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On some evolution equations of subdifferential operators

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1. Introduction

In this paper we are concerned with nonlinear evolution equations of a form

$$\frac{du}{dt} + \partial\psi^t u(t) + A(t)u(t) \ni f(t), \quad 0 \leqq t \leqq T \qquad (1.1)$$

in a real Hilbert space H. Here for each fixed t, $\partial\psi^t$ is subdifferential of a lower semicontinuous convex function ψ^t from H into $(-\infty, \infty]$, $\psi^t \not\equiv \infty$ and $A(t)$ is a monotone, single valued and hemicontinuous operator which is perturbation in a sense. The effective domain of ψ^t defined by $\{u \in H : \psi^t(u) < +\infty\} = D$ is independent of t. We denote the inner product and the norm in H by $(\ ,\ )$ and $\|\ \|$ respectively. Let T be a positive constant.

We assume the following conditions for ψ^t and $A(t)$.

A - (1) For every $r > 0$ there exist a positive constant $L_1(r)$ such that

$$|\psi^t(u) - \psi^s(u)| \leqq L_1(r)|h(t) - h(s)|\{\psi^t(u) + 1\}$$

hold if $0 \leqq s, t \leqq T$, $u \in D$ and $\|u\| \leqq r$, where $h(t)$ is continuous function with bounded total variation.

A - (2) If $u(t) \in D$ is absolutely continuous on $[a, b]$ $(0 \leqq a < b \leqq T)$ then $A(t)u(t)$ is strongly measurable on $[a, b]$

and for any fixed $t_0 \in [a, b]$ $A(t_0)u(t)$ is also strongly measurable on $[a, b]$. For any fixed $u \in D$, $A(t)u$ is continuous on $[0, T]$.

A - (3) There are Riemann integrable functions $w_r^2(t)$ on $[0, T]$ and a constant $0 < k_r < 1/2$ such that

$$\|A(t)u\| \leqq k_r \|\partial\psi^t u\| + w_r(t) \quad \text{for any} \quad \|u\| \leqq r.$$

A -(4) If $u(t)$ is absolutely continuous and $|\psi^t(u)| + \|u(t)\| \leqq r$, then $A(t)u(t) \leqq W_r(t)$.

Under the above assumptions we consider the uniqueness and existence of the solution of (1-1) where the solution is defined as follows:

Definition 1 - 1: We say that $u(t)$ is a solution of (1-1) if and only if $u(t)$ is continuous on $[0, T]$ and absolutely continuous on $(0, T]$ and if (1-1) holds almost everywhere on $[0, T]$.

Theorem 1 - 1. Suppose that the assumptions stated above are satisfied. Then we hold the unique solution of (1-1) where $f \in L_2[0, T ; H]$ and the initial date $u_0 \in \bar{D}$.

Remark 1 -1. The continuty assumption A-(1) is weaker than those of J. Watanabe [3] and H. Attouch and A. Damlamian [1].

2. The outline of the proof.

Using $\psi^0(a) \geqq C'\|a\| + D'$ and A-(1), we get the following lemma.

Lemma 2 - 1 There exist constants C_1 and C_2 which are independent of t and α such that

$$\psi^t(\alpha) \geq C_1\|\alpha\| + C_2 \quad \text{for any} \quad \alpha \in H.$$

We take a sequence $\{t_i\}_{i=1}^n$ such that $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ and $t_i \in I$ for any $i = 0, 2, \cdots, n$ and $|t_i - t_{i-1}| \longrightarrow 0$ as $n \longrightarrow \infty$ for any $i = 1, 2, \cdots, n$.
We denote by

$$\psi_n^t(u) = \psi^{t_i}(u),\ A_n(t) = A(t_i),\ \text{for} \quad t_i \leq t < t_{i+1}.$$

We consider the following evolution equations

$$\begin{cases} \frac{d}{dt}u_n^i + (\partial\psi_n^t + A_n(t))u_n^i(t) \ni f(t) & t_i \leq t < t_{i+1} \\ u_n^i(t_i) = u_n^{i-1}(t_i) \text{ and } u_n^0(0) = u_0 \in D \text{ for } i = 0, 1 \cdots \\ \cdots n-1 \text{ and } f(t) \in L^2[0, T : H]. & (2\text{-}1) \end{cases}$$

The solution of (2-1) is defined inductively by the solution of a operator with constant coeffícents. For the sake of simplicity we wright $u_n(t) = u_n^i(t)$.

