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Compact Kähler Manifolds of Nonnegative Bisectional Curvature

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§ 0. It is conjectured that a compact positively curved Kähler manifold is biholomorphically equivalent to a complex projective space. This conjecture is animportant subject to solve in the field of Kähler manifolds. Positivity of curvature in the conjecture is replaced by the condition that holomorphic bisectional curvature is positive for any pair of holomorphic planes. These are true for the following cases: the manifold is 2-dimensional, the Kähler metric is Einstein, the holomorphic transformation group acts transitively on the manifold.

We can also discuss the problem to classify compact Kähler manifolds of nonnegative holomorphic bisectional curvature. It is expected to solve the classification problem under condition of isometrically biholomorphic equivalence. We consider here the problem except for biholomorphic equivalence.

Gauss-Bonnet formula shows that such a manifold of 1-dimensional is biholomorphically equivalent to the complex projective line $P_1(C)$ or 1-dim complex torus.

Howard and Smyth [5] classified 2-dim compact Kähler manifolds of nonnegative bisectional curvature by the aid of classification theorems of compact complex surfaces:

<u>THEOREM</u> (Howard and Smyth [5]) Let M be a compact Kähler surface of which bisectional curvature is nonnegative everywhere. Then one of the following holds:

(i) M is biholomorphically equivalent to the complex projective plane $P_2(C)$.

(ii) M is biholomorphically equivalent to the complex hyperquadric $Q_2(C)$.

(iii) M is flat.

(iv) M is a ruled surface (i.e., $P_1(C)$-bundle) over an elliptic curve. And the universal covering space of M is $C \times P_1(C)$ endowed with the product of the flat metric on C and a metric of nonnegative bisectional curvature on $P_1(C)$.

Relative to Einstein Kähler manifolds, Matsushima's result is known:

<u>THEOREM</u> (Matsushima [76]) Let M be an Einstein Kähler surface of nonnegative bisectional curvature. If the Ricci tensor is nondegenerate, then M is hermitian symmetric space, i.e., isometrically biholomorph. equivalent to $P_2(C)$ or $Q_2(C)$.

We shall consider the case where the manifolds are higher dimensional. It is natural to restrict the problem to the case where the manifolds are irreducible in the sense of holonomy from the following structure theorem together with Theorem 1.1 in §1:

<u>THEOREM</u> (Howard and Smyth [5]) Let M be an n-dim compact Kähler manifold of nonnegative bisect. curvature, and r the

maximal rank of the Ricci tensor. Then there exist a flat Kähler manifold N of (n-r)-dim and a holomorphic fibering $\pi : M \to N$ such that the metric on M is locally a product compatible with the fibering. Moreover, the Ricci tensor of the fiber F has maximal rank r, and under the de Rham decomposition, $F = F_1 \times \cdots \times F_q$, each F_j is simply connected and has second Betti number equal to one.

Irreducible Hermitian symmetric spaces of compact type are the examples of compact Kähler manifolds of nonnegative bisect. curvature.

In this paper, we shall discuss 3-dim compact Kähler manifolds of nonnegative bisect. curvature.

§ 1. Definitions and An Auxiliary Theorem

In the following, manifolds are assumed to be connected, and a Kähler manifold (M, g) with a Kähler metric g is abbreviated as M, unless otherwise stated.

Let P and P′ be two planes which are invariant by the complex structure J of a Kähler manifold M. Holomorphic bisectional curvature H(P,P′) of P and P′ is defined as H(P,P′) = g(R(X,JX)JY,Y), where R is the curvature tensor, X and Y are unit vectors of P and P′ respectively (see Kobayashi and Nomizu [9]). Holomorphic sectional curvature of a holomorphic plane P coincides with H(P,P) by the definition. We have from Bianchi's identity, $H(P,P') = |X \vee Y| \cdot K(\{X,Y\}) + |X \vee JY| \cdot$

$K(\{X,JY\})$, where $|X\vee Y|$ denotes the area of the parallelogram, $K(\{X,Y\})$ sectional curvature of the plane spanned by X and Y. Hence, the nonnegativity of the sectional curvature (resp. the bisectional curvature) implies that the bisect. curvature (resp. holomorphic curvature) is nonnegative.

