数理解析研究所講究録
第 261 巻 1976 年 29-40

AN ASYMPTOTIC FORMULA FOR FUNCTION SPACE INTEGRALS WITH RESPECT TO GAUSSIAN MEASURES

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1. A Laplace asymptotic formula for integrals in one dimension is of the form

$$\lim_{\lambda\to 0} \int_{-\infty}^{\infty} G(x)\exp[\lambda^{-2}F(x)]dx / \int_{-\infty}^{\infty}\exp[\lambda^{-2}F(x)]dx = G(\xi),$$

where F is a continuous function having a unique global maximum at ξ and G is continuous at ξ.

In [8], M. Schilder proved the following analogue of the above formula for integrals with respect to Wiener measure μ on the space $C = C[0,1]$ of continuous functions on [0,1]:

$$(*) \qquad \lim_{\lambda\to 0} \int_C G(\lambda x)\exp[\lambda^{-2}F(\lambda x)]\mu(dx) / \int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx) = G(\phi),$$

where F and G are real functionals on C satisfying certain conditions and ϕ is a maximizing point at which the functional $F(x) - (1/2)\int_0^1 (x'(t))^2 dt$ attains a global maximum over the space H of absolutely continuous functions on [0,1] vanishing at the origin and having square integrable derivatives. Schilder proved further some related asymptotic formulas and gave some applications of (*) to analysis.

M. Pincus [7] generalized the result of Schilder to a large class of Gaussian processes and showed a close connection with Hammerstein integral equations. Note also a comment in [3]. In a recent paper [4], R. Marcus has indicated that the results of Schilder and Pincus can be obtained by the method of that paper.

The object of this note is to prove the asymptotic formula (*) for integrals with respect to general Gaussian measures on C, making use of a Freidlin-Wentzell type estimate given in [5] (see Theorem 1, [6]) and the arguments used there. The proof is along the lines of that of Schilder, but it may be noted that it is considerably shorter than those of Schilder and Pincus. Furthermore, the result can be extended to abstract Wiener spaces using similar arguments and the extension is expected to have some applications to nonlinear functional equations in Banach spaces. The details will be published elsewhere ([2]).

2. Let $C = C[0,1]$ be the space of all real continuous functions on $[0,1]$ with the sup norm $||\cdot||_\infty$, and let $\mathcal{A}$ be the σ-field of Borel subsets of C. Let μ be a Gaussian measure on $(C, \mathcal{A})$ with mean zero and covariance function $R(s,t)$, and let $H = H(R)$ be the reproducing kernel Hilbert space (RKHS) with reproducing kernel R, whose norm is denoted by $||\cdot||_H$. Note that $H \subset C$ and the RKHS associated with Wiener measure is the space H in the Introduction.

<u>Theorem 1</u>. Suppose F and G are real measurable functionals on C satisfying the following conditions:

(a) $F(\psi) - (1/2)||\psi||_H^2$, $\psi \in H$, attains its unique global maximum over H at $\phi \in H$,

(b) $F(x) < a_1 + a_2||x||_\infty$ for all $x \in C$, where a_1 is any positive constant, $a_2 < (4M)^{-1}$ and $M = \sup_{0\le t\le 1} R(t,t)$,

(c) F is uniformly continuous on the set $E = \{ x \in C \mid ||x||_\infty \le 2rM^{1/2}\}$, where $r^2 > 4\{a_1 - F(\phi) + (1/2)||\phi||_H^2\}$,

(d) $|G(x)| < b_1 \exp(b_2||x||_\infty^2)$ for all $x \in C$, where b_1 and b_2 are any positive constants, and G is continuous at ϕ.

Then

$$(*)\quad \lim_{\lambda\to 0} \{\int_C G(\lambda x)\exp[\lambda^{-2}F(\lambda x)]\mu(dx)/\int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx)\} = G(\phi).$$

Remark. It follows from (b) and (c) that $r^2 > ||\phi||_H^2 \ge 0$. Note also that $\phi \in E$, since $||\phi||_\infty \le M^{1/2}||\phi||_H < M^{1/2}r$.

