

$$\Delta(x, y; h) = \begin{cases} \dfrac{z(x+h)-z(x)}{h} & (h \neq 0) \\ f(x, y) & (h = 0), \end{cases}$$

where the function $z(x)$ is a solution of the differential equation $z' = f(x, z)$, $z(x_0) = z_0$.

Then the solution of the differential equation

$$\begin{cases} y' = f(x, y) \\ y(x_0) = z_0 \end{cases}$$

is given by the equation

$$(3.2)\qquad y(x_{n+1}) = y(x_n) + h\Delta(x_n, y(x_n); h).$$

If the approximate value are calculated from (3.1), we have the relation, saltracting (3.2) from (3.1),

$$y_{n+1} - y(x_{n+1}) = y_n - y(x_n) + h\{\Phi_f(x_n, y_n; h) - \Delta(x_n, y_n; h)\} + R_{n+1}$$

and by setting

$$e_n = y_n - y(x_n)$$

$$(3.3)\qquad e_{n+1} = e_n + h\{\Phi_f(x_n, y_n; h) - \Delta(x_n, y(x_n); h)\} + R_{n+1}$$

where, we may notice that we may write

$$\Phi(x_n, y_n; h) - \Delta(x_n, y_n; h)$$
$$= \{\Phi(x_n, y_n; h) - \Phi(x_n, y(x_n); h) + \Phi(x_n, y(x_n); h) - \Delta(x_n, y_n; h)\}$$

If $\Phi_y(x, y; h)$ is continuous

$$(3.4) \quad \Phi(x_n, y_n; h) - \Phi(x_n, y(x_n); h) = \Phi_y(x_n, \eta_n; h) e_n$$

where η_n is between y_n and Y_n.

We now state the definition of approximation.

Definition

The formula (3.1) will be said to be of degree p if there exists a function such that

$$(3.5) \quad \Phi(x_n, y(x_n); h) - \Delta(x_n, y(x_n); h) = h^p \varphi(x_n, y(x_n)) + O(h^p) \equiv T_{n+1}.$$

From (3.4), (3.5), (3.6), we have

$$e_{n+1} = e_n + h\{\Phi_y(x_n, \eta_n; h) e_n + T_n\} + R_{n+1}$$

or

$$(3.6) \quad \Delta e_n = \rho_n e_n + \{h\Phi_y(x_n, \eta_n; h) - \rho_n\} e_n + E_{n+1}$$

where

$$E_{n+1} = h T_{n+1} + R_{n+1}.$$

And we shall consider the asymptotic behavior of the differential equation (3.6).

Theorem [5].

Consider the difference equation (3.6) under the assumptions

(1) $\exists \psi(x, h)$, $\exists C > 0$, $\exists L > 0$, $\exists h_1 > 0$,

$$\Phi_y(x; y, h) \le \psi(x, h) \qquad (-\infty < y < +\infty)$$

$$\sum_{\nu=0}^{\infty} h\psi(x_0 + \nu h, h) \le \frac{C-L}{C} \qquad (C > L) \text{ for } 0 < h \le h_1,$$

(2) $\sum_{\nu=0}^{\infty}\{|T_\nu| + |R_\nu|\} \leq L$ for $0<h\leq h_2$,

(3) $e_0 = 0$,

then we have

$$|e_n| \leq C \qquad (n=1,2,\cdots\cdots).$$

Proof.

The proof is derived by mathematical induction.

Let us assume

$$|e_\nu| \leq C \qquad (\nu=1,2,\cdots,n-1),$$

and we shall show

$$|e_n| \leq C.$$

From (3.6) we have

$$\begin{aligned} |e_n| &\leq \left|\sum_{\nu=0}^{n-1} h\Phi_y(x_\nu, \eta_\nu, h)\right||e_\nu| + \sum_{\nu=0}^{n-1}|E_{\nu+1}| \\ &\leq \sum_{\nu=0}^{n-1} h\,|\psi(x_0+\nu h, h)|\,C + \sum_{\nu=0}^{n-1}|E_{\nu+1}| \\ &= C\cdot\frac{C-L}{C} + L \\ &= C. \end{aligned}$$

Q. E. D.

Next we consider the solution of difference equation (3.6), more explicitly.