Let $e_1,\dots,e_n, e_1^*,\dots,e_n^*$, be an orthonormal basis of a tangent space of M, where $e_\alpha^* = Je_\alpha$, $\alpha = 1,\dots,n = \dim_C M$. We define complex vectors Z_α, $\overline{Z}_\alpha$ ($\alpha = 1,\dots,n$) by

$$Z_\alpha = \frac{1}{2}(e_\alpha - \sqrt{-1}e_\alpha^*), \quad \overline{Z}_\alpha = \frac{1}{2}(e_\alpha + \sqrt{-1}e_\alpha^*), \quad \text{and}$$

denote $g(R(Z_\alpha,\overline{Z}_\beta)Z_\gamma,\overline{Z}_\delta)$ by $R_{\overline{\delta}\gamma\alpha\overline{\beta}}$. Then by an easy calculation, $H(P,P') = R_{\overline{\beta}\beta\alpha\overline{\alpha}}$ for $P = \{e_\alpha, e_\alpha^*\}$ and $P' = \{e_\beta, e_\beta^*\}$. $S(Z_\alpha,\overline{Z}_\alpha) = \sum_\beta R_{\overline{\alpha}\alpha\beta\overline{\beta}}$ from the definition of the Ricci tensor S.

Hence the nonnegativity of the bisectional curvature implies that the Ricci tensor is positive semi-definite.

The following theorem plays an auxiliary role in the argument of main theorems in § 2.

<u>THEOREM 1.1.</u> Let M be a compact irreducible Kähler manifold. If M has nonnegative holomorphic bisectional curvature, then

(i) M is simply connected.

(ii) the first Chern class $c_1(M)$ of M is positive.

(iii) $h^{p,0}(M) = h^{0,p}(M) = 0$ $(p \geqq 1)$ and $h^{1,1}(M) = 1$.

Here we denote by $h^{p,q}(M)$ the dimension of the vector space of (p,q)-harmonic forms on M.

NOTE. M is algebraic from (ii), by the aid of Kodaira's imbedding theorem. The arithmetic genus of M, $\sum_{p=0}^{n}(-1)^p h^{p,0}(M)$ is equal to one.

The combination of the following assertions completes the proof of the theorem.

ASSERTION 1.2. $h^{1,1}(M) = 1$.

What we show is that an arbitrary (1,1)-harmonic form is proportional to the so-called Kähler form. A (1,1)-harmonic form ϕ is a sum of the real part of ϕ, $\phi' = \frac{1}{2}(\phi + \overline{\phi})$ and the imaginary part $\sqrt{-1}\,\phi'' = \frac{1}{2}(\phi - \overline{\phi})$: $\phi = \phi' + \sqrt{-1}\,\phi''$. ϕ' and ϕ'' are real forms, hence they can be written locally as

$$\phi' = \sqrt{-1}\sum_{\alpha,\beta}\phi'_{\alpha\bar\beta}\,dz^\alpha\wedge d\bar z^\beta, \qquad \phi'' = \sqrt{-1}\sum_{\alpha,\beta}\phi''_{\alpha\bar\beta}\,dz^\alpha\wedge d\bar z^\beta.$$

$(\phi'_{\alpha\bar\beta})$ and $(\phi''_{\alpha\bar\beta})$ are Hermitian symmetric matrices. The complex Laplacian $\square$ satisfies $\overline{\square} = \square$. ϕ' and ϕ'' are then also harmonic. It suffices to show that an arbitrary real (1,1)-harmonic form is proportional to the Kähler form. We apply the formula in [12] which is concerned with harmonic forms to a real harmonic form $\phi = \sqrt{-1}\sum_{\alpha,\beta}\phi_{\alpha\bar\beta}\,dz^\alpha\wedge d\bar z^\beta$. We have

$$\int_M (\nabla_{\bar\gamma}\phi_{\alpha\bar\beta}\cdot\overline{\nabla^{\gamma}\phi^{\bar\alpha\beta}})dv = \int_M (-g^{\bar\tau\sigma}S_{\sigma\bar\beta}\phi_{\alpha\bar\tau}\overline{\phi^{\bar\alpha\beta}} + g^{\bar\tau\sigma}g^{\bar\delta\gamma}R_{\bar\tau\alpha\gamma\bar\beta}\phi_{\sigma\bar\delta}\overline{\phi^{\bar\alpha\beta}})dv.$$