Proof. (i) By the condition (d), for any $\varepsilon > 0$, there is a $\delta > 0$ such that $||x - \phi||_\infty < \delta$ implies $|G(x) - G(\phi)| < \varepsilon/2$. Hence

$$\left| \int_C G(\lambda x)\exp[\lambda^{-2}F(\lambda x)]\mu(dx)/\int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx) - G(\phi) \right|$$

$$< \varepsilon/2 + \int_{||\lambda x-\phi||_\infty \ge \delta} |G(\lambda x)-G(\phi)|\exp[\lambda^{-2}F(\lambda x)]\mu(dx)/\int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx).$$

We shall show that the second term is $< \varepsilon/2$ for all sufficiently small λ.

(ii) Consider first the denominator $\int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx)$. By the condition (c) and the remark, for any $\eta > 0$, there is a $\delta' > 0$ such that $||\lambda x - \phi||_\infty < \delta'$ implies $|F(\lambda x) - F(\phi)| < \eta/2$ and hence $F(\phi) - \eta/2 < F(\lambda x)$. Therefore, using a Freidlin-Wentzell type estimate, we obtain

$$\begin{aligned}\int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx) &\ge \int_{||\lambda x-\phi||_\infty<\delta'} \exp[\lambda^{-2}F(\lambda x)]\mu(dx) \\ &\ge \exp[\lambda^{-2}(F(\phi)-\eta/2)]\cdot\mu\{ ||\lambda x - \phi||_\infty < \delta' \} \\ &\ge \exp[\lambda^{-2}(F(\phi)-\eta/2)]\cdot\exp[-(1/2)\lambda^{-2}(||\phi||_H^2+\eta)]\end{aligned}$$

$$= \exp[\lambda^{-2}\{F(\phi)-(1/2)||\phi||_H^2-\eta\}]$$

for any $\eta > 0$, if λ is sufficiently small.

(iii) Let $K_r = \{ \psi \in H | \ ||\psi||_H \leq r \}$. Since K_r is compact in C and the set $\{ x \ | \ ||x - \phi||_\infty \geq \delta/2 \}$ is closed,

$$D = D(r,\delta) = K_r \cap \{ x \ | \ ||x - \phi||_\infty \geq \delta/2 \}$$

is compact in C. $||\psi||_H$ is lower semi-continuous in the sup norm topology, and since $D \subset K_r \subset E$, $F(\psi) - (1/2)||\psi||_H^2$ is upper semi-continuous in D. By the condition (a), ϕ is the unique maximizing point, and hence there is an $\eta' = \eta'(r,\delta) > 0$ such that

$$\max_{\psi \in D} \{ F(\psi) - (1/2)||\psi||_H^2 \} < F(\phi) - (1/2)||\phi||_H^2 - \eta'.$$

Choose $\delta'' > 0$ small enough so that $\delta'' < \min(\delta/2, rM^{1/2})$ and if $||x - y||_\infty < \delta''$ for $x, y \in E$, $|F(x) - F(y)| < \eta'/3$, which is possible by the condition (c).

Let $\{\psi_j\}$ be a complete orthonormal system in H and let $\{\xi_j(x)\}$ be a sequence of independent standard Gaussian random variables defined by

$$\psi_j(t) = \int_C \xi_j(x)x(t)\mu(dt), \quad 0 \leq t \leq 1.$$

Put

$$x_n(t) = \sum_{j=1}^{n} \xi_j(x)\psi_j(t), \quad 0 \leq t \leq 1.$$

Then

$$\sigma_n^2 = \sup_{0\leq t\leq 1} \int_C \{x(t)-x_n(t)\}^2\mu(dx) = \sup_{0\leq t\leq 1} \{R(t,t) - \sum_{j=1}^{n} \psi_j^2(t)\} \to 0 \quad \text{as } n \to \infty.$$

Let N be an integer such that $1/(2\sigma_N^2) > \gamma = r^2/(2(\delta'')^2)$. Put

$$A = \{ x \mid ||\lambda x - \phi||_\infty \geq \delta,\ ||\lambda x - \lambda x_N||_\infty < \delta",\ ||\lambda x_N||_H \leq r \}$$

and

$$B = \{ x \mid ||\lambda x - \lambda x_N||_\infty \geq \delta" \} \cup \{ x \mid ||\lambda x_N||_H > r \}.$$

Then $\{ x \mid ||\lambda x - \phi||_\infty \geq \delta \} \subset A \cup B$.

