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Vanishing cycles on complex analytic sets.

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In this talk we are going to give a summary of some results about the topology of complex analytic sets. We shall deal especially about a theorem of Zariski-Lefschetz type proved recently by Mitsuyoshi Kato and the author.

1. The fibration theorem.

Let $X \subset U \subset \mathbb{C}^N$ be a complex analytic subset of an open set U of $\mathbb{C}^N$. We shall consider the topology of X in the neighbourhood of a point $x \in X$.

Let $f : X \to \mathbb{C}$ be an analytic function defined on X (actually we only need it is defined in a neighbourhood of $x \in X$). For simplicity we shall assume that $f(x) = 0$.

First we notice that, because of a theorem of Łojasiewicz - Giesecke (cf [15] or [6]), X is triangulable, thus we obtain:

Lemma (1.1) If $\varepsilon > 0$ is small enough, B_ε being the ball of $\mathbb{C}^N$ of radius $\varepsilon > 0$ centered at x, $B_\varepsilon \cap X$ is contractible.

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Actually, from the theorem of Łojasiewicz we may obtain:

<u>Lemma</u> (1.2) If ε and ε', $\varepsilon \geq \varepsilon' > 0$, are small enough, then $B_{\varepsilon'} \cap X$ is a retract by deformation of $B_\varepsilon \cap X$ homeomorphic to $B_\varepsilon \cap X$. In fact for any finite family $(Y_i)_{i \in I}$ of analytic subsets of X such that $x \in Y_i$, the pair $(B_{\varepsilon'} \cap X, B_{\varepsilon'} \cap Y_i)_{i \in I}$ is a retract by deformation of $(B_\varepsilon \cap X, B_\varepsilon \cap Y_i)_{i \in I}$ which is moreover homeomorphic to $(B_\varepsilon \cap X, B_\varepsilon \cap Y_i)_{i \in I}$.

We recall a definition due to D. Prill [19]:

<u>Definition</u> (1.3) Let X be a topological space, $x \in X$ and $Y \subset X$ a topological subspace of X. A neighbourhood U of x is called a <u>good neighbourhood of X with regard to Y</u> if there exists a fundamental system of neighbourhoods $(U_\alpha)_{\alpha \in A}$ of x in X such that $U_\alpha - Y$ is a retract by deformation of $U - Y$.

Thus from the lemma (1.2) we may say that for any analytic subset Y of X such that $x \in Y$, sufficiently small balls B_ε centered at x define good neighbourhoods $B_\varepsilon \cap X$ of x with regard to Y. Moreover it is clear that for any good neighbourhood U of x with regard to Y the homotopy type of $U - Y$ is the same.

Now we are going to state a fibration theorem which generalizes a theorem of Milnor (cf [17]) when $X = \mathbb{C}^N$ and a theorem of H. Hamm when X is a complete intersection and $X - \{f = 0\}$ is smooth (cf [7]).

<u>Theorem</u> (1.4) (Fibration theorem) If $\varepsilon > 0$ is small enough and $\varepsilon \gg \eta > 0$, then, if D_η is the open disk of $\mathbb{C}$ centered at 0 with radius η, f induces a topological fibration $B_\varepsilon \cap X \cap f^{-1}(D_\eta - \{0\}) \xrightarrow{\varphi_{\varepsilon,\eta}} D_\eta - \{0\}$ which is locally trivial. Moreover all these fibrations are fiber homeomorphic.

<u>Remark</u> In [5], P. Deligne quoted that such a theorem should be true but gave an incomplete hint of the proof. We are not going to prove this theorem here but just give a hint of proof a bit more complete. But first we want to define the vanishing cycles of f at x.

Let $\varepsilon > 0$ be small enough and $\varepsilon \gg \eta > 0$ as in the theorem (1.4). It is easy to see that $f^{-1}(D_\eta) \cap B_\varepsilon$ retracts by deformation on $f^{-1}(0) \cap B_\varepsilon$. Thus $f^{-1}(D_\eta) \cap B_\varepsilon$ is contractible. But in general if $t \in D_\eta - \{0\}$, $f^{-1}(t) \cap B_\varepsilon$ is not contractible. The cycles of $f^{-1}(t) \cap B_\varepsilon$ are called <u>vanishing cycles of f at x</u>.

<u>Example</u>: From Milnor's work in [17] it is known that when $X = \mathbb{C}^n$ and f has an isolated critical point at 0, $f^{-1}(t) \cap B_\varepsilon$, for t small enough, $\varepsilon \gg |t| > 0$, has the homotopy type of a bouquet of real spheres of dimension $n-1$.

