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GFAPH AND PERMANENT

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The permanent of a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} \qquad (m \leq n) \tag{1}$$

has been definend as :

$$\mathrm{per}(A) = \sum a_{1\alpha_1} a_{2\alpha_2} \cdots\cdots a_{m\alpha_m} \tag{2}$$

where the sum is taken over all permutation $(\alpha_1, \alpha_2, \ldots \alpha_n)$ of the set $\{1,2,\ldots,n\}$, and its value is also denoted by

$$\overset{+}{|} A \overset{+}{|} = \overset{+}{\left|\begin{matrix} a_{11} & \cdots\cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots\cdots & a_{mn} \end{matrix}\right|}^{+} \tag{3}$$

though the permanent may seem familiar to the idea of the determinant, the concept is very differrent. And the evaluation of the permanents are in general very hard.

The work is usuary given by the next theorem (Cf.[1])

<u>Theorem 1</u> (Ryser) Let A be a matrix of $m\times n$ $(m\leq n)$ type. And let A denote a matrix obtained from A by repracing r colums of A by zeros.

Let $S(A_r)$ denote the product of the row sums of A_r, and let $\sum S(A_r)$ denote the sums of the $S(A_r)$ over all of the choice for A_r. Then

$$\begin{aligned}\mathrm{per}(A)=&\sum S(A_{n-m})-\binom{n-m+1}{1}\sum S(A_{n-m+1})\\&+\binom{n-m+2}{2}\sum S(A_{n-m+2})-\cdots\cdots\\&\cdots\cdots+(-1)^{m-1}\binom{n-1}{m-1}\sum S(A_{n-1})\end{aligned}\tag{4}$$

And, as a corollary of this theorem, we have

<u>Corollary</u>. Let A be a matrix of order n, then

$$\begin{aligned}\mathrm{per}(A)=&\ S(A)-\sum S(A_1)+\sum S(A_2)-\cdots\cdots\\&\cdots\cdots+(-1)^{n-1}\sum S(A_{n-1}).\end{aligned}\tag{5}$$

For example,

$$\mathrm{per}\ (I)=\left|\begin{matrix}1 & & 0\\ & 1 & \\ & & \ddots & \\ 0 & & & 1\end{matrix}\right|^{+}_{+}=1 \qquad (I=I_n)\tag{6}$$

$$\mathrm{per}(J)=\left|\begin{matrix}1 & \cdots & 1\\ \vdots & & \\ 1 & \cdots & 1\end{matrix}\right|^{+}_{+}=n! \qquad (J=J_n)\tag{7}$$

$$\mathrm{per}(J-I)=\mathrm{per}(A(K_n))=\left|\begin{matrix}0 & 1 & 1 & \cdots & 1\\ 1 & 0 & 1 & \cdots & 1\\ 1 & 1 & 0 & \cdots & 1\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 1 & 1 & 1 & \cdots & 0\end{matrix}\right|^{+}$$

$$=n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\cdots\cdots+(-1)^n\frac{1}{n!}\right)\tag{8}$$

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{vmatrix}^{+} = S(A_4) - \binom{5}{1} \sum S(A_5)$$

$$+ \binom{6}{2} \sum S(A_6) - \binom{7}{3} S(A_7) = 136 \qquad (9)$$

Let, $G(V,L)$ be abipartete graph, where

$$V = \{u_1, u_2, \ldots\ldots, u_m; v_1, v_2, \ldots\ldots, v_n\}$$
$$= \{U_m; V_n\} \quad \text{say.}$$

And, let B be its bipartete matrix

$$B = B(G) = \begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{array} \!\! \begin{array}{c} \begin{array}{ccc} v_1 & v_2 \cdots & v_n \end{array} \\ \begin{pmatrix} b_{11} & b_{12} \cdots & b_{1n} \\ b_{21} & b_{22} \cdots & b_{2n} \\ \cdot & \cdot \quad \cdot & \cdot \\ b_{m1} & b_{m2} \cdots & b_{mn} \end{pmatrix} \end{array}$$