(iv) By the conditions (b), (d) and Schwarz's inequality, we have

$$\int_B |G(\lambda x)| \exp[\lambda^{-2} F(\lambda x)] \mu(dx)$$

$$< \int_B b_1 \exp(b_2 ||\lambda x||_\infty^2) \cdot \exp[\lambda^{-2}(a_1 + a_2 ||\lambda x||_\infty^2)] \mu(dx)$$

$$\leq b_1 \exp(a_1 \lambda^{-2}) \{ \int_C \exp[2(b_2 \lambda^2 + a_2) ||x||_\infty^2] \mu(dx) \}^{1/2} \cdot (\mu(B))^{1/2}.$$

By the condition (b), $2(b_2\lambda^2 + a_2) < (2M)^{-1}$ for all sufficiently small λ, and by Fernique-Marcus-Shepp's theorem ([1], [3]) on the supremum of Gaussian processes,

$$\int_C \exp[2(b_2\lambda^2 + a_2) ||x||_\infty^2] \mu(dx) \leq \text{const.} < \infty.$$

Now

$$\mu(B) \leq \mu\{ ||x - x_N||_\infty > \delta/\lambda \} + \mu\{ ||x_N||_H > r/\lambda \}.$$

By Fernique-Marcus-Shepp's theorem,

$$\text{the first term} < \text{const.} \cdot \exp(-\gamma(\delta")^2 \lambda^{-2})$$

$$\leq \text{const.} \cdot \exp[-2\lambda^{-2}\{a_1 - F(\phi) + (1/2)||\phi||_H^2 + h\}],$$

for any small $h > 0$ such that $r^2 > 4\{a_1 - F(\phi) + (1/2)||\phi||_H^2\} + 5h$, and by the condition (c), for all sufficiently small λ,

$$\text{the second term} = \mu\{ \sum_{j=1}^{N} \xi_j^2(x) > r^2/\lambda^2 \}$$

$$\leq \text{const.}\cdot\exp[-(1/2)\lambda^{-2}(r^2 - h)]$$

$$\leq \text{const.}\cdot\exp[-2\lambda^{-2}\{a_1 - F(\phi) + (1/2)||\phi||_H^2 + h\}].$$

Hence we have, for all sufficiently small λ,

$$\int_B |G(\lambda x)|\exp[\lambda^{-2}F(\lambda x)]\mu(dx) \leq \text{const.}\cdot\exp[\lambda^{-2}\{F(\phi) - (1/2)||\phi||_H^2 - h\}].$$

Similarly we have

$$\int_B |G(\phi)|\exp[\lambda^{-2}F(\lambda x)]\mu(dx) \leq \text{const.}\cdot\exp[\lambda^{-2}\{ F(\phi) - (1/2)||\phi||_H^2 - h\}].$$

(v) Let $x \in A$. Then $\lambda x_N \in K_r \subset E$ and

$$||\lambda x||_\infty < ||\lambda x - \lambda x_N||_\infty + ||\lambda x_N||_\infty < \delta'' + rM^{1/2} < 2rM^{1/2}.$$

Since $||\lambda x - \lambda x_N||_\infty < \delta''$ and $\lambda x, \lambda x_N \in E$, $F(\lambda x) < F(\lambda x_N) + \eta'/3$. Furthermore, since

$$||\lambda x_N - \phi||_\infty \geq ||\lambda x - \phi||_\infty - ||\lambda x - \lambda x_N||_\infty \geq \delta - \delta'' \geq \delta/2,$$

$$F(\lambda x_N) \leq (1/2)||\lambda x_N||_H^2 + F(\phi) - (1/2)||\phi||_H^2 - \eta',$$

and hence

$$F(\lambda x) \leq (1/2)||\lambda x_N||_H^2 + F(\phi) - (1/2)||\phi||_H^2 - (2/3)\eta'.$$