Our notation of the components of the curvature is different in the sign from that in [12]. We denote by $-U(\phi,\phi)$ the integrand of the right hand side, namely

$$U(\phi,\phi) = \sum g^{\bar\tau\sigma}S_{\sigma\bar\beta}\phi_{\alpha\bar\tau}\overline{\phi^{\bar\alpha\beta}} - \sum g^{\bar\delta\gamma}g^{\bar\tau\sigma}R_{\bar\tau\alpha\gamma\bar\beta}\phi_{\sigma\bar\delta}\overline{\phi^{\bar\alpha\beta}}.$$

We shall show $U(\phi,\phi) \geqq 0$ at any point p of M. We can choose a suitable local coordinate around p such that $g_{\alpha\bar\beta}(p) = \delta_{\alpha\beta}$, $\phi_{\alpha\bar\beta}(p) = \phi_\alpha\delta_{\alpha\beta}$ $\alpha,\beta = 1,\dots,n$ for some real numbers $\phi_1,\dots,\phi_n$. Put $A_{\alpha\beta} = S_{\alpha\bar\beta}\delta_{\alpha\beta} - R_{\bar\alpha\alpha\beta\bar\beta}$, then $U(\phi,\phi) = \sum_{\alpha,\beta} A_{\alpha\beta}\phi_\alpha\phi_\beta$. From the condition on the bisectional curvature, $(A_{\alpha\beta})$ is symmetric and satisfies $A_{\alpha\beta} \leqq 0$ for $\alpha \neq \beta$ and $\sum_\beta A_{\alpha\beta} = 0$. From Lemma 1.3 below, $(A_{\alpha\beta})$ is positive semi-definite. As p is an arbitrary point, we have

$$0 \leqq \int_M (\nabla_{\bar\gamma}\phi_{\alpha\bar\beta}\cdot\overline{\nabla^{\gamma}\phi^{\bar\alpha\beta}})dv = -\int_M U(\phi,\phi)\,dv \leqq 0,$$

that is $\nabla_{\bar\gamma}\phi_{\alpha\bar\beta} = 0$. From Hermitian symmetricity of $(\phi_{\alpha\bar\beta})$, we have $\nabla_\gamma\phi_{\alpha\bar\beta} = \overline{\nabla_{\bar\gamma}\phi_{\beta\bar\alpha}} = 0$. Hence $\nabla\phi = 0$ (ϕ is parallel). The irreducibility of M implies that ϕ is proportional to the Kähler form.

LEMMA 1.3. Let $A = (A_{\alpha\beta})$ be a real symmetric $n \times n$ matrix which satisfies $A_{\alpha\beta} \leqq 0$ for $\alpha \neq \beta$ and $\sum_\beta A_{\alpha\beta} \geqq 0$ for any α. Then A is positive semi-definite.

PROOF. It is sufficient to show that $\sum_{\alpha\,\beta} A_{\alpha\beta} x^\alpha x^\beta \geqq 0$ for any real x^α. Set $\overline{A}_{\alpha\beta} := -A_{\alpha\beta}$ $\alpha \neq \beta$ and $\overline{A}_{\alpha\alpha} := \sum_\beta A_{\alpha\beta}$. Then $(\overline{A}_{\alpha\beta})$ satisfies: $\overline{A}_{\alpha\beta} = \overline{A}_{\beta\alpha}$, $\sum_\beta \overline{A}_{\alpha\beta} = A_{\alpha\alpha}$ and $\overline{A}_{\alpha\beta} \geqq 0$ for any α, β.