Before stating Theorem, we shall give a Lemma.

Lemma [3.1].

The solution of the equation ;

$$\Delta Z(x_0+nh) = A_n Z(x_0+nh) + B(x_0+nh) Z(x_0+nh) + w(x_0+(n+1)h)$$

$$Z(x_0) = Z_0$$

is given by

$$Z(x_0+nh) = Z_1 + \sum_{\nu=0}^{n-1} Y(x_0+nh) Y^{-1}(x_0+\nu h) B(x_0+\nu h) Z(x_0+\nu h)$$
$$+ \sum_{\nu=0}^{n-1} Y(x_0+nh) Y^{-1}(x_0+\nu h) W(x_0+(\nu+1)h),$$

where $Y(x)$ is a solution of the following equation:

$$\begin{cases} \Delta Y(x_0+nh) = A_n Y(x_0+nh) & (A_n \neq -1) \\ Y(x_0) = 1 \end{cases}$$

Proof

The proof is derived by the variation of parameters.

Let
$$Z(x_0+nh) = Y(x_0+nh) U(x_0+nh)$$

then
$$\Delta Z(x_0+nh) = \Delta Y(x_0+nh) U(x_0+nh)$$
$$= U(x_0+nh) \Delta Y(x_0+nh) + Y(x_0+(n+1)h) \Delta U(x_0+n).$$

and
$$\Delta Y(x_0+nh) = A_n Y(x_0+nh).$$

Thus

$$\Delta Z(x_0+nh) = A_n U(x_0+nh) Y(x_0+nh) + Y(x_0+(n+1)h) \Delta U(x_0+nh)$$
$$= A_n Z(x_0+nh) + B(x_0+nh) Z(x_0+nh)$$
$$+ w(x_0+(n+1)h),$$

From the above equation, we have

$$U(x_0+nh) = U(x_0) + \sum_{\nu=0}^{n-1} \{ Y^{-1}(x_0+\nu h) B(x_0+\nu h) Z(x_0+\nu h)$$
$$+ Y^{-1}(x_0+\nu h) w(x_0+(\nu+1)h) \}.$$

Thus we have the solution;

$$\Xi(x_0+nh) = Y(x_0+nh)\,u(x_0+nh)$$

$$= \Xi(x_0)Y(x_0+nh) + Y(x_0+nh)\sum_{\nu=0}^{n-1}Y^{-1}(x_0+\nu h)B(x_0+\nu h)\Xi(x_0+\nu h)$$

$$+ Y(x_0+nh)\sum_{\nu=0}^{n-1}Y^{-1}(x_0+\nu h)\,w(x_0+(\nu+1)h)\}.$$

Q. E. D.

Theorem [6].

Suppose that there exist constants $^{\exists}M>0$, $^{\exists}h_0>0$, $^{\exists}h_1>0$, $^{\exists}\{\lambda_\nu\}$, $^{\exists}\{a_\nu\}$, $^{\exists}L_1>0$, $^{\exists}L_2>0$, $^{\exists}L_3>0$, $(L_1+L_2<1)$, which satisfy the following conditions

(1) $|1+\rho_\nu| \leq e^{\lambda_{\nu+1}}$ $\quad(\nu=1,2,\cdots\cdots)$,

(2) $^{\exists}\psi(x,h)$, $\quad ^{\exists}h_0>0$,

$$\Phi_y(x,y,h) \leq \psi(x,h)$$

$$\sum_{\nu=0}^{\infty} h\psi(x_0+\nu h, h) \leq (1-L_1-L_2) \quad \text{for } 0<h\leq h_0,$$

(3) $|E_\nu| \leq a_\nu L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_\nu}$ $\quad(\nu=1,2,\cdots\cdots)$.

for $0<h\leq h_1$,

$$\sum_{\nu=0}^{n-1} a_\nu \leq L_2$$

then we have

$$|e_n| \leq L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_n} \qquad (n=1,2,\cdots\cdots)$$

for. $0<h\leq \min\{h_0, h_1\}$.

Proof.

The proof is carried out by mathematical induction.

