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On Regular Surfaces of General Type II.

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1. Introduction. In this paper a <u>surface</u> shall mean a compact complex manifold of dimension 2. We denote by $|mK_S|$ ($m \in N$) a <u>pluricanonical</u> <u>system</u> on a surface S and by Φ_{mK_S} the associated rational map (the <u>pluri-canonical</u> <u>map</u>), assuming that $|mK_S|$ is not empty. A surface S is called of <u>general</u> <u>type</u> if $\Phi_{mK_S}(S)$ in the projective space P^N ($N = \dim\ mK_S$) for a large number m is a variety of dimension 2. If S is a surface of general type the following results are well-known.

Theorem 1 (Mumford [ ]). If m is sufficiently large, Φ_{mK_S} is a birational morphism and $\Phi_{mK_S}(S) \cong X = \mathrm{Proj} \bigoplus_r H^0(S, \underline{O}(rK_S))$. X is a normal variety with only a finite number of rational double points as singularities. If S is a minimal surface, then S is the minimal resolution of X.

Theorem 2 (Mumford [ ]). Assume that S is minimal. Then we have $H^1(S, \underline{O}(mK_S)) = 0$, for $m \neq 0, 1$, $m \in Z$.

Theorem 3 (Riemann-Roch Theorem for pluricanonical systems). Letting $\bar{c}_1^{\,2}$ be the self intersection number for the canonical divisor of the minimal model of S, we have

$$\dim H^0(S,\underline{O}(mK_S)) = \chi(\underline{O}) + (\bar{c}_1{}^2/2)\, m(m-1),$$

where $\chi(\underline{O})$ denotes the Euler characteristic of the structure sheaf $\underline{O}_S$ of S.

Theorem 4 (Iitaka [ ]). The m-genus $P_m(S) = \dim H^0(S,\underline{O}_S(mK_S))$ is deformation-invariant.

As an immediate corollary to Theorems 3 and 4, we obtain the following

Theorem 5 (Deformation Invariance of the Minimality). If S is minimal, then any deformation of S is also a minimal surface of general type.

From now on, we denote by S a minimal surface of general type with the following numerical conditions:

$$* \begin{cases} p_g(S) = \dim H^0(S,\underline{O}(K_S)) = 0, \\ q(S) = \dim H^1(S,\underline{O}) = 0, \\ K_S{}^2 = 2. \end{cases}$$

A surface of this type shall be called a numerical Campedelli surface.

In section 2, we study the property of the tricanonical system $|3K_S|$ on a numerical Campedelli surface. In spite of Bombieri's comprehensive work [ ] on pluricanonical maps, the tricanonical system on S was not completely surveyed. ~~But~~ And there remains still an open problem: Is the tricanonical map of S is a birational morphism?

It is an interesting but, in general, a very difficult problem to determine the complex structures on a given underlying differentiable manifold. In our case the problem is rather easy under some conditions. In section 3, we shall determine the structure of S under the condition that the fundamental group of S is a direct sum of three copies of the cyclic group of order 2.

2. Regularity of the tricanonical maps.

Let S be a numerical Campedelli surface. Then we have the following

Theorem 5 (Regularity of tricanonical maps). The tricanonical system $|3K_S|$ on S is free from base points and fixed components.

For the proof we need some results .

Definition. An effective divisor D on a surface F is called 1-connected if

$$D_1 \cdot D_2 > 0,$$

for any non-trivial decomposition $D = D_1 + D_2$, $D_i > 0$.

Theorem 6 (Ramanujam vanishing theorem [ ]). If an effective divisor D on a regular surface (i.e. $q(F) = 0$) is 1-connected , then $H^1(F,\underline{O}(-D)) = 0$.

Theorem 7 (Bombieri [ ]). Let F be a minimal surface of general type and P a point on F. Let $p: \tilde{F} \rightarrow F$ denote a quadric transformation at P and E the exceptional curve over P. If an effective divisor D is numerically equivalent to $2p^*K_F - 2E$, then D is 1-connected except in the case where $K_F^2 = 1$.