$$\begin{aligned}\sum_{\alpha,\beta} A_{\alpha\beta} x^\alpha x^\beta &= \sum_\alpha A_{\alpha\alpha} (x^\alpha)^2 + 2\sum_{\alpha<\beta} A_{\alpha\beta} x^\alpha x^\beta \\ &= \sum_\alpha \Big(\sum_\beta \overline{A}_{\alpha\beta} \Big) (x^\alpha)^2 - 2\sum_{\alpha<\beta} \overline{A}_{\alpha\beta} x^\alpha x^\beta \\ &= \sum_{\alpha=1}^{n} \overline{A}_{\alpha\alpha} (x^\alpha)^2 + \sum_{\alpha<\beta} \overline{A}_{\alpha\beta} (x^\alpha - x^\beta)^2 \geqq 0,\end{aligned}$$

since $\overline{A}_{\alpha\beta} \geqq 0$ for all α, β.

ASSERTION 1.4. The first Chern class $c_1(M)$ of M is positive.

Note that the first Chern class $c_1(E)$ of a holomorphic vector bundle E is called positive (resp. negative) if and only if there is a real d-closed (1,1)-form $\psi = \sqrt{-1} \sum \psi_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar{z}^\beta$ such that ψ represents $c_1(E)$ and $(\psi_{\alpha\bar\beta})$ is positive (resp. negative) definite Hermitian symmetric. And a line bundle L is positive (resp. negative) if and only if $c_1(L)$ is positive (resp. negative).

PROOF of ASSERTION 1.4. The first Chern class $c_1(M)$ of M is represented by $\frac{1}{2\pi}\sigma$, $\sigma = \sqrt{-1}\sum_{\alpha,\beta} S_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}}$ (see [8] p.83). Under the Hodge decomposition, we have $\frac{1}{2\pi}\sigma = c\,\omega + \sqrt{-1}\,\partial\bar{\partial} f$ for some real constant c, from $h^{1,1}(M) = 1$, and ω is the Kähler form, f is a real smooth function on M. The positivity of constant c completes the proof. Suppose that c is not positive. Hence,

$$\left(\frac{\partial^2 f}{\partial z^{\alpha} \partial z^{\bar{\beta}}}\right) = \frac{1}{2\pi}\left(S_{\alpha\bar{\beta}} \right) + (-c)\left(g_{\alpha\bar{\beta}} \right)$$

is positive semi-definite from the nonnegativity of curvature. This means that f is a plurisubharmonic function, hence f is constant from the compactness. $(S_{\alpha\bar{\beta}}) = c\,(g_{\alpha\bar{\beta}})$ for non-positive c implies that the Ricci tensor is flat. $S(Z_\alpha, \bar{Z}_\alpha) = R_{\bar{\alpha}\alpha\alpha\bar{\alpha}} + \sum_{\beta \neq \alpha} R_{\bar{\alpha}\alpha\beta\bar{\beta}}$ for an arbitrary Z_α, all the holomorphic sectional curvatures vanish, that is, the curvature tensor is flat. This contradicts the irreducibility of M.

ASSERTION 1.5. The maximal rank of the Ricci tensor is equal to the complex dimension of M.

Here the rank of the Ricci tensor S at a point p is defined by the rank of the matrix $(S_{\alpha\bar{\beta}})$ at p.

PROOF. Suppose that the maximal rank of the Ricci tensor S is less than the complex dimension of M. Then we have $\det(S_{\bar{\alpha}\beta})$

$= 0$ for any point p. We have from the argument in Assertion 1.4

$$\left(\frac{1}{2\pi}\right)^n \int_M \sigma^n = c^n \int \omega^n > 0.$$

By the way, $\sigma^n = (\sqrt{-1})^n\, n!$

$\det(S_{\alpha\bar{\beta}})\, dz^1 \wedge d\bar{z}^{\bar{1}} \wedge \ldots = 0$ at any point p. This is a contradiction.

ASSERTION 1.6. Let M be a compact Kähler manifold. If the Ricci tensor is positive semi-definite at any point and positive definite at some point. Then $h^{p,0}(M) = h^{0,p}(M) = 0, \quad p \geq 1$.

This is a direct consequence of [13] p.93.

ASSERTION 1.7. Under the same situation as the above assertion, the manifold M is simply connected.