Consider now the infinite cyclic covering $\widetilde{D}_\eta \xrightarrow{\pi} D_\eta - \{0\}$ of $D_\eta - \{0\}$. Pulling back $\varphi_{\varepsilon,\eta}$ by π we obtain:

$$\begin{array}{ccccc}
\widetilde{X}_{\varepsilon,\eta} & \xrightarrow{\widetilde{\pi}} & X_{\varepsilon,\eta} & = & B_\varepsilon \cap X \cap f^{-1}(D_\eta - \{0\}) \\
\widetilde{\varphi}_{\varepsilon,\eta} \downarrow & & \downarrow \varphi_{\varepsilon,\eta} & & \\
\widetilde{D}_\eta & \xrightarrow{\pi} & D_\eta - \{0\} & &
\end{array}$$

Then $\widetilde{\varphi}_{\varepsilon,\eta}$ is a locally trivial topological fibration and $\widetilde{\pi}$ is ~~a cyclic~~ infinite cyclic covering of $X_{\varepsilon,\eta}$. But, because $\widetilde{D}_\eta$ is contractible $\widetilde{X}_{\varepsilon,\eta}$ is homeomorphic to $F \times \widetilde{D}_\eta$, where F is a fiber of $\varphi_{\varepsilon,\eta}$ and $\widetilde{\varphi}_{\varepsilon,\eta}$ is fiber homeomorphic to $F \times \widetilde{D}_\eta \to \widetilde{D}_\eta$ defined by the second projection. The Galois group of $\widetilde{\pi}$

operates on the homology (or the cohomology of F). This defines what is called the monodromy of f at x.

Notice. In the case where $X = \mathbb{C}^N$ this geometric situation has been studied using methods of Analysis. We hope that the extensions of known results to more general geometric situations will draw the attention of analysts to these problems.

Now let us introduce the basic tools we need to ~~study~~ prove the theorem (1.4).

First we need to stratify X. We use the notion introduced by H. Whitney (cf [22]) (compare with R. Thom in [20]).

Definition (1.5) Let X be a semi-analytic subset of an open set $U \subset \mathbb{R}^N$. We shall say that $(X_i)_{i \in I}$, family of subsets of X, is a stratification of X if:

1) X_i is a locally finite partition of X;

2) X_i are analytic manifolds;

3) $\overline{X_i}$, $\overline{X_i} - X_i$ are semi-analytic subsets of X;

4) (frontier property) $X_i \cap \overline{X_j} \neq \emptyset \Rightarrow X_i \subset \overline{X_j}$.

But for a stratification to be of some interest in topology we usually requires the Whitney condition or the A_f property (good stratifications).

<u>Definition</u> (1.6) Let M and N be non singular semi-analytic subsets of an open set $U \subset \mathbb{R}^N$. Suppose that $N \subset \overline{M}$. Let $x \in N$. We say that M satisfies <u>the Whitney condition at x along N</u> if :

for any sequence of points (x_n) of M which converges to x such that the sequence of tangent spaces $T(x_n, M)$ converges to T in the corresponding Grassmann manifold and for any sequence of points (y_n) of N which converges to x such that the sequence of directions $\overline{y_n x_n}$ converges to ℓ, we have:

$$\ell \subset T.$$

Sometimes the Whitney condition above is refered as (b) - Whitney condition. If it is satisfied for any $x \in N$, we simply say that M satisfies the Whitney condition along N.

We say <u>the stratification $(X_i)_{i \in I}$ of X satisfies the Whitney condition</u> if for any (X_i, X_j) such that $X_j \subset \overline{X_i}$, X_i satisfies the Whitney condition along X_j.

Sometimes we may use the following property too:

<u>Definition</u> (1.7) We use the same notations as in (1.6). We say that <u>M satisfies the (a)-Whitney condition</u> at x along N if:

for any sequence of points (x_n) of M, which converges to x, such that the sequence of tangent spaces $T(x_n, M)$ converges to T in the corresponding Grassmann manifold, ~~at~~ we have:

$$T(x, N) \subset T .$$

In the case where X is ~~a~~ complex analytic subset of $U \subset \mathbb{C}^N$, we shall require that, if $(X_i)_{i \in I}$ is a stratification, X_i are complex analytic manifolds, and $\overline{X_i}$, $\overline{X_i} - X_i$ are complex analytic subsets of X.

Now we introduce a new condition on a stratification of X in order to prove the fibration theorem (1.4) (cf [8], [9], [14]).

