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On Formal Fractions Associated with the Symmetric Groups

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"What is 1 divided by 2 divided by 3?" Two answers to this question are possible according as we interpret it as "(1 divided by 2) divided by 3" or "1 divided by (2 divided by 3)". The ambiguity of the question of this sort increases as the number of "divided by" in the question gets larger. The value of "x_0 divided by x_1 divided by x_2 divided by ... divided by x_n" can be uniquely determined provided the precedence (i.e., total order) between these n "divided by" is given. Thus, by assigning the precedence in all possible ways (n! ways), we shall study how the value of "x_0 divided by ... divided by x_n" behaves as the precedence varies. We formalize this argument in the following way.

Let x_0, x_1, x_2, ... be distinct denumerable formal variables. Let G be the free commutative group generated by these variables. Let S_n be the set of all permutations on $\{1, 2, ..., n\}$, i.e., the symmetric group of degree n. (We put $S_0 = \{1\}$.) As is customary, we denote the elements of S_n by

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}.$$

For any $\sigma \in S_n$ and $K \subseteq \{1, 2, ..., n\}$ whose cardinality $|K| = k$, we define $\sigma_K \in S_k$ as follows. Let f be the order isomorphism from K to $\{1, 2, ..., k\}$, and g be the order isomorphism from $\sigma(K)$ to $\{1, 2, ..., k\}$. We put $\sigma_K = g \circ \sigma \circ f^{-1}$, where $\circ$ denotes the composition of mappings.

We now define the mapping $V_n : S_n \to G$ as follows.

(i) $V_0(1) = x_0$, where 1 is the sole element of S_0.

(ii) For $n>0$, $V_n(\sigma)=V_{k-1}(\sigma_{\{1,\dots,k-1\}})(V_{n-k}(\sigma_{\{k+1,\dots,n\}}))^{-1}$, where $k=\sigma^{-1}(n)$.

It is easily seen that the above defined mapping gives the evaluation of the following fraction:

$$\begin{array}{cl}
x_0 & \\
\rule{6em}{0.4pt} & \sigma(1) \\
x_1 & \\
\rule{3em}{0.4pt} & \sigma(2) \\
x_2 & \\
\vdots & \\
x_{k-1} & \\
\rule{12em}{0.4pt} & \sigma(k) = n \\
x_k & \\
\vdots & \\
x_{n-1} & \\
\rule{7em}{0.4pt} & \sigma(n) \\
x_n &
\end{array} ,$$

where $\sigma(j)$ denotes the length of the bar between x_{j-1} and x_j. We call this fraction the fraction associated with σ. For example, if $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$, $V_4(\sigma)$ is calculated thus:

$$V_4(\sigma) = \begin{array}{cl}
x_0 & \\
\rule{4em}{0.4pt} & 3 \\
x_1 & \\
\rule{2em}{0.4pt} & 1 \\
x_2 & \\
\rule{8em}{0.4pt} & 4 \\
x_3 & \\
\rule{3em}{0.4pt} & 2 \\
x_4 &
\end{array} = \left(\begin{array}{cl}
x_0 & \\
\rule{4em}{0.4pt} & 2 \\
x_1 & \\
\rule{2em}{0.4pt} & 1 \\
x_2 &
\end{array} \right) \left(\begin{array}{cl}
x_3 & \\
\rule{2em}{0.4pt} & 1 \\
x_4 &
\end{array} \right)^{-1}$$

$$= (x_0(x_1/x_2)^{-1})(x_3x_4^{-1})^{-1} = (x_0(x_1x_2^{-1})^{-1})(x_3^{-1}x_4)$$

$$= x_0x_1^{-1}x_2x_3^{-1}x_4 \ .$$

Evidently, $V_n(\sigma)$ can be written in the form

$$V_n(\sigma) = \prod_{i=0}^{n} x_i^{\varepsilon_i},$$

where $\varepsilon_i = \pm 1$.

Hence, instead of V_n, we may consider the mapping

$$\varphi_n : S_n \to \{\pm 1\}^{n+1},$$

defined by $\varphi_n(\sigma) = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) = \varepsilon_0\varepsilon_1\ldots\varepsilon_n$.