PROOF. The universal covering space $\tilde{M}$ of M is isometric to $\bar{M} \times R^k$ (Riemannian product) from Cheeger and Gromoll [1]. Here $\bar{M}$ is compact and R^k an Euclidean space. As the Ricci tensor is locally positive definite, R^k is reduced to be a point, namely, the fundamental group $\pi_1(M)$ is finite. We have shown that the arithmetic genus of M is equal to one. Hence, M is simply connected by the aid of the argument in the proof of Theorems in Kobayashi [7].

§ 2. Main Theorems

In this section we consider 3-dim compact irreducible Kähler manifolds which have nonnegative bisectional curvature. To determine these manifolds, we shall impose some conditions on them. One is a condition relative to cohomology rings, another to infinitesimal isometries of them.

First we shall show a theorem which is considered as characterizations of the 3-dim complex projective space $P_3(C)$ and the 3-dim complex hyperquadric $Q_3(C)$.

<u>THEOREM 2.1.</u> Let M be a 3-dim compact irreducible Kähler manifold of nonnegative bisectional curvature. If M satisfies $\hat{H}^*(M; Z) \cong \hat{H}^*(P_3(C); Z)$ (resp. $\hat{H}^*(Q_3(C); Z)$) (ring-isomorphic). Then M is biholomorphically equivalent to $P_3(C)$ (resp. $Q_3(C)$).

We denote here by $\hat{H}^*(M; Z)$ the subring $\sum_k H^{2k}(M; Z)$ of Z-cohomology ring $H^*(M; Z) := \sum_k H^k(M; Z)$.

Moreover, we have the following in the case where M is Einstein:

<u>COROLLARY 2.2.</u> Let (M, g) be a 3-dim compact irreducible Einstein Kähler manifold of nonnegative bisect. curvature. If $\hat{H}^*(M; Z) \cong \hat{H}^*(P_3(C); Z)$ (resp. $\hat{H}^*(Q_3(C); Z)$) (ring-isomorph), then (M, g) is biholomorphically homothetic to $P_3(C)$ (resp. $Q_3(C)$) endowed with the canonical Kähler metric

$\bar{g}$, that is, there exist a biholomorphic map $\varphi : M \longrightarrow P_3(C)$ (resp. $Q_3(C)$) and a positive constant c such that $g = c\varphi^*\bar{g}$.

PROOF of COROLLARY 2.2. Above theorem shows that M is homogeneous in the sense of the complex structure. The metric on M induced by the canonical metric $\bar{g}$ has the scalar curvature constant, hence is Einstein from $H^2(M; Z) \cong Z$ and the so-called Ricci form being harmonic. Matsushima [16] Theorem 3 shows that the metric g is equivalent to the induced metric, in other words, (M, g) is biholomorphically homothetic to ($P_3(C)$, $\bar{g}$) (resp. ($Q_3(C)$, $\bar{g}$)).

PROOF of THEOREM 2.1. Since M is of nonnegative bisectional curvature, the tangent bundle of M is semi-positive in the sense of Kobayashi and Ochiai [10] Theorem 6.4. Therefore, by Theorem 4.1 in [10] we have

$$\{c_1{}^3(M) - 2\, c_1(M)\cdot c_2(M) + c_3(M)\}[M] \geqq 0,$$

where $c_k(M)$ is k-th Chern class of M, and $[M]$ the fundamental homology class.

$$c_3(M)[M] = \text{the Euler number of } M = \sum_{p,q} (-1)^{p+q}\, h^{p,q}(M) =$$

$4 - \sum_{p+q=3} h^{p,q}(M) \leqq 4$. $c_1(M)\cdot c_2(M)[M] = 24$, since the arithmetic genus of M coincides with $\frac{1}{24}\, c_1(M)\cdot c_2(M)[M]$ from theorem of Riemann-Roch-Hirzebruch [3]. We have then

$$c_1{}^3(M)[M] \geqq 2\, c_1(M)\cdot c_2(M)[M] - c_3(M)[M] \geqq 44.$$

We have $\dim H^p(M; \mathcal{O}) = h^{0,p}(M) = 0$, $p= 1,2$ from Theorem 1.1 in §1, where $\mathcal{O}$ denotes the sheaf of germs of holomorphic functions on M. Hence the multiplicative group of line bundles over M is isomorphic to $H^2(M; Z) \cong Z$ in assigning its Chern class $c_1(L)$ to a line bundle L. Then we have a positive line bundle F such that $c_1(F)$ is a positive generator of $H^2(M; Z) \cong Z$.