For the case $n=1$, we have

$$|e_1| \leq |e_0| + h|\Phi_y(x_0, \eta_0, h)||e_0| + |R_0|$$

$$\leq a_1 L_3 e^{\lambda_1} < L_3 e^{\lambda_1}.$$

Let us assume

$$|e_m| \leq L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_m} \qquad (m=1,2,\cdots,n-1)$$

and we shall show that the above inquality holds for $m=n$ as well.

From Lemma [3.1] we have

$$|e_n| \leq \left|\sum_{\nu=0}^{n-1} Y(x_0+nh)Y^{-1}(x_0+\nu h)(h\Phi_y(x_0+\nu h,\eta_\nu,h)-\rho_\nu)e_\nu\right|$$
$$+\left|\sum_{\nu=0}^{n-1} Y(x_0+\nu h)Y^{-1}(x_0+\nu h)E(x_0+(\nu+1)h)\right|.$$

From the condition (1), (2) and the hyphothesis of induction.

$$\sum_{\nu=0}^{n-1}|Y(x_0+nh)Y^{-1}(x_0+\nu h)(h\Phi_y(x_0+\nu h,\eta_\nu,h)-\rho_\nu)e_\nu|$$
$$\leq h\sum_{\nu=0}^{n-1}|Y(x_0+nh)Y^{-1}(x_0+\nu h)\Psi(x_0+\nu h,h)e_\nu|$$
$$+\sum_{\nu=0}^{n-1}|Y(x_0+nh)Y^{-1}(x_0+\nu h)\rho_\nu e_\nu|$$
$$\leq (1-L_1-L_2)L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_{n-1}}+L_1L_3e^{\lambda_1+\lambda_2+\cdots+\lambda_{n-1}}.$$

And by making (1), (3)

$$\left|\sum_{\nu=0}^{n-1} Y(x_0+nh)Y^{-1}(x_0+\nu h)E_{\nu+1}\right|$$
$$\leq L_2L_3e^{\lambda_1+\lambda_2+\cdots+\lambda_{n-1}}.$$

Hence we have the inequality

$$|e_n| \leq L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_n}.$$

Q. E. D.

§4

We intend, in this section, to generalize the result of §3. We shall now consider the general difference method;

$$(4.1)\quad \sum_{\nu=0}^{k} \alpha_\nu y_{n+\nu} = h\sum_{\nu=0}^{k} \beta_\nu f(x_{n+\nu}, y_{n+\nu}) - T_{n+1} \qquad (n=0,1,\cdots,N-k),\ (\alpha_k \neq 0)$$

with the initial conditions

$$y_K = y_{0,K} \qquad (K=0,1,\cdots,k-1)$$

where T_n implyes the trucation error.

And since (4.1) determine y_n in terms of k preceding value y_j, it is refered to as k-th order difference formula. In this meaning, we can say in othere words that one step method is a first order-difference formula. With the same notation of §1, §2, and §3, the calculated value of y_n will be given by the formula;

$$(4.2)\quad \sum_{\nu=0}^{k} \alpha_\nu Y_{n+\nu} = h\sum_{\nu=0}^{k} \beta_\nu f(x_{n+\nu}, Y_{n+\nu}) + R_{n+1},$$

where R_n is a round-off error.

If we substract (4.2) from (4.1) and write

$$e_n = Y_n - y_n, \quad E_{n+1} = T_{n+1} + R_{n+1},$$

then we have the following equation,

$$(4.3)\quad \sum_{\nu=0}^{k} \alpha_\nu e_{n+\nu} = h\sum_{\nu=0}^{k} \beta_\nu \{ f(x_{n+\nu}, Y_{n+\nu}) - f(x_{n+\nu}, y_{n+\nu})\} + E_{n+1},$$

and we notice first that we may write

$$f(x_{n+\nu}, Y_{n+\nu}) - f(x_{n+\nu}, y_{n+\nu}) = f_y(x_{n+\nu}, \eta_{n+\nu})(Y_{n+\nu} - y_{n+\nu})$$

If $\frac{\partial f}{\partial y}(x,y)$ exist and we may write (4.3) in the following form:

$$(4.4)\quad \sum_{\nu=0}^{k} \alpha_\nu e_{n+\nu} = h\sum_{\nu=0}^{k} \beta_\nu e_{n+\nu} f_y(x_{n+\nu}, \eta_{n+\nu}) + E_{n+1}.$$

or

$$\alpha_k e_{n+k} = -\sum_{\nu=0}^{k-1} \alpha_\nu e_{n+\nu} + h\sum_{\nu=0}^{k} \beta_\nu e_{n+\nu} f_y(x_{n+\nu}, \eta_{n+\nu}) + E_{n+1}$$

where $\eta_{n+\nu}$ is between $y_{n+\nu}$ and $Y_{n+\nu}$.