Let β be a number such that $0 < \beta < (2/3)\eta' r^{-2}$. Then $(1/2)\beta||\psi||_H^2 < \eta'/3$ for all $\psi \in K_r$. Therefore, $(1/2)\beta||\lambda x_N||_H^2 < \eta'/3$ and hence

$$F(\lambda x) \leq (1/2)(1 - \beta)||\lambda x_N||_H^2 + F(\phi) - (1/2)||\phi||_H^2 - \eta'/3.$$

Thus

$$\int_A |G(\lambda x)|\exp[\lambda^{-2}F(\lambda x)]\mu(dx) < b_1\exp(b_2\lambda^2\cdot 4Mr^2)\cdot\int_A \exp[\lambda^{-2}F(\lambda x)]\mu(dx)$$

$$\leq \text{const.}\cdot\exp[\lambda^{-2}\{F(\phi)-(1/2)||\phi||_H^2-\eta'/3\}]\cdot\int_C \exp[(1/2)(1-\beta)||x_N||_H^2]\mu(dx).$$

But

$$\int_C \exp[(1/2)(1-\beta)||x_N||_H^2]\mu(dx) = \int_C \exp[(1/2)(1-\beta)\cdot\sum_{j=1}^{N}\xi_j^2(x)]\mu(dx)$$

$$= (2\pi)^{-N/2}\cdot\prod_{j=1}^{N}\int_{-\infty}^{\infty}\exp[-(\beta/2)s^2]ds < \infty.$$

Therefore

$$\int_A |G(\lambda x)|\exp[\lambda^{-2}F(\lambda x)]\mu(dx) < \text{const.}\cdot\exp[\lambda^{-2}\{F(\phi) - (1/2)||\phi||_H^2 - \eta'/3].$$

Similarly we have

$$\int_A |G(\phi)|\exp[\lambda^{-2}F(\lambda x)]\mu(dx) < \text{const.}\cdot\exp[\lambda^{-2}\{F(\phi) - (1/2)||\phi||_H^2 - \eta'/3].$$

(vi) From (iv) and (v) we get

$$\int_{||\lambda x-\phi||_\infty\geq\delta} |G(\lambda x)-G(\phi)|\exp[\lambda^{-2}F(\lambda x)]\mu(dx)$$

$$< \text{const.}\cdot\{\exp[\lambda^{-2}\{F(\phi)-(1/2)||\phi||_H^2-\eta'/3\}]$$

$$+ \exp[\lambda^{-2}\{F(\phi)-(1/2)||\phi||_H^2-h\}]\},$$

Choose $\eta > 0$ (in (ii)) small enough so that $\eta < \min(\eta'/3, h)$. Then

$$\int_{||\lambda x-\phi||_\infty\geq\delta} |G(\lambda x)-G(\phi)|\exp[\lambda^{-2}F(\lambda x)]\mu(dx)/\int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx)$$

$$< \text{const.}\cdot\{\exp[-\lambda^{-2}(\eta'/3-\eta)] + \exp[-\lambda^{-2}(h-\eta)]\} < \varepsilon/2$$

for all sufficiently small λ. This completes the proof.

3. For the integrability of $G(\lambda x)\exp[\lambda^{-2}F(\lambda x)]$, it is enough to assume $a_2 < (2M)^{-1}$, and it is desirable to weaken the restriction $a_2 < (4M)^{-1}$ in the condition (b). This can be done if the condition (c) is replaced by a stronger condition, and we have the following (perhaps better formulation of) result.

Theorem 2. Assume the conditions (a) and (d) of Theorem 1 and the following conditions:

(b') $F(x) < a_1 + a_2||x||_\infty$ for all $x \in C$, where a_1 is any positive constant, $a_2 < (2M)^{-1}$ and $M = \sup_{0\le t\le 1} R(t,t)$, and

(c') F is uniformly continuous on any bounded set in C.