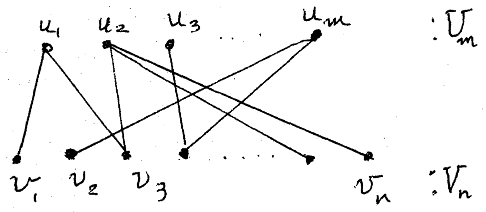


fig. 1

where

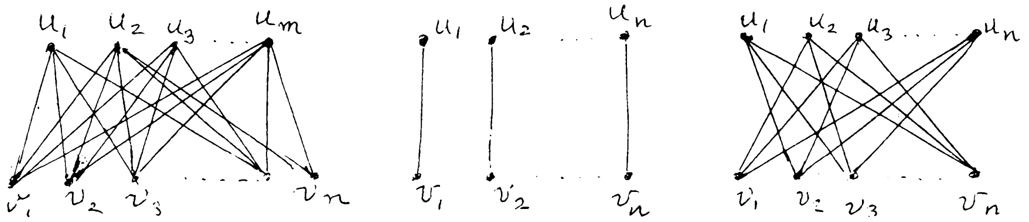
$$b_{ij} = \begin{cases} 1 & (u_i\ v_j \text{ are adjacent}) \\ 0 & (u_i; v_j \text{ are not adjacent}) \end{cases}$$

For example

$$B(K_{m,n}) = \overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdots & 1 \end{bmatrix}}^{n} \Bigg\} m \qquad (10)$$

$$B(\ B_S\ ) = \begin{bmatrix} 1 & 0 & 0 \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \cdots & 1 \end{bmatrix} \Bigg\} n \qquad (11)$$

$$B(K_{m,n} - B_S) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix} \Bigg\} n \qquad \text{(Cf. fig 2)} \quad (12)$$



$K_{m,n}$ B_S $K_{n,n}-B_s$

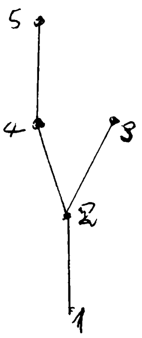
fig.2

We will call $K_{m,n}-B_S$ as semi-complete $m\times n$ bipartete graph, and this coresponds to the n-th complete graph K_n.

In general, we obtain an adjacent (square) matrix $A(G)$ to a given graph G. And by regarding $A(G)$ as a $n\times n$ bipartite matrix, we can construct a bipartite graph B_G from the linear graph G. We will call this process the Bipartete Transformation. That is

$$\varphi :\ G \xrightarrow{A(G)} B_G \qquad (A(G)=B(B_G)) \qquad (13)$$

For example



G

$$A(G)=B(B_G)=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

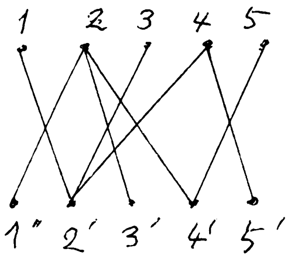


fig.3

For the permutation $\{1,2,3,\ldots\ldots n\}\to\{i_1,i_2,\ldots\ldots i_n\}$, we can concider a bipartete graph, as the vertexes

$$V=\{1,2,\ldots\ldots,n;\ 1',2',\ldots\ldots,n'\}$$

and, where ν is adjacent to i_ν, and ν is not adjacent to i_μ ($\nu \neq \mu$). This graph is a 1-factor graph, and we call this graph as the Permutation graph.

And the bipartete matrixP for permutation graph is called permutation matrix , which satisfy the condition

$$PP^t = I \tag{13}$$

The value of permanent of permutation graph is equals to 1, and the converse is also valid; that is

$$\mathrm{per}(P)=1 \iff P \text{ is permutation matrix}$$

And let A be a square matrix, and P_1,P_2 be permutation matrixes, then we can easily prove the relation

$$\mathrm{per}(P_1AP_2)= \mathrm{per}(A) \ . \tag{14}$$

A linear graph is maped into a bipartete graph by the bipartete transformation, but its converse is not true (see fig.4).

A separarated linear graph is maped into a separate bipartete graph (see fig.5), and abipartete (contains the tree graph) is maped into a separated bipartete graph by suitably labelling. (here linear graph means simple gf.)