Further if we set

$$e_{n+p} = e_n^{(p+1)}$$

and

$$\mathbf{e}_n = (e_n^{(1)}, e_n^{(2)}, \cdots\cdots, e_n^{(k)}).$$

Then (4.4) may be rewritten

$$(4.5) \qquad \mathbf{e}_{n+1} = A\mathbf{e}_n + B(x_n, \eta_n; f_y)\mathbf{e}_{n+1} + C(x_n, f_y)\mathbf{e}_n + \mathbf{E}_{n+1}$$

where the matrix A, $B(x_n, f_y)$, $C(x_n, f_y)$ and $\mathbf{E}_n$ are the following formula;

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & \cdots & 0 & 1 \\ -\frac{\alpha_0}{\alpha_k} & -\frac{\alpha_1}{\alpha_k} & \cdots & \cdots & -\frac{\alpha_{k-1}}{\alpha_k} \end{pmatrix}, \qquad B(x_n, f_y) = \begin{pmatrix} & & 0 \\ & 0 & \vdots \\ & & 0 \\ 0 & \cdots\cdots 0 & a \end{pmatrix},$$

$$\left(a = \frac{\beta_k}{\alpha_k} h f_y(x_{n+k}, \eta_{n+k})\right)$$

$$C(x_n, f_y) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ -\frac{h}{\alpha_k}\beta_0 g_0, & -\frac{h}{\alpha_k}\beta_1 g_1 & \cdots\cdots & -\frac{h}{\alpha_k}\beta_{k-1} g_{k-1} \end{pmatrix} \quad (g_\mu = f_y(x_{n+\mu}, \eta_{n+\mu})),$$

$$\mathbf{E}_{n+1} = (0, \cdots\cdots, 0, E_{n+1}/\alpha_k).$$

If we use difference operator, (4.5) may be written in the form

$$(4.6) \qquad \Delta \mathbf{e}_n = (A - E)\mathbf{e}_n + B(x_n, f_y)\mathbf{e}_{n+1} + C(x_n, f_y)\mathbf{e}_n + \mathbf{E}_{n+1}.$$

Corresponding to the Theorem [5], and Theorem [6], we now state the Theorem refered to the difference formula (4.6). Proceding now to state the Theorem, we will present several Lemmas.

Lemma [4. I].

If all roots ζ_i of the characteristic polynomial of matrix A

(4.7) $$\rho(\zeta) = \sum_{\nu=0}^{k} \alpha_\nu \zeta^\nu$$

are such that $|\zeta_i| < 1$ and those roots for $|\zeta_i| = 1$ are simple. Then, there exists a constant G which depends only on the coefficients of (4.7) such that

$$|A^n| \leqq G \qquad (n = 0, 1, 2, \cdots\cdots),$$

where $|\ |$ denotess the norms and the norm may be taken in any definition, especially if for all roots of (4.7) one has $|\zeta_i| < 1$, then there exists constants G and r $(0 < r < 1)$ such that

$$|A^n| \leqq G r^n$$

Proof.

It is easily proved by using Jordans Canonical form for the matrix A.

When the difference formula (4.I) is explicite, ie $\beta_k = 0$ the difference formula (4.6) becomes the following formula;

(4.8) $$\Delta e_n = (A - E) e_n + C(x_n, f_y) e_n + E_{n+1}.$$

For matrix $C(x_n, f_y)$, one may derive

$$|C(x_n, f_y)| = h \sum_{i=0}^{k} \left| \frac{\beta_i g_i}{\alpha_k} \right|.$$

then if $g_i = f_y(x_{n+i}, \eta_{n+i})$ is bounded, we shall obtain the following results.