We show first the following:

<u>ASSERTION 2.3.</u> If $\hat{H}^*(M; Z) \cong \hat{H}^*(P_3(C); Z)$ (ring isomorphic), then M is biholomorphically equivalent to $P_3(C)$.

<u>PROOF.</u> It is sufficient to show $c_1(M) - 4c_1(F) \geqq 0$ from Kobayashi and Ochiai [11] which deal with characterizations of a complex projective space and a complex hyperquadric. Z-cohomology ring of $P_n(C)$ is, as well known, generated by an element of $H^2(P_n(C); Z)$. Hence by the assumption, we have

$$\hat{H}^*(M; Z) = Z + Z\alpha + Z\alpha^2 + Z\alpha^3, \quad \alpha = c_1(F)$$

α, $\alpha^2 = \alpha\cdot\alpha$ and $\alpha^3 = \alpha^2\cdot\alpha$ are generators of $H^{2k}(M; Z)$, $k= 1,2,3$. Put $c_1(M) = r\alpha$ for some positive integer r. Then $r^3 = r^3\alpha^3[M] = c_1(M)^3[M] \geqq 44$, which shows $r \geqq 4$. Q. E. D.

Now assume that $\hat{H}^*(M; Z) \cong \hat{H}^*(Q_3(C); Z) = H^*(Q_3(C); Z)$. We have $\hat{H}^*(M; Z) = Z + Z\alpha_1 + Z\alpha_2 + Z\alpha_3$ from Morrow [17] p320, where α_k is a generator of $H^{2k}(M; Z)$, $k = 1,2,3$, $\alpha_1 = c_1(F)$, $\alpha_1{}^2 = 2\alpha_2$ and $\alpha_1{}^3 = 2\alpha_3$. If we set $c_1(M) = r\alpha_1$ in the same manner as above, then $44 \leqq c_1{}^3(M)[M] = r^3\alpha_1{}^3[M] = 2r^3\alpha_3[M]$

$= 2r^3$, which leads $r \gtrsim 3$. Suppose $r \geqq 4$, then we obtain that M is biholomorphically equivalent to $P_3(C)$ from the above assertion, which contradicts the assumption of the cohomology. Hence we have $r = 3$, i.e., $c_1(M) = 3\, c_1(F)$. We can conclude from [11]:

ASSERTION 2.4. If $\hat{H}^*(M; Z) \cong \hat{H}^*(Q_3(C); Z)$, then M is biholomorphically equivalent to $Q_3(C)$.

These assertions completes the proof of Theorem 2.1.

NOTE. (i) The irreducibility of M is redundant, since $H^2(M; Z) \cong Z$.

(ii) $P_3(C)$ and $Q_3(C)$ are the only known examples of 3-dim compact irreducible Kähler manifolds which have nonnegative bisect. curvature.

(iii) Theorem 2.1 is considered to be a generalization of Howard [4]:

THEOREM (Howard) Let M be a 3-dim compact Kähler manifold of positive bisectional curvature. If $H^*(M; Z) \cong H^*(P_3(C); Z)$ (ring isomorphic), then M is biholomorphically equivalent to $P_3(C)$.

Next we shall consider that to what extent the existence of infinitesimal isometries restricts the cohomology of M. We have the following by the aid of Howard and Smyth[5] together with Frankel[2]:

THEOREM 2.5. Let M be a 3-dim compact irreducible Kähler manifold of nonnegative bisectional curvature. If M admits a nontrivial infinitesimal isometry, then relative to Z-cohomology group,

$$H^k(M; Z) \cong Z \quad \text{for even } k,$$
$$H^1(M; Z) \cong H^5(M; Z) \cong 0,$$
$$H^3(M; Z) \cong 0 \quad \text{or} \quad Z \oplus Z.$$

COROLLARY 2.6. Let M be a 3-dim compact irreducible Kähler manifold which has nonnegative bisect. curvature and positive holomorphic curvature. If M admits a nontrivial infinitesimal isometry, then $H^*(M; Z) \cong H^*(P_3(C); Z)$ (group isomorphic).