<u>Definition</u> (1.8) Let X be our original complex analytic subset of $U \subset \mathbb{C}^n$. Let M and N be complex submanifolds of X, such that $N \subset \overline{M}$. We suppose for simplicity that $f(N) = 0$. Let $x \in N$. Then we say that <u>M satisfies A_f - condition</u> at x along N if:

for any sequence of points (x_n) which converges to x, such that the sequence of tangent spaces $T(x_n, M(x_n))$ of $M(x_n) = M \cap f^{-1}(f(x_n))$ at x_n converges to T in the corresponding Grassmann manifold, we have:

$$T(x, N) \subset T.$$

Now we have the following <u>existence theorems</u>:

<u>Theorem</u> (1.9) (H. Whitney)(cf [22]) Let $X \subset U \subset \mathbb{C}^N$ be a complex analytic subset of U open set in $\mathbb{C}^N$. Then there exists a stratification of X which satisfies the Whitney condition.

<u>Theorem</u> (1.10) (H. Hironaka) (cf [9]). Let $X \subseteq U \subset \mathbb{C}^N$ be a complex analytic subset of U - open set of $\mathbb{C}^N$. Let $f : X \longrightarrow \mathbb{C}$ be an analytic function on X and $(X_\alpha)_{\alpha \in A}$ a locally finite family of closed complex analytic subsets of X. Then there exists a stratification $(X_i)_{i \in I}$ of X such that:

1) $(X_i)_{i\in I}$ satisfies the Whitney condition;

2) for any $i \in I$, either $f(X_i)$ is a point or f induces a smooth morphism from X_i into $\mathbb{C}$;

3) If $X_j \subset \overline{X_i}$, $f(X_j)$ is a point and f induces a smooth morphism from X_j into $\mathbb{C}$ then X_i satisfies the A_f-condition along X_j at every point of X_j;

4) Any Y_α is a union of strata of $(X_i)_{i\in I}$.

Now the theorem (1.4) is proved by using:

Theorem (1.11) (Thom-Mather first isotopy theorem) (cf [20] and [16])

Let $\varphi : X \longrightarrow Y$ a proper analytic morphism from the complex analytic subset X of U, open in $\mathbb{C}^N$, onto a complex analytic manifold Y. Let $(X_i)_{i\in I}$ be a stratification of X which satisfies the Whitney condition. Then if the restriction of φ to each X_i is submersive then φ is locally trivial topological fibration.

But to get the transversality of f on $S_\varepsilon \cap X$, where $S_\varepsilon = \partial B_\varepsilon$ is the boundary of B_ε we need to use the property A_f. We have used this fact in [8] where

$X = \mathbb{C}^N$ and the stratifications with the A_f property were called <u>good stratifications</u>.

2. A theorem of Zariski-Lefschetz type.

~~We~~ use the notations quoted at the beginning of §1.

To study the vanishing cycles of f at 0 or the monodromy of f at 0 we can use:

1) the resolution of singularities (Clemens-Griffiths [4], A. Landman [11], P. Deligne [5], A'Campo [1],...)

2) the theory of deformations (E. Brieskorn [2], F. Pham [18], Lê Dũng Tráng [13])

3) the theory of Lefschetz (A. Landman (implicitly) [11], Lê Dũng Tráng [12], [14], H. Hamm-Lê Dũng Tráng [7]).

In this lecture we shall only deal with the theory of Lefschetz from a Zariski viewpoint (cf [23]).

The theory of Lefschetz compares the situation obtained by making a hyperplane section and the original situation. In the case we are interested to, we have to choose a hyperplane section sufficiently general so that:

1) it is in general position relatively to a well-chosen stratification of X;

2) the corresponding polar curve is reduced.

Then first we fix a stratification $(X_i)_{i \in I}$ of X as in the theorem (1.10) with $Y_\alpha = f^{-1}(0)$. We suppose each X_i connected.

Now let ℓ be a linear form of $\mathbb{C}^N$. It defines a function on X we still denote by $\ell : X \longrightarrow \mathbb{C}$. Thus we have a mapping $\Phi_\ell : X \longrightarrow \mathbb{C}^2$ defined by f and ℓ. Then we have the following theorem:

Theorem (2.1) There is an open dense Zariski set Ω_0 of the space of hyperplanes of $\mathbb{C}^N$ passing through x such that for any $H \in \Omega_0$ and any linear form ℓ defining H, there is an open neighbourhood V of x in X such that if C the critical locus of the restriction of Φ_ℓ to the smooth part $X - \Sigma$ of X, then $(C - \{f=0\}) \cap V$ is either void or a non singular curve. We denote $\Gamma_X = \overline{C - \{f=0\}}$

The proof of this theorem mainly uses the Bertini theorem about singular points of linear systems on germ of analytic sets (which is proved in a similar way as the one on algebraic varieties).