Theorem [7].

Consider the difference equation (4.8) under the assumptions;

(1) $|\frac{\partial f}{\partial y}(x, y)| \leq K \qquad (a \leq x, -\infty < y < +\infty)$,

(2) the polynomial (4.7) are such that for all roots one has $|\xi_i| < 1$,

(3) $\| E(x_0 + \nu h) \| \leq \frac{1-r}{Gr} C \qquad$ for $0 < h \leq h_1$.

then we have

$$\| e_n \| \leq 2C$$

for $0 < h \leq \min\{h_1, \frac{1-r}{2MGr}\}$.

Proof.

The proof is derived by mathematical induction.

Let us assume

$$\| e_\nu \| \leqq 2C \qquad (\nu = 1, 2, \cdots, n-1).$$

and we shall show

$$\| e_n \| \leqq 2C.$$

From Lemma [3.1], we have

$$\| e_n \| \leqq \| \sum_{\nu=0}^{n-1} Y(x_0 + nh) Y^{-1}(x_0 + \nu h) B(x_0 + \nu h) e_\nu \|$$
$$+ \| \sum_{\nu=0}^{n-1} Y(x_0 + nh) Y^{-1}(x_0 + \nu h) E(x_0 + (\nu+1)h) \|$$

where $Y(x)$ is a solution of the difference equation

$$\begin{cases} \Delta Y(x_0 + nh) = (A - E) Y(x_0 + nh) \\ Y(x_0) = E \end{cases}$$

From the condition (1), (2), (3) and the hypothes of induction, we have

$$\| e_n \| \leq 2C \| \sum_{\nu=0}^{n-1} A^{n-\nu} B_\nu \| + \sum_{\nu=0}^{n-1} \| A^{n-\nu} E(x_0+(\nu+1)h) \|$$

$$\leq 2hCM \frac{Gr}{1-r} + \frac{1-r}{Gr} \cdot C \cdot \frac{r}{1-r} G$$

and taking h small, we have

$$\| e_n \| \leq 2C .$$

When all roots ζ_i of the polynomial (4.7) are such that $|\zeta_i|<1$ and those roots for $|\zeta_i|=1$ are simple, the same difficulties are arose. But we may derive easily the condition, under which the solution of the difference equation (4.6) is uniformly bounded with respect to n. In fact, the results are similar to those of Theorem [5], and it does not seem of enough interest to warrant detailed discussion. Thus we shall be content with this few remarks.

At last we shall state the results corresponding to the Theorem [6]. (4.8) may be written in the form;

$$(4.9) \quad \Delta e_m = (A - E - P_m) e_n + (\mathbb{B}(x_m, f_y) + P_n) e_m + \mathbb{E}_{m+1},$$

and we discuss the asymptotic behavior of the solution of (4.9).

Theorem [8].

Suppose that there exist constants $^\exists h_0>0$, $^\exists h_1>0$, $^\exists\{\lambda_\nu\}$, $^\exists\{a_\nu\}$, $^\exists L_1>0$, $^\exists L_2>0$, $^\exists L_3$, $(L_1+L_2<1)$ which satisfy the following conditions,

(1) $\| A - P_\nu \| \leq e^{\lambda_{\nu+1}}$ $\quad (\nu = 1, 2, 3, \cdots)$

$\| \sum_{\nu=0}^{\infty} P_\nu \| \leq L_2$,

(2) $\exists \Phi(x)$ $\quad (-\infty < y < +\infty)$

$\| B(x_0 + \nu h ; \eta_\nu , h) \| \leq \Phi(x_0 + \nu h , h)$

$\sum_{\nu=0}^{\infty} \Phi(x_0 + \nu h , h) \leq (1 - L_1 - L_2)$ for $0 < h \leqq h_1$,

(3) $\| E_\nu \| \leqq a_\nu L_3 e^{\lambda_1 + \lambda_2 + \cdots + \lambda_\nu}$ $\quad (\nu = 1, 2, \cdots\cdots)$

for $0 < h \leqq h_1$

$\sum_{\nu=0}^{\infty} a_\nu \leqq L_1$

then we have

$\| e_m \| \leqq L_3 e^{\lambda_1 + \lambda_2 + \cdots + \lambda_m}$

for $0 < h \leq \min\{h_1, h_2\}$.