Then the asymptotic formula (*) holds.

Proof. The condition $a_2 < (4M)^{-1}$ is used only in the step (iv) of the proof of Theorem 1. It hence suffices to make the following small changes in the arguments. Choose a number $p > 1$ close enough to 1 so that $pa_2 < (2M)^{-1}$ and put $q = p/(p-1)$. Let r be a number such that

$$r^2 > 2q\cdot\{a_1 - F(\phi) + (1/2)||\phi||_H^2\} + (2q+1)h$$

for some $h > 0$. Use Hölder's inequality instead of Schwarz's in the step (iv) in the proof. Then

$$\int_B |G(\lambda x)|\exp[\lambda^{-2}F(\lambda x)]\mu(dx)$$

$$< b_1\exp(a_1\lambda^{-2})\cdot\{\int_C \exp[p(b_2\lambda^2+a_2)||x||_\infty^2]\mu(dx)\}^{1/p}\cdot(\mu(B))^{1/q}.$$

Since $p(b_2\lambda^2+a_2) < (2M)^{-1}$ for all sufficiently small λ,

$$\int_C \exp[p(b_2\lambda^2+a_2)||x||_\infty^2]\mu(dx) \le \text{const.} < \infty,$$

and by the same reasoning as in (iv) we get, for sufficiently small λ,

$$\mu(B) < \text{const.}\cdot\exp[-q\lambda^{-2}\{a_1 - F(\phi) + (1/2)||\phi||_H^2 + h\}].$$

Hence, for all sufficiently small λ,

$$\int_B |G(\lambda x)|\exp[\lambda^{-2}F(\lambda x)]\mu(dx) < \text{const.}\cdot\exp[\lambda^{-2}\{F(\phi)-(1/2)||\phi||_H^2-h\}].$$

The rest of proof does not require any changes.

Remark. The uniform continuity of F is a technical assumption. Note that Pincus [3] assumed the uniform Hölder continuity of F in the L^2-norm.

In (ii) of the proof of Theorem 1 it has been shown that

$$\int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx) > \exp[\lambda^{-2}\{F(\phi) - (1/2)||\phi||_H^2 - \eta\}]$$

for any $\eta > 0$, if λ is sufficiently small. In fact, the arguments used in the proof yield the following asymptotic estimate for $\int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx)$.

Theorem 3. Suppose F satisfies the conditions (a), (b) and (c) (or (b') and (c')). Then

$$\lim_{\lambda\to 0} \lambda^2\cdot\log \int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx) = F(\phi) - (1/2)||\phi||_H^2.$$

Proof. In view of the above result in (ii), it suffices to show that

$$\int_C \exp[\lambda^{-2}F(\lambda x)]\mu(dx) < \exp[\lambda^{-2}\{F(\phi) - (1/2)||\phi||_H^2 + \varepsilon\}]$$

for any $\varepsilon > 0$, if λ is sufficiently small. Put

$$A = \{\ x \mid ||\lambda x - \lambda x_N||_\infty < \delta,\ ||\lambda x_N||_H \leq r\ \},$$

where $\delta > 0$ is a number to be specified later and N is an integer such that $(2\sigma_N^2)^{-1} > r^2(2\delta^2)^{-1}$ (see (iii)). Then, in exactly the same way as in (iv) of the proof of Theorem 1 and in the proof of Theorem 2, it can be shown that, for any $\delta > 0$,

$$\int_{C-A}\exp[\lambda^{-2}F(\lambda x)]\mu(dx) < \text{const.}\cdot\exp[\lambda^{-2}\{F(\phi) - (1/2)||\phi||_H^2 - h\}]$$

$$< \exp[\lambda^{-2}\{F(\phi) - (1/2)||\phi||_H^2 - h'\}]$$

for a sufficiently small $h' > 0$, if λ is sufficiently small.