For any $\varepsilon \in \{\pm\}^{n+1}$ (instead of $+1$ (-1) we simply write $+$ $(-)$), we put

$$N_n(\varepsilon) = |\varphi_n^{-1}(\varepsilon)|.$$

Namely, $N_n(\varepsilon)$ is the number of permutations σ such that $\varphi_n(\sigma) = \varepsilon$. We also put

$$M_n = \max\ \{N_n(\varepsilon) | \varepsilon \in \{\pm\}^{n+1}\}.$$

The meanings of N_n and M_n might be interpreted as follows: Let us fix some n, and consider the set of all fractions associated with σ's in S_n. We call it the fraction language of degree n. Each element of the fraction language is said to be a sentence. Then V_n may be considered as giving a semantical interpretation (in G) of each sentence. Thus, the fact that $\varphi_n(\sigma) = \varphi_n(\tau)$ means that the sentences σ and τ are talking about the (semantically) same thing though they may be syntactically distinct. Hence, for any $\sigma \in S_n$, $N_n(\varphi_n(\sigma))$ gives the cardinality of the class semantically equivalent to σ. M_n denotes the cardinality of the largest class in this sense.

In the following, we investigate the values of M_n. First note that for any $\sigma \in S_n$ $(n \geqq 1)$, we have $\varepsilon_0\varepsilon_1 = +-$, where $\varphi_n(\sigma) = \varepsilon = \varepsilon_0\varepsilon_1\ldots\varepsilon_n$. Later it is also proved that $N_n(\varepsilon) \neq 0$ if $\varepsilon_0\varepsilon_1 = +-$. For small n, M_n is easily calculated directly as follows:

$$M_0 = M_1 = M_2 = 1,\ M_3 = 2,\ M_4 = 5,\ M_5 = 16.$$

We also list the values of N_5:

$$N_5(\ +-----\ ) = N_5(\ +-++++\ ) = 1,$$

$$N_5(\ +----+\ ) = N_5(\ +-+++-\ ) = 4,$$

$$N_5(\ +---+-\ ) = N_5(\ +-++-+\ ) = 9,$$

$$N_5(\ +---++\ ) = N_5(\ +-++--\ ) = 6,$$

$$N_5(\ +--+--\ ) = N_5(\ +-+-++\ ) = 9,$$

$$N_5(\ +--+-+\ ) = N_5(\ +-+-+-\ ) = 16,$$

$$N_5(\ +--++-\ ) = N_5(\ +-+--+\ ) = 11,$$

$$N_5(\ +--+++\ ) = N_5(\ +-+---\ ) = 4,$$

$$N_5(\varepsilon) = 0 \qquad \text{(otherwise)}.$$

Since n is dependent on the length of ε $(n = |\varepsilon|-1$, where $|\varepsilon|$ is the length of $\varepsilon)$, we henceforth write simply $N(\varepsilon)$ in place of $N_n(\varepsilon)$. We shall deduce two recursive equations for $N(\varepsilon)$. To do so, we define two sets of indices. Let $\varepsilon = \varepsilon_0\varepsilon_1\ldots\varepsilon_n$ $(n \geqq 1)$, where $\varepsilon_i \in \{\pm\}$. We put,

$$I(\varepsilon) = \{i \mid \varepsilon_i\varepsilon_{i+1} = -+,\ i < n\} \cup \{i = n \mid \varepsilon_n = -\}. \quad (1)$$

$$J(\varepsilon) = \{j \mid \varepsilon_{j-1} = \overline{\varepsilon}_j\}, \text{ where } \overline{+} = - \text{ and } \overline{-} = +. \quad (2)$$

Now, let us consider the fraction associated with some $\sigma \in \varphi_n^{-1}(\varepsilon)$. Let $k = \sigma^{-1}(n)$, then by definition of $V_n(\sigma)$, we must have $\varepsilon_k = -$ and, if $k \neq n$, $\varepsilon_{k+1} = +$. This means $k \in I(\varepsilon)$. Hence we have the first recursive equation for $N(\varepsilon)$:

$$N(\varepsilon) = \sum_{i \in I(\sigma)} \binom{n-1}{i-1} N(\varepsilon_0 \varepsilon_1 \ldots \varepsilon_{i-1}) N(\overline{\varepsilon_i \ldots \varepsilon_n}). \quad (3)$$