It is expected to eliminate $H^3(M; Z) \cong Z \oplus Z$ from the result in Theorem 2.5.

NOTE. If M in Theorem 2.5 or Corollary 2.6 is furthermore Einsteinian, then the existence of infinitesimal isometries is redundant. It is necessarily guaranteed by Matsushima [14] and Kobayashi and Ochiai [10].

PROOF of THEOREM 2.5. Let X be a given infinitesimal isometry. $\mathrm{Zero}(X) = \{ p \in M;\ X = 0 \text{ at } p \}$ is not empty from the following consideration. M is algebraic (that is, a Hodge manifold) from NOTE for Theorem 1.1 in §1 and the first Betti number $b_1(M)$

$= h^{1,0} + h^{0,1} = 0$. $X-\sqrt{-1}\,JX$ is a holomorphic field and $\mathrm{Zero}(X) = \mathrm{Zero}(X-\sqrt{-1}\,JX)$. Suppose that X has no zero points. Matsushima's theorem [15] shows that the first Betti number is not equal to zero. This is a contradiction.

Let $\mathrm{Zero}(X) = \bigcup_i N_i$ be the decomposition of $\mathrm{Zero}(X)$ into its connected components. Each N_i is a closed totally geodesic submanifold from Kobayashi [6]. And we have the following due to Frankel [2]: Each N_i is moreover Kähler submanifold and there exists a real smooth function f on M such that $df = J\xi$ where ξ is the 1-form which corresponds to X, and $\mathrm{Zero}(X)$ coincides with the critical point set of f and in particular each N_i is a nondegenerate critical manifold. And Frankel showed the following:

<u>THEOREM</u> (Frankel)

(1) $$b_k(M) = \sum_i b_{k-\lambda_i}(N_i),$$

where λ_i is the index, i.e., the number of the negative eigenvalues of the Hessian of f at any point of N_i, hence is an even integer not greater than $2\cdot\mathrm{codim}_{\mathbb{C}} N_i$.

(2) $\mathrm{Zero}(X)$ has torsion part if and only if M has torsion part.

(3) $H_{2i+1}(\mathrm{Zero}(X); Z) = 0$ for all i if and only if $H_{2i+1}(M;Z) = 0$ for all i.

Each N_i is totally geodesic, hence is of nonnegative bisect. curvature. We consider three cases according to dim of N_i:

(1) N_i is 0-dim: N_i consists of a single point.

(2) N_i is 1-dim: From the Gauss-Bonnet formula N_i is bihol. equivalent to $P_1(C)$ or is a 1-dim complex torus of flat metric.

(3) N_i is 2-dim: In this case, one of the following holds [5]: N_i is a) bihol. equivalent to $P_2(C)$, b) bihol. equivalent to $Q_3(C)$, c) of flat metric, d) a ruled surface over an elliptic curve.

In d), the ruled surface has positive first Betti number ([5]). Compact flat Kähler manifolds have trivial Chern classes. Hence, 2-dim compact flat Kähler manifolds have the arithmetic genera $1 - h^{1,0} + h^{2,0} = 0$ from Theorem of Riemann-Roch-Hirzebruch. We have $b_1(M) = b_3(M) = 2\ h^{1,0} = 2 + 2\ h^{2,0}$. A 2-dim compact flat Kähler manifold has positive odd Betti numbers. Paying a regard to these results, we shall prove the theorem.

If M has no torsion, then Z-cohomology group is isomorphic to Z-homology group. It is shown in § 1 that $b_k = 1$ for even k, $b_1 = b_5 = 0$. Hence, what we show is that M has no torsion and $b_3 = 0$ or 2.