Now we proceed to the proof of Theorem 1. Let $p: \tilde{S} \rightarrow S$ be the quadric transformation at a point P and E the associated exceptional curve. Let us consider the following natural exact sequence of sheaves:

$$0 \longrightarrow \underline{O}_{\tilde{S}}(3p^*K_S - E) \longrightarrow \underline{O}_{\tilde{S}}(3p^*K_S) \longrightarrow \underline{O}_E \longrightarrow 0.$$

Then it is obvious that $|3K_S|$ is free from base point at P if and only if $H^1(\tilde{S},\underline{O}(3p^*(K_S - E)) = 0$. By the Serre duality we have

$$\dim H^1(\tilde{S},\underline{O}(3p^*K_S - E)) = \dim H^1(\tilde{S},\underline{O}(2E - 2p^*K_S)).$$

Hence Theorem 7 yields the vanishing of the cohomology group under the condition that $|2p^*K_S - 2E| \neq \emptyset$.
Now assume that $|2p^*K_S - 2E| = \emptyset$. Since $\dim H^0(S,\underline{O}(2K_S)) = 3$, this implies that the rational map Φ_{2K_S} associated with the bicanonical system $|2K_S|$ is a local isomorphism at P. Therefore there exists an effective divisor $D \in |2p^*K_S - E|$ such that D is irreducible in a neighbourhood of E and that the unique irreducible component D_0 which simply intersects E satisfies $D_0{}^2 \geq 0$. Now we shall take the following exact sequence of cohomology groups:

$$0 \longrightarrow H^0(\tilde{S},\underline{O}(2E - 2p^*K_S)) \longrightarrow H^0(\tilde{S},\underline{O}(E)) \longrightarrow H^0(D, \underline{O}_D(E))$$
$$\rightarrow H^1(\tilde{S},\underline{O}(2E - 2p^*K_S)) \longrightarrow H^1(\tilde{S},\underline{O}(E)).$$

Note that $H^0(\tilde{S},\underline{O}(2E - 2p^*K_S)) = 0$ and that

$$\dim H^1(\tilde{S},\underline{O}(E)) = \dim H^1(\tilde{S},\underline{O}(p^*K_S)) = \dim H^1(S,\underline{O}(K_S))$$
$$= q(S) = 0.$$

Hence, for the proof of Theorem 5, it is sufficient to show the equality

$$\dim H^0(D,\underline{O}(E)) = \dim H^0(\tilde{S},\underline{O}(E)) = 1.$$

On the other hand we have the following natural commutative diagram

$$\begin{array}{ccccccc} 0 \to & H^0(D,\underline{O}) & \to & H^0(D,\underline{O}(E)) & \xrightarrow{r} & H^0(D\cdot E,\underline{O}) & \\ & \downarrow & & \downarrow & & \downarrow & \text{identity} \\ 0 \to & H^0(D_0,\underline{O}) & \to & H^0(D_0,\underline{O}(E)) & \xrightarrow{r} & H^0(D.E,\underline{O}) & \end{array}$$

of which the rows are exact. But it is obvious that the virtual genus of D_0 is not 0. Since the degree of the divisor E on D_0 is 1, the restriction map r is the zero-map. This implies that

$$\dim H^0(D,\underline{O}(E)) = \dim H^0(D,\underline{O}).$$

Moreover we have $\dim H^0(D,\underline{O}) = 1$. In fact, there exists the following natural exact sequence

$$0 \longrightarrow H^0(\tilde{S},\underline{O}(E - 2p^*K_S)) \longrightarrow H^0(\tilde{S},\underline{O}) \longrightarrow H^0(D,\underline{O})$$
$$\longrightarrow H^1(\tilde{S},\underline{O}(E - 2p^*K_S)),$$

where $\dim H^1(\tilde{S},\underline{O}(E - 2p^*K_S)) = \dim H^1(\tilde{S},\underline{O}(3p^*K_S))$ $= \dim H^1(S,\underline{O}(3K_S)) = 0$. Thus $\dim H^0(D,\underline{O}) = \dim H^0(S,\underline{O})$ $= 1$ and the assertion is proved.