The second recursive equation for $N(\varepsilon)$ can be deduced by paying attention to the index $k = \sigma^{-1}(1)$. Then, again by definition of V_n, we must have $\varepsilon_{k-1} = \overline{\varepsilon}_k$, that is, $k \in J(\varepsilon)$. Thus we have the second recursive equation for $N(\varepsilon)$:

$$N(\varepsilon) = \sum_{j \in J(\varepsilon)} N(\varepsilon_0 \varepsilon_1 \ldots \varepsilon_{j-1} \varepsilon_{j+1} \ldots \varepsilon_n). \quad (4)$$

For any $\delta \in \{\pm\}^n$ $(n \geqq 0)$, we define $\tilde{\delta} \in \{\pm\}^{n+2}$ by $\tilde{\delta} = +-\delta$. Then the following lemma holds.

Lemma 1. <u>$N(\tilde{\delta}) = N(\tilde{\bar{\delta}})$, for any δ.</u>

<u>Proof</u>. By induction on n, where $n = |\delta|+2$. Suppose that $n \geqq 2$ and that the lemma holds for smaller n. Let $\delta = \delta_2 \delta_3 \ldots \delta_n$ $(\delta_i \in \{\pm\},\ n \geqq 2)$, so that $\tilde{\delta} = +-\delta_2\delta_3\ldots\delta_n$ and $\tilde{\bar{\delta}} = +-\overline{\delta_2\delta_3\ldots\delta_n}$. Without losing generality, we may assume that $\delta_2 = +$. Then we see that $J(\tilde{\delta}) = J(\tilde{\bar{\delta}}) \cup \{2\}$ and $2 \notin J(\tilde{\bar{\delta}})$. Let us compare $J(\tilde{\delta})$ and $J(\tilde{\bar{\delta}})$ by using (4). If $j > 2$ then $j \in J(\tilde{\delta})$ if and only if $j \in J(\tilde{\bar{\delta}})$ and $N(+-\delta_2\ldots\delta_{j-1}\delta_{j+1}\ldots\delta_n)$

$= N(+-\overline{\delta_2 \ldots \delta_{j-1}\delta_{j+1} \ldots \delta_n})$ by induction hypothesis. If $j = 2$ then $2 \in J(\tilde{\delta})$ but $2 \notin J(\tilde{\tilde{\delta}})$ and the term $N^{(2)}(\tilde{\delta}) = N(+-\delta_3\delta_4 \ldots \delta_n)$ contributes to $N(\tilde{\delta})$. If $j = 1$ then $1 \in J(\tilde{\delta})$ and $1 \in J(\tilde{\tilde{\delta}})$. In this case the contribution of the corresponding term to $N(\tilde{\delta})$ is $N(+\delta_2\delta_3 \ldots \delta_n) = 0$, since $\delta_2 = +$. The contribution to $N(\tilde{\tilde{\delta}})$ is $N(+\overline{\delta_2\delta_3 \ldots \delta_n}) = N(+-\overline{\delta_3 \ldots \delta_n}) = N^{(2)}(\tilde{\delta})$ by induction hypothesis. By appealing to (4) we see that the lemma holds for n.

The Eq. (3), together with this lemma, enables us to calculate $N(\alpha^n)$ and $N(\beta^n)$, where α^n and β^n are defined thus:

Let $\alpha_0 = \beta_0 = +$ and $\alpha_1 = \beta_1 = -$. For $i \geq 1$, we put $\alpha_{2i} = \beta_{2i+1} = +$ and $\alpha_{2i+1} = \beta_{2i} = -$. We then define $\alpha^n = \alpha_0\alpha_1 \ldots \alpha_n$ and $\beta^n = \beta_0\beta_1 \ldots \beta_n$.