Using that $\{u_n(t)\}$ are the solutions of (2-1) and lemma 1 we get the following lemma.

Lemma 2 - 2 There is a constant γ independent of n and t such that

$$\|u_n(t)\| \leq \gamma.$$

On the other hand since we get

$$\frac{d}{dt}\psi_n^t(u_n) + \|\frac{d}{dt}u_n\|^2 = (f(t) - A_n(t)u_n, \frac{d}{dt}u_n) \qquad \text{a.e.t}$$

from H. Brezis [2], $u_n(t)$ is a strong solution of (2-1) and A-(3) we see

$$\psi_n^t(u_n(t)) + \delta\int_{t_i}^{t}\|\frac{d}{dt}u_n\|^2 dt \leq \psi_n^{t_i}(u_n(t_i))$$

$$+ \int_{t_i}^{t} C_\delta(\|f\| + w_r)^2 ds \qquad (2\text{-}2)$$

from our assumption A-(3) where δ and C_δ are positive conctants independent of n, t and t_i. Combining (2-2) and A-(1) we see

$$\psi_n^{t_i}(u_n(t_{i+1})) \leq \psi_n^{t_i}(u_n(t_i))\{1+L_1(\gamma)|h(t_{i-1}) - h(t_i)|\}$$

$$+ \int_{t_i}^{t_{i+1}} C_\delta(\ f(s) \ +W\ (t_i))^2 ds$$

$$+ L_1(\gamma)|h(t_{i-1}) - h(t_i)|. \qquad (2\text{-}3)$$

We put

$$K = \{\int_0^T 2C_\delta\|f\|^2 ds + 2\int_0^T w_\gamma^2(t)dt + L_1(\gamma)V(h) + |\psi^0(u_0)|+1\}$$

then from (2-3) we see

$$|\psi_n^t(u_n(t))| < 3Ke^{KL_1(\gamma)V(h)} \qquad (2\text{-}4)$$

where $V(h)$ = tolal variation of h on $[0, T]$.

Combining (2-3) and (2-4) we get the following lemma.

Lemma 2 - 3 We know

$$|\psi_n^t(u_n(t))| + \int_0^t\|\frac{du_n}{dt}\|^2 dt \leq C_3$$

where C_3 is a constant independent of n and t.

From the above lemma we know that there exists subsequence $\{\frac{d}{dt}u_{n_j}\}$ which is L_2-weakly convergent. For the sake of simplicity we put $u_n = u_{n_j}$. Thus we see that $u_n(t)$ is weak convegence to $u(t)$ and $u(t)$ is absolutely continuous on $[0, T]$. On the other hand since $u_n(t)$ is the solution of (2-1) we find

$$\int_0^T \psi_n^s(v(s))ds - \int_0^T \psi_n^s(u_n(s))ds \geqq \int_0^T (f(s) - A_n(s)u_n(s)$$

$$- \frac{d}{ds}u_n(s),\ v(s) - u_n(s))ds \geqq$$

$$\int_0^T (f(s) - A_n(s)v(s) - \frac{d}{ds}v(s),\ v(s) - u_n(s))ds +$$

$$+ 1/2\|u_0 - v(0)\|^2.$$

Then

$$\int_0^T (\psi^s(v(s)) - \psi^s(u(s)))ds \geqq \int_0^T (f(s) - A(s)v(s) - \frac{d}{dt}v(s),$$

$$v(s) - u(s))ds + 1/2\|u_0 - v(0)\|^2.$$

Next we put $v(t) = pu(t) + (1-p)w(t)$ where $w(t) \in D$ and is absolutely continuous.