Given $\varepsilon > 0$, choose $\delta > 0$ small enough so that if $x \in A$, λx, $\lambda x_N \in E$, and if $||\lambda x - \lambda x_N||_\infty < \delta$, then $F(\lambda x) < F(\lambda x_N) + \varepsilon/3$. Since

$$F(\lambda x_N) \leq F(\phi) - (1/2)||\phi||_H^2 + (1/2)||\lambda x_N||_H^2,$$

$$\int_A \exp[\lambda^{-2}F(\lambda x)]\mu(dx)$$

$$< \exp[\lambda^{-2}\{F(\phi)-(1/2)||\phi||_H^2+\varepsilon/3\}]\cdot\int_A\exp[(1/2)\lambda^{-2}||\lambda x_N||_H^2]\mu(dx).$$

Just as in (v) of the proof of Theorem 1, let $\beta > 0$ be a number such that $\beta < (2/3)\varepsilon r^{-2}$. Then

$$\int_A\exp[(1/2)\lambda^{-2}||\lambda x_N||_H^2]\mu(dx)$$

$$\leq \exp((\varepsilon/3)\lambda^{-2})\cdot\int_A\exp[(1/2)(1-\beta)||x_N||_H^2]\mu(dx)$$

$$< \text{const.}\cdot\exp((\varepsilon/3)\lambda^{-2})$$

$$< \exp((2/3)\varepsilon\lambda^{-2})$$

for sufficiently small λ. Hence

$$\int_A\exp[\lambda^{-2}F(\lambda x)]\mu(dx) < \exp[\lambda^{-2}\{F(\phi) - (1/2)||\phi||_H^2 + \varepsilon\}]$$

if λ is sufficiently small. The proof is complete.

Remark. More precise results can be obtained, if F is assumed to have derivatives at ϕ.

If F is continuous on C, then the condition (b') (and (b)) implies that $F(\psi) - (1/2)||\psi||_H^2$ attains its supremum over H at a point in H, and hence we need only assume the uniqueness of its maximizing point ϕ in H.

Indeed, since

$$F(\psi) - (1/2)||\psi||_H^2 < a_1 + a_2||\psi||_\infty^2 - (1/2)||\psi||_H^2$$
$$\leq a_1 + (a_2 M - (1/2))||\psi||_H^2 \leq a_1,$$

$a_1 > \sup_{\psi \in H} \{F(\psi) - (1/2)||\psi||_H^2\} = d_0$, say, and, given any $d > 0$, if $||\psi||_H^2 > (a_1 - d_0 - d)/((1/2) - a_2 M) = c^2$, then $F(\psi) - (1/2)||\psi||_H^2 < d_0 - d$. Since $F(\psi) - (1/2)||\psi||_H^2$ is upper semi-continuous in $||\cdot||_\infty$ and $K_c = \{ \psi \in H \mid ||\psi||_H \leq c \}$ is compact in C, $F(\psi) - (1/2)||\psi||_H^2$ attains its supremum on K_c, and hence on H, in $K_c \subset H$.

References

[1] X. Fernique: Régularité de processus gaussiens. Inventiones math., 12 (1971), 304-320.

[2] G. Kallianpur and H. Oodaira: paper in preparation.

[3] M. B. Marcus and L. A. Shepp: Sample behavior of Gaussian processes. Proc. Sixth Berkeley Sympos. Math. Statist. Prob. 2, Univ. of California Press (1972), 423-441.

[4] R. Marcus: Parabolic Itô equations. Trans. Amer. Math. Soc., 198 (1974), 177-190.

[5] H. Oodaira: On Freidlin-Wentzell's estimates for Gaussian processes. Preprint (1975).

[6] H. Oodaira: Tail probabilities of some continuous functionals of Gaussian processes. Preprint (1975). (This report)

[7] M. Pincus: Gaussian processes and Hammerstein integral equations. Trans. Amer. Math. Soc., 134 (1968), 193-214.

[8] M. Schilder: Some asymptotic formulas for Wiener integrals. Trans. Amer. Math. Soc., 126 (1966), 63-85.