By Lemma 1, for $n \geq 0$, we have $N(\alpha^n) = N(\beta^n)$. By the definition of α^n and β^n, we have

$$I(\alpha^n) = \{i | 1 \leq i \leq n,\ i\text{: odd}\} \quad (\text{if } n \geq 1), \tag{5}$$

$$I(\beta^n) = \{i | 1 \leq i \leq n,\ i\text{: even}\} \quad (\text{if } n \geq 2). \tag{6}$$

Note that for $n = 1$ we have $I(\beta^1) = \{1\}$. Now, we calculate:

$$N(\alpha^n) = \sum_{i \in I(\alpha^n)} \binom{n-1}{i-1} N(\alpha_0\alpha_1 \ldots \alpha_{i-1}) N(\overline{\alpha_i\alpha_{i+1} \ldots \alpha_n})$$

$$= \sum_{i \in I(\alpha^n)} \binom{n-1}{i-1} N(\alpha^{i-1}) N(\alpha^{n-i}). \tag{7}$$

$$N(\beta^n) = \sum_{i\in I(\beta^n)} \binom{n-1}{i-1} N(\beta_0\beta_1\cdots\beta_{i-1}) N(\overline{\beta_i\beta_{i+1}\cdots\beta_n})$$

$$= \sum_{i\in I(\beta^n)} \binom{n-1}{i-1} N(\beta^{i-1}) N(\alpha^{n-i}). \tag{8}$$

Let us assume $n \geqq 2$. Then, by adding (7) and (8), we have, using (5) and (6),

$$2N(\alpha^n) = \sum_{i=1}^{n} \binom{n-1}{i-1} N(\alpha^{i-1}) N(\alpha^{n-i}) \qquad (\text{if } n \geqq 2). \tag{9}$$

Or, equivalently,

$$2N(\alpha^{n+1}) = \sum_{i=0}^{n} \binom{n}{i} N(\alpha^{i}) N(\alpha^{n-i}) \qquad (\text{if } n \geqq 1). \tag{10}$$

Now, we can prove the following theorem.

Theorem 2. <u>If $\varepsilon \neq \alpha^n, \beta^n$ then $N_n(\varepsilon) < N(\alpha^n) = N(\beta^n)$.</u>

<u>Proof.</u> By induction on n.

(I) If $n = 0, 1, 2$, the theorem is vacuously true.

(II) Suppose that $n \geqq 3$, and that the theorem holds for any smaller n. Let $\varepsilon = \varepsilon_0\varepsilon_1\ldots\varepsilon_n$ be any element in $\{\pm\}^{n+1}$ other than α^n or β^n. We want to show that $N(\varepsilon) < N(\alpha^n)$. If $\varepsilon_0\varepsilon_1 \neq +-$ then we have $N(\varepsilon) = 0 < N(\alpha^n)$. So we assume that $\varepsilon_0\varepsilon_1 = +-$. We put $\delta = \varepsilon_2\varepsilon_3\ldots\varepsilon_n$, so that we have

$$N(\varepsilon) = N(\tilde{\delta}) = N(\overline{\tilde{\delta}}).$$

It is easy to see that $I(\tilde{\delta}) \cap I(\tilde{\tilde{\delta}}) = \emptyset$. Since $n \geqq 3$ and $\varepsilon \neq \alpha^n$ or β^n, there must be some $k \geqq 2$ such that $\varepsilon_k = \varepsilon_{k+1}$. Then, clearly, $k \notin I(\tilde{\delta}) \cup I(\tilde{\tilde{\delta}})$. Hence,

$$I(\tilde{\delta}) \cup I(\tilde{\tilde{\delta}}) \subsetneqq \{1, 2, \ldots, n\}.$$

We calculate $2N(\varepsilon) = N(\tilde{\delta}) + N(\tilde{\tilde{\delta}})$, by using (3):

$$2N(\varepsilon) = \sum_{i \in I(\tilde{\delta})} \binom{n-1}{i-1} N(\varepsilon_0 \varepsilon_1 \cdots \varepsilon_{i-1}) N(\overline{\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_n})$$

$$+ \sum_{i \in I(\tilde{\tilde{\delta}})} \binom{n-1}{i-1} N(\varepsilon_0 \varepsilon_1 \overline{\varepsilon_2 \cdots \varepsilon_{i-1}}) N(\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_n)$$