First we show:

<u>ASSERTION 2.6</u>. There exist neither 2-dim flat manifolds nor ruled surfaces among N_is'.

<u>PROOF</u>. Suppose that some 2-dim N_i is flat. The index $\lambda_i = 0$ or 2, since λ_i is not greater than $2\ \mathrm{codim}_C N_i$.

i) $\lambda_i = 0$: Set $k = 1$ in the formula (1) of Theorem

(Frankel), then $b_1(M) = b_1(N_i) + \sum_{j \neq i} b_{1-\lambda_j}(N_j) \geqq 1.$

ii) $\lambda_i = 2$: Set $k = 5$, $b_5(M) = b_3(N_i) + \sum_{j \neq i} b_{5-\lambda_j}(N_j)$

$\geqq 1.$

These are contrary to $b_1(M) = b_5(M) = 0$. In the same way we can also eliminate ruled surfaces from N_is'. Hence the assertion is obtained.

Now we assume that some N_i is a 1-dim torus. The possible values of the index λ_i of N_i are 0, 2 and 4. We set $\lambda_i = 0$, $k = 1$ and $\lambda_i = 4$, $k = 5$ resp. in Frankel's formula (1) to obtain

$$b_1(M) = b_1(N_i) + \sum_{j \neq i} b_{1-\lambda_j}(N_j) \geqq 1,$$

$$b_5(M) = b_1(N_i) + \sum_{j \neq i} b_{5-\lambda_j}(N_j) \geqq 1.$$

These also contradict the requirement on the Betti numbers of M. If $\lambda_i = 2$, then we have, setting $k = 2$,

$$b_2(M) = b_0(N_i) + \sum_{j \neq i} b_{2-\lambda_j}(N_j),$$ which shows the following.

<u>ASSERTION 2.7.</u> If Zero(X) has a 1-dim complex torus as its component, then it has the index $\lambda = 2$, and Zero(X) has no other 1-dim tori.

Furthermore we can conclude:

ASSERTION 2.8. If there is a 1-dim complex torus N of $\lambda = 2$ among the components of $\text{Zero}(X)$, then $\text{Zero}(X) = N_1 \cup N_2 \cup N_3$ where $N_1 = \{p\}$ of $\lambda_1 = 0$, $N_2 = N$ and $N_3 = \{q\}$ of $\lambda_3 = 6$ for some points p and q.

PROOF. Put $\text{Zero}(X) = \bigcup_{i_0} N^0_{i_0} \cup \bigcup_{i_1} N^1_{i_1} \cup \bigcup_{i_2} N^2_{i_2}$ under the decomposition of its components, where $N^m_{i_m}$ is of m-dim. Put $k = 0, 2, 4$ and 6 into the formula:

$$b_k(M) = \sum_{i_0} b_{k-\lambda_{i_0}}(N^0_{i_0}) + \sum_{i_1} b_{k-\lambda_{i_1}}(N^1_{i_1}) + \sum_{i_2} b_{k-\lambda_{i_2}}(N^2_{i_2}).$$

Since $\lambda_{i_1} = 0, 2$ or 4 and $\lambda_{i_2} = 0$ or 2, and the even Betti numbers do not vanish, $\text{Zero}(X) = \bigcup N_i$ contains only N as its components of dim 1 and 2. And the possible values of the index λ of 0-dim N^0 are 0 and 6. Hence we obtain the above assertion.

We proceed with the proof of Theorem 2.5.

M has no torsion from (2) in Frankel's theorem, since none of the components of $\text{Zero}(X)$ has a torsion.

If there exist no 1-dim complex tori, we can conclude $b_3(M) = 0$ by (3) in Frankel's theorem, since there are no components of positive odd Betti numbers among N_is' from Assertion 2.8.

Provided that there exists a 1-dim complex torus, we have $b_3(M) = 2$ from Assertion 2.7 together with (1) of Frankel's theorem. Therefore we obtain the requirement on the Z-cohomology of M.

Corollary 2. 6 is easily obtained. By the assumption on the holomorphic curvature, the Gauss-Bonnet formula eliminates the 1-dim torus of $\lambda = 2$ from Assertion 2.7.

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