From the preceding theorem we can define an open dense Zariski subset Ω_1 of the space of hyperplanes of $\mathbb{C}^N$ going through x, such that for any $H \in \Omega$, there is an open neighbourhood V of x in X such that for any $\underbrace{X_i}_{(i=1,\dots,k)}$, such that $0 \in \overline{X_i}$, $\Gamma_{\overline{X}_i} \cap V$ is void or a curve. We shall call $\Gamma = \bigcup_{i=1}^{k} (\Gamma_{\overline{X}_i} \cap V)$ the <u>polar curve of f at x relatively to the stratification $(X_i)_{i \in I}$</u>.

Let $H \in \Omega_1$ and $l = 0$ defining H. It is then clear that the restriction of Φ_l to Γ in a neighbourhood of x is finite when Γ is non-void. Then if Γ is non void, $\Phi_l(\Gamma)$ is an analytic curve (if V is chosen small enough) in an open set of $\mathbb{C}^2$. We call this curve the <u>Cerf's diagramm of f at x relatively to the stratification $(X_i)_{i \in I}$</u>. We shall denote it by Δ.

We want now to choose the hyperplane to be in general position relatively to the stratification $(X_i)_{i \in I}$ of X.

✱ For this purpose we have:

Lemma (2.2) There is an open Zariski dense set Ω_2 of the space of hyperplanes of $\mathbb{C}^N$ passing through x such that for any $H \in \Omega_2$ there is an open neighbourhood V of x in $\mathbb{C}^N$ such that H is transverse in V to all the strata X_i such that $x \in \overline{X_i}$ except maybe $\{x\}$ if it is a stratum.

Finally we shall choose H in $\Omega_1 \cap \Omega_2 = \Omega$. But before going on we have to change a bit our original situation. Namely instead of doing our proof in B_ε we shall do it in a neighbourhood $P = D_1 \times \ldots \times D_k \times \widetilde{B}_r$ where D_i are disks and $\widetilde{B}_r$ a ball in a linear space of dimension $N-k$ equal to the dimension of X at x.

Such a neighbourhood is built by induction such that:

1) the boundary is in general position relatively to the stratification of X;

2) For $\eta > 0$ sufficiently small we have a topological fibration locally trivial $\psi_{P,\eta} : P \cap f^{-1}(D_\eta - \{0\}) \longrightarrow D_\eta - \{0\}$ induced by f, fiber homeomorphic to $\varphi_{\varepsilon,\eta}$ of (1.4)

We do as in [14] and we build this neighbourhood by induction on k, codimension of X in $\mathbb{C}^N$. We call such neighbourhoods <u>privilege mixed polydiscs of X at x relatively to f and the stratification</u> $(X_i)_{i\in I}$.

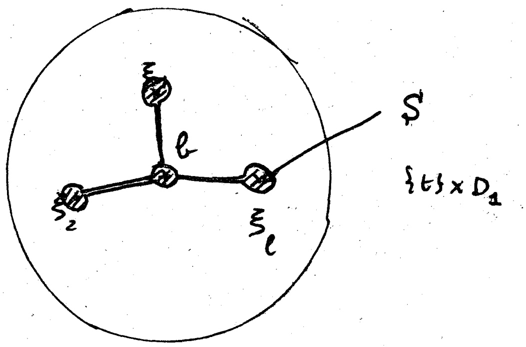
Then let us choose $H \in \Omega = \Omega_1 \cap \Omega_2$. We may suppose that $z_1 = 0$ defines H. The consider in H a privilege mixed polydisc P_1 of $X \cap H$ at x relatively to f restricted to $X \cap H$ and to the stratification defined by $(X_i \cap H)_{i\in I}$. As in [14] we may find D_1 such that $P = D_1 \times P_1$ is a privilege mixed polydisc of X at x relatively to f and to the stratification $(X_i)_{i\in I}$.