$$\leqq \sum_{i \in I(\tilde{\delta})} \binom{n-1}{i-1} N(\alpha^{i-1}) N(\alpha^{n-i})$$

$$+ \sum_{i \in I(\tilde{\tilde{\delta}})} \binom{n-1}{i-1} N(\alpha^{i-1}) N(\alpha^{n-i})$$

$$< \sum_{i=1}^{n} \binom{n-1}{i-1} N(\alpha^{i-1}) N(\alpha^{n-i})$$

$$= 2N(\alpha^n).$$

In the above calculation, the first inequality ($\leqq$) is derived by the induction hypothesis, and the second one ($<$) follows since $\binom{n-1}{i-1} N(\alpha^{i-1}) N(\alpha^{n-i}) > 0$ for any $i (1 \leqq i \leqq n)$. Hence we conclude that the theorem holds for this n.

By this theorem, the Eq. (10) becomes:

$$2M_{n+1} = \sum_{i=0}^{n} \binom{n}{i} M_i M_{n-i} \qquad (n \geqq 1). \tag{11}$$

We put:

$$Q_n = \frac{M_n}{n!} \qquad (n \geqq 0). \tag{12}$$

Then we have

$$\left.\begin{array}{l} Q_0 = Q_1 = 1, \\ 2(n+1)Q_{n+1} = \sum_{i=0}^{n} Q_i Q_{n-i} \qquad (n \geqq 1). \end{array}\right\} \tag{13}$$

We consider the generating function of Q_n:

$$q(x) = \sum_{n=0}^{\infty} Q_n x^n. \tag{14}$$

From (13) and (14) we obtain the following differential equation:

$$\left.\begin{array}{l} q(0) = 1, \\ 2q'(x) = q^2(x) + 1. \end{array}\right\} \tag{15}$$

The solution of (15) is:

$$q(x) = \tan\left(\tfrac{1}{2}x + \tfrac{\pi}{4}\right)$$

$$= \frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}$$

$$= \frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}$$

$$= \frac{1 + 2\cos\frac{x}{2}\sin\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \frac{1 + \sin x}{\cos x}$$

$$= \sec x + \tan x. \qquad (16)$$

Now, for $|x| < \frac{\pi}{2}$, we have

$$\sec x = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n}, \qquad (17)$$

and

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1}, \qquad (18)$$

where E_{2n} and B_{2n} denote Euler and Bernoulli numbers respectively. (See e.g.[2])

From (12), (15), (16), (17) and (18), we finally obtain

Theorem 3.

$$\underline{M_{2n} = E_{2n}} \qquad \underline{(\text{if } n \geqq 0),}$$

$$\underline{M_{2n-1} = \frac{2^{2n}(2^{2n}-1)}{2n} B_{2n}} \qquad \underline{(\text{if } n \geqq 1).}$$

The values of M_n for $n \leq 10$ is as follows:
$M_0 = M_1 = M_2 = 1$, $M_3 = 2$, $M_4 = 5$, $M_5 = 16$, $M_6 = 61$, $M_7 = 272$, $M_8 = 1385$, $M_9 = 7936$, $M_{10} = 50521$.

We now consider the asymptotic behavior of M_n. First, we quote the following theorem due to König (see [1]).

Theorem 4. Let $h(z) = \sum_{i=0}^{\infty} c_i z^i$ $(c_0 \neq 0)$ be meromorphic for $|z| < R$. Suppose that h has only one singular point $z = z_r$ in $|z| < R$, and that z_r is a simple pole. Then, for any ρ such that $|z_r| < \rho < R$, we have

$$\frac{c_k}{c_{k+1}} = z_r\{1 + O(\left|\frac{z_r}{\rho}\right|^{k+1})\}.$$

Since the singular points of $q(z)$ are $\frac{\pi}{2} + 2n\pi$ (n: integer), and they are all simple poles, we have by the above theorem that

$$\lim_{n\to\infty} \frac{Q_n}{Q_{n+1}} = \frac{\pi}{2} .$$

In other words:

Theorem 5.

$$\lim_{n\to\infty} \frac{M_n}{nM_{n-1}} = \frac{2}{\pi}.$$

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