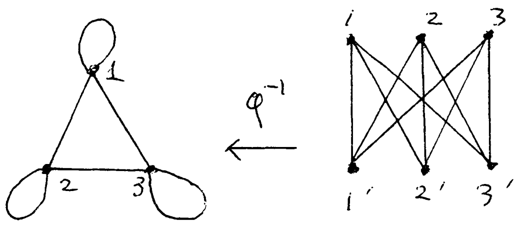


fig. 4

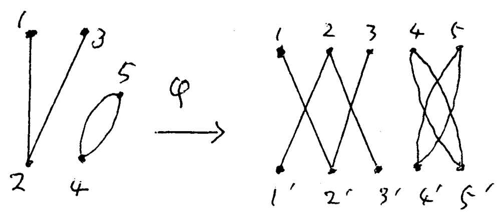


fig.5

Now the permanent of matrix A is the number of different 1-factor of G(A)([4]).that is

$$\operatorname{per}(A)=\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}^{+} = \text{the number of different 1-factors of } B(A)$$

This is one of the graphical meaning of permanent with respect to the bipartete graph B(A) from A. And we can extend this fact a little more;

Theorem 2 Let $B_{m,n}$ (m n) be a bipartete graph, where

$$V=\{U_m, V_n\}=\{u_1, u_2, \ldots, u_m; v_1, v_2, \ldots, v_n\}$$

and $B(B_{m,n})$ be its bipatete matrix, then the value of $\operatorname{per}(B(B_{m,n}))$ is equals to number of all 1-factor graph which pass the points of U_m. And their join of the all 1-factor graphs constitute the first bipartete graph.

The proof is immediate from definition of permanent of binary bipartete matrix.

It is clear that the concept of permanent of bipartete matrix relates to the enumeration in the Matching theory.

Erom the fact, let D_n, U_n be the number of Derangement and Ménage respectively, then we have

$$D_n=\operatorname{per}(J-I)=\begin{vmatrix} 0 & & 1 \\ & 0 & & \\ & & \ddots & \\ 1 & & & 0 \end{vmatrix}^{+} \quad \text{(fig.6, see(8))}$$

$$U_n=\operatorname{per}(J-I-C)=\begin{vmatrix} 0 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 1 & \cdots & 1 \\ \cdots & & & & & \\ 1 & 1 & 1 & 1 & \cdots 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots 1 & 0 \end{vmatrix}^{+} \quad \text{(fig.7)} \qquad (15)$$

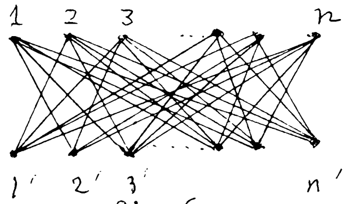


fig.6

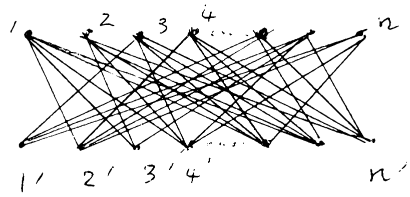


fig.7

Generally, for the substitution $1 \to \nu+1,\ 2 \to \nu+2,\ \cdots\ n \to n+\nu \pmod{n}$, the value of permanent of the corespondence bipartete matrix will be called ν-th Derangement number, and is denoted by $D_n^{(\nu)}$, and specially $D_n^{(1)} \equiv D_n$, $D_n^{(2)} \equiv U_n$; that is

$$D_n = \begin{array}{c} \overbrace{\begin{array}{ccccccccc} 1 & & & & \nu+1 & & & & n \end{array}}^{n} \\ \left|\begin{array}{ccccccccc} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & & 1 \\ 1 & 0 & \cdots & & 0 & 1 & \cdots & & 1 \\ 1 & 1 & \cdots & & 0 & 0 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \end{array}\right| \end{array} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\end{array}}\right\} n \qquad (16)$$

But the evaluation of $D_n^{(\nu)}$ is the other things.

Next, we will tell the meaning of the permanent;

A linear subgraph of directed graph D is a spanning subgraph(pass all the points of D) in which every point has both indegree and outdegree 1. Then per(A) is the number of different linear subgraphs of D(A). This is clearly a rephrasing of the definition of D(A) where A is a adjacent matrix of directed graph D. And, if A is regarded as an incdence matrix with rows as sets and columns as elements, then it is known that per(A) is the number of S.D.R.'s (systems of distinct representatives).That is;

<u>Theorem 3</u> If A is the incidence matrix for an S.D.R. probrem, then the value of per(A) equals to the number

of S.D.R.'s for that probrem.