Now consider the mapping $\Phi_1 : X \longrightarrow \mathbb{C}^2$ defined by f and X_1. Actually for $\eta > 0$ small enough we shall consider the mapping $\Phi : X \cap P \cap f^{-1}(D_\eta) \to D_\eta \times D_1$ induced by Φ_1. Actually if one composes Φ with the projection $D_\eta \times D_1 \longrightarrow D_\eta$ we obtain $\psi_{P,\eta}$. Let $t \in D_\eta - \{0\}$, $\Phi^{-1}(\{t\} \times D_1)$ is then a fiber F_t of this fibration. We notice that Φ induces a mapping $\Phi_t : F_t \longrightarrow \{t\} \times D_1$. Let $\{\xi_1, \dots, \xi_\ell\} = (\{t\} \times D_1) \cap \Delta$

where Δ is the Cerf diagramm. Now using the Thom-Mather isotopy theorem (cf (1.11)) and the properties of A_f-stratifications, we find that Φ_t induces a topological fibration of $F_t - \Phi_t^{-1}(\{\xi_1,\dots,\xi_\ell\})$ onto $\{t\}\times D_1 - \{\xi_1,\dots,\xi_\ell\}$.

Because the polar curve Γ has a finite number of points over each point of Δ, the "bad" fibers $\Phi_t^{-1}(\xi_i)$ has isolated points $x_{1i},\dots,x_{r_i i}$ such that in a neighbourhood on F_t of any point of $\Phi_t^{-1}(\xi_i)$ outside $\{x_{1i},\dots,x_{r_i i}\}$, Φ_t induces a local topological fibration.

fig. (2.3)



we choose $b \in \{t\}\times D_1$ outside $\{\xi_1,\dots,\xi_\ell\}$ we may shrink $\{t\}\times D_1$ on the "star" S in dark in the picture above. Because Φ_t is a fibration outside $\{\xi_1,\dots,\xi_\ell\}$, F_t has the same homotopy type as $\Phi_t^{-1}(S)$. Now it is easy to see that $\Phi_t^{-1}(b)$ is homeomorphic to $F_t \cap H$, hyperplane section of F_t with the general hyperplane H (defined here by $z_1 = 0$). Thus we may give a theorem of Lefschetz type for F_t. To do this we have to introduce another notion:

<u>Definition</u> (2.4) Let Y be a complex analytic subset of an open set U of $\mathbb{C}^N$. Let $y \in Y$. Let ℓ a sufficiently general linear form of $\mathbb{C}^N$. Let us call $\ell : Y \to \mathbb{C}$ the function induced by ℓ. Let $\varepsilon > 0$ be small enough and B_ε be the ball of $\mathbb{C}^N$ centered at y of radius $\varepsilon > 0$. ℓ induces a topological fibration $\varphi_{\varepsilon,\eta}(\ell) : Y \cap B_\varepsilon \cap \ell^{-1}(D_\eta - \{0\}) \to D_\eta - \{0\}$ when $\alpha\,\eta \ll \varepsilon$ (cf (1.4)). If the fiber of this projection is k-connected but not $(k+1)$-connected, we say that Y has the <u>topological depth</u> k at y. If it is contractile we say that the topological depth is ∞.

This notion may be compared to the <u>k-regularity</u> of M. Kato in [10].

Now we may state our theorem:

<u>Theorem</u> (2.5) We have $\pi_i(F_t, F_t \cap H, x) = 0$ with $x \in F_t \cap H$ when $i \leq \inf(d-1, k)$ where d is the smallest dimension among the ones of the components of X at x and k is the smallest topological depth among the ones of the points x_{ij} $(j = 1, \ldots, \ell\ ,\ i = 1, \ldots, r_j)$

<u>Remark</u> Because of the fibration theorem, with this theorem we may compare the homotopy groups of $B_\varepsilon \cap X - \{f=0\}$ and $B_\varepsilon \cap X \cap H - \{f=0\}$ when H is sufficiently general (as described above) and $\varepsilon > 0$ small enough. Thus this allows us to compare the homotopy groups of $\mathfrak{X} - \{\tilde{f}=0\}$, when $\mathfrak{X} \subset \mathbb{P}^{N-1}$ is a projective variety and $\tilde{f}=0$ a hypersurface of $\mathbb{P}^{N-1}$, and the ones of $(\mathfrak{X} - \{\tilde{f}=0\}) \cap \mathcal{H}$, when $\mathcal{H}$ is a sufficiently general hyperplane of $\mathbb{P}^{N-1}$. This result is obtained by looking at the cone over $\mathfrak{X}$ in $\mathbb{C}^N$. Thus the theorem (2.5) generalizes the result of O. Zariski in [23], proved by H. Hamm and Lê Dũng Tráng in [8] (see Varchenko [21] and Chéniot [3] too).

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