Thus we obtain the following inequality

$$\int_0^T (\psi^s(w(s)) - \psi^s(u(s)))ds \geqq \int_0^T (f(s) - A(s)u(s) - \frac{d}{dt}u(s),$$

$$w(s) - u(s))ds.$$

Next for any fixed $\xi \in D$ and $0 \leqq t_1 < t_2 \leqq T$ we put

$$w(t) = \begin{cases} \xi & : \; t_1 + \varepsilon \leq t \leq t_2 - \varepsilon \\ pu(t_1) + q\xi & : \; t = pt_1 + q(t_1 + \varepsilon) \\ u(t) & : \; 0 \leq t \leq t_1, \; t_2 \leq t \leq T \\ pu(t_2) + q\xi & : \; t = pt_2 + (t_2 - \varepsilon)q \end{cases}$$

where $p + q = 1$ $p > 0$, $q > 0$ and $\varepsilon > 0$.

If $\varepsilon \longrightarrow 0$ we get

$$\int_{t_1}^{t_2} \psi^t(\xi)\,dt - \int_{t_1}^{t_2} \psi^t(u(t))\,dt \geq \int_{t_1}^{t_2} (f(t) - A(t)u(t) - \tfrac{d}{dt}u(t),$$

$$- u(t))\,dt.$$

For any Lebesque points of $\psi^t u(t)$, $f(t)$ $A(t)u(t)$, $\frac{d}{dt}u(t)$, and $u(t)$ we know

$$\psi^t(\xi) - \psi^t u(t) \geq (f(t) - A(t)u(t) - \tfrac{d}{dt}u(t),\; \xi - u(t)).$$

Considering that $\partial\psi^t + A(t)$ is monotone opeator we can show the uniquniss of (1-1). If $u_0 \in D$ we can proved the theorem.

Next if $u_0 \in \bar{D}$. We put $u_{m,0} = (1 + 1/m\partial\psi^0)^{-1} u_0$. We denote by $u_m(t)$ the solution of (1-1) of initial date $u_{m,0}$. Since $\partial\psi^t + A(t)$ is monotone operator we see that $u_m(t)$ is uniformly convergent on $[0, T]$ then $\lim_{m\to\infty} u_m(t) = u(t)$.

Using that $u_m(t)$ are strong solutions of (1-1) and A-(3) we know for any $0 < \delta < T$,

$$\int_0^{\delta} \psi^t(u_m(t))\,dt \leq C_4$$

where C_4 is a constant independent of δ and m.
There exist $0 < \delta_m < \delta$ $m = 1, 2, \cdots$ such that

$$\psi^{\delta_m}(u_m(\delta_m)) \leq \frac{1}{\delta}\int_0^{\delta}\psi^t(u_m(t))dt \leq \frac{C_4}{\delta} = C_5.$$

We denote by $V_m(t)$ the solution of (1-1) for the initial date $V(\delta_m) = u_m(\delta_m) \in D$ on $[\delta_m, T]$. Then we find $v_m(t) = u_m(t)$ on $[\delta_m, T]$ from the uniqueness of the solution of (1-1).
On the other hand noting the method of Lemma 2-3 we get

$$|\psi_n^{t_n}(V_m^n(t))| \leq C_6 \quad \text{for} \quad t \in [\delta_m, T]$$

where C_6 is independent of n and m.
Thus we get

$$\int_{\delta}^{T}\left\|\frac{du_m}{dt}(t)\right\|^2 dt \leq \int_{\delta_m}^{T}\left\|\frac{dV_m}{dt}(t)\right\|^2 dt \leq C_7$$

Using the above same method on $[\delta, T]$ we can prove the Theorem.

Bibliography

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