Moreover , as for the magnitude of permanent, we can prove next formula;

Theorem 4(Marcus-May) If A is an n-square matrix with eigenvalues $\alpha_1, \alpha_2, \cdots, \alpha_n$, then

$$|\mathrm{per}(A)| \leq \frac{1}{n} \sum_{i=1}^{n} |\alpha_i|^{e} \qquad \text{(Cf. [5])} \tag{17}$$

Now we will consider the permanent by using the linear forms with bases (the operation are usual summation +, multiplication $\overset{+}{\wedge}$)

$$e_1, \ e_2, \ \cdots, \ e_n \qquad (e_i \overset{+}{\wedge} e_j = e_j \overset{+}{\wedge} e_i)$$

where they have the subsidery conditions;

$$e_i \overset{+}{\wedge} e_i = 0 \qquad \text{(Or simply denote } e_i \cdot e_i = e_i^2 = 0) \tag{18}$$

that is, we will consider the sets of linear forms

$$a_i = a_{i1} e_1 + a_{i2} e_2 + \cdots\cdots + a_{in} e_n \qquad (a_{ij} \in R) \tag{19}$$

in n-dimensional linear vector space, which next basis;

$$e_0 \ ; \ e_1, \ e_2, \cdots, e_n \ ; \ e_1 \overset{+}{\wedge} e_2, \ \cdots\cdots, e_{n-1} \overset{+}{\wedge} e_n ;$$
$$e_1 \overset{+}{\wedge} e_2 \overset{+}{\wedge} e_3, \ \cdots\cdots e_{n-2} \overset{+}{\wedge} e_{n-1} \overset{+}{\wedge} e_n ; \ \cdots ; \ e_1 \overset{+}{\wedge} e_2 \overset{+}{\wedge} \cdots \overset{+}{\wedge} e_n \tag{20}$$

Then we will define the permanent of A as follows;

$$\mathrm{per}(A) = (a_1 \overset{+}{\wedge} a_2 \overset{+}{\wedge} \cdots \overset{+}{\wedge} a_m) / (e_1 \overset{+}{\wedge} e_2 \overset{+}{\wedge} \cdots \overset{+}{\wedge} e_n) \tag{21}$$

when m=n , and

$$\mathrm{per}(A) = \{(a_1 \overset{+}{\wedge} a_2 \overset{+}{\wedge} \cdots \overset{+}{\wedge} a_m) \overset{+}{\wedge} (\textstyle\sum e_{i_1} \overset{+}{\wedge} e_{i_2} \overset{+}{\wedge} \cdots \overset{+}{\wedge} e_{i_{n-m}})\}$$
$$\div (e_1 \overset{+}{\wedge} e_2 \overset{+}{\wedge} \cdots \overset{+}{\wedge} e_n) \tag{22}$$

when $m < n$.

From this definition, we can easily deduce next formulas ;

I. per(A)=Per(A^t when m=n (A^t:transposed matrix of A)

(23)

II.

$$\operatorname{per}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{bmatrix}\right) = \operatorname{per}\left(\begin{bmatrix} a \\ \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}\right) \qquad (i \neq j) \tag{24}$$

III.

$$\operatorname{per}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_i + a_i' \\ \vdots \\ a_n \end{bmatrix}\right) = \operatorname{per}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}\right) + \operatorname{per}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_i' \\ \vdots \\ a_n \end{bmatrix}\right) \tag{25}$$

IIII.

$$\operatorname{per}\left(\begin{bmatrix} a_1 \\ \vdots \\ k a_i \\ \vdots \\ a_n \end{bmatrix}\right) = k \operatorname{per}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}\right) \qquad (k \in \mathbb{R}) \tag{26}$$

V. Similar formula of the cofactor expansion by row in determinant theory, that is

$$\begin{array}{c} + \qquad\qquad\qquad + \\ \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{vmatrix} \end{array} = \sum_{\nu=1}^{n} a_{1\nu} \begin{array}{c} + \qquad\qquad \nu \qquad\qquad + \\ \begin{vmatrix} a_{21} & \cdots & a_{2\nu} & \cdots & a_{2n} \\ \vdots & & \vdots & & \\ a_{m1} & \cdots & a_{m\nu} & \cdots & a_{mn} \end{vmatrix} \end{array} \tag{27}$$

It is considerable convinient to compute the value of permanent of binary (matrix with 0's and 1's as entry, especially when it has many 0's) matrix by the formula (27).

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