Proof.

The proof can be drived in a similar fashion as of Theorem [6].

Numerical illustration

1. Consider the initial-value problem.

$$\begin{cases} y' = \dfrac{y}{(x-1)^2} \\ y(4) = \exp\left(\dfrac{2}{3}\right) \end{cases},$$

the true solution is

$$y(x) = \exp\left\{1 - \frac{1}{(x-1)}\right\}.$$

We shall compare the actual error with the bound given by Theorem [2] . In Theorem [2] if we set constants $\exists\{A_\nu\}, \exists\{a_\nu\}, \exists\{\lambda_\nu\}$ $\exists L_1$, $\exists L_2$, $\exists L_3$, $\exists R_\nu$, $\exists h$, as follows,

$$a_1 = \frac{2h}{3(1+h)}, \quad a_\nu = \frac{a_1}{(1+h)^{\nu-1}} \qquad (\nu = 2, 3, \cdots)$$

$$e^{\lambda_1} = \max\left\{\frac{(h+1)^2(5+2h)}{h(h+2)(3+h)^4} Y_1 + \frac{R_1}{a_1 L_3}, 1\right\}$$

$$e^{\lambda_2} = \max\left\{\frac{(3+h)^4(5+4h)(1+h)}{(5+2h)(3+2h)^4} \frac{Y_2}{Y_1} + \frac{R_2(1+h)}{a_2 L_3 e^{\lambda_1}}, 1\right\}$$

$$e^{\lambda_n} = \max\left\{\frac{(3+(n-1)h)^4(5+2nh)(1+h)}{(5+2(n-1)h)(3+nh)^4} \cdot \frac{Y_n}{Y_{n-1}} + \frac{R_n(1+h)}{a_{n-1} L_3 e^{\lambda_1+\lambda_2+\cdots+\lambda_{n-1}}}, 1\right\}$$

$$A_\nu = \frac{-\varepsilon}{2^\nu} \qquad \varepsilon\text{; any small positive constant.}$$

$$R_\nu = R \qquad (\nu = 1, 2, 3, \ldots)$$

$$L_1 \cong 0, \quad L_2 \cong \frac{2}{3}, \quad L_3 = \frac{3}{4} h^2 \frac{(h+2)}{(h+1)}$$

and $R = 5 \times 10^{-16}$

$y(x_1) \cong Y_1$

$\dfrac{y(x_{n-1})}{y(x_n)} \simeq \dfrac{Y_{n-1}}{Y_n}$

$h = 0.001$

x_n	$y(x_n)$	e_n	$L_3e^{\lambda_1+\lambda_2+\ldots+\lambda_n}$
4.002	1.94816	0.12015×10^{-6}	0.90204×10^{-4}
4.003	1.94838	0.18016×10^{-6}	0.90220×10^{-4}
⋮			
4.010	1.94989	0.59907×10^{-6}	0.90333×10^{-4}
⋮			
4.020	1.95203	0.11939×10^{-5}	0.90497×10^{-4}
⋮			
4.030	1.95417	0.17845×10^{-5}	0.90662×10^{-4}
⋮			
4.100	1.96879	0.58046×10^{-5}	0.91881×10^{-4}
⋮			
4.500	2.04272	0.25441×10^{-4}	0.10097×10^{-3}
⋮			
5.000	2.11700	0.43903×10^{-4}	0.11796×10^{-3}
⋮			
5.500	2.17662	0.57730×10^{-4}	0.14265×10^{-3}
⋮			
6.000	2.22554	0.68366×10^{-4}	0.17747×10^{-3}
⋮			
6.500	2.26637	0.76736×10^{-4}	0.22611×10^{-3}
⋮			
7.000	2.30097	0.83453×10^{-4}	0.29393×10^{-3}
⋮			
7.500	2033066	0.88935×10^{-4}	0.38877×10^{-3}
⋮			
8.000	2.35641	0.93475×10^{-4}	$0.\cancel{25}^{52}196\times10^{-3}$
⋮			
9.000	2.3988	0.10051×10^{-3}	0.97715×10^{-3}

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