3. The structure of Campedelli surfaces.

In this section we shall study numerical Campedelli surfaces of special type.

Definition (cf. Campedelli [ ]). A numerical Campedelli surface is called a Campedelli surface if its fundamental group is isomorphic to $Z/(2) + Z/(2) + Z/(2)$.

If S is a Campedelli surface, the universal covering $\bar{S}$ of S has the following numerical characters:

$$\begin{cases} \chi(\bar{S},\underline{O}_{\bar{S}}) = 8\chi(S,\underline{O}_S) = 8, \\ q(\bar{S}) = 0, \\ p_g(\bar{S}) = \chi(\bar{S},\underline{O}_{\bar{S}}) - q(\bar{S}) - 1 = 7, \\ K_{\bar{S}}^2 = 8\,K_S^2 = 16. \end{cases}$$

The fundamental group G of S acts on $\bar{S}$ as the covering transformation group of the unramified covering $e: \bar{S} \longrightarrow S$, and G naturally operates on the vector space $H^0(\bar{S},\underline{O}(K_{\bar{S}}))$ as linear transformations. Hence we obtain a canonical representation $k: G \longrightarrow GL(7,C)$ and the induced representation $k': G \longrightarrow PGL(6,C)$.

Lemma 1. k' is a faithful representation.

Proof. Let $g \in G$ be an element of $\ker k'$. Since $g^2 = id$, $k(g) = \pm\, id$. Hence $p_g(\bar{S}/\langle g\rangle) = 7$ or 0. But $p_g(\bar{S}/\langle g\rangle) = 3$, if g is of order 2. Hence $g = id$.

Let V denotes the image of $\bar{S}$ by the canonical map $\Phi_{K_{\bar{S}}}$ associated with the canonical system $|K_{\bar{S}}|$.

Then $k'(g)$ $(g \in G)$ induces an automorphism of V. Thus we obtain a natural homomorphism $a: G \longrightarrow Aut(V)$, where $Aut(V)$ denotes the automorphism group of V.

Lemma 2. a is injective.

Proof. A trivial consequence of Lemma 1. ~~Assume that $g \in G$ induces the identity on V. Then V is contained in an eigenspace of $k'(g)$. Since V is not contained in any proper linear subspace of P^6, this implies that $k'(g) = id$. Lemma 1 yields the equality $g = id$.~~

Lemma 3. The canonical system $K_{\bar{S}}$ of $\bar{S}$ is not composed of a pencil.

Proof. Assume that V is a curve. Since $q(\bar{S}) = 0$, V must be a (possibly singular) rational curve. An automorphism of V induces a unique automorphism of the non-singular model P^1 of V. Hence, in virtue of the above lemma, we infer that there exists a faithful representation $a': G \longrightarrow PGL(1,C)$. On the other hand, it is obvious that $PGL(1,C)$ does not contain a subgroup isomorphic to $(Z/(2))^3$. This is a contradiction.

Since G is a commutative group, we may assume that $k(G)$ is contained in the diagonal subgroup of $GL(7,C)$. Let $w_1,\dots,w_7$ be a basis of $H^0(\bar{S},\underline{O}(K_{\bar{S}}))$ such that $g^*(w_j) = \pm w_j$ for any $g \in G$.

Lemma 4. The linear subspace W of $H^0(\bar{S},\underline{O}(2K_{\bar{S}}))$ spanned by $w_1^2, w_2^2,\dots, w_7^2$ is 3-dimensional.

Proof. Lemma 3 implies that the transcendental degree over C of the field $C(w_2/w_1,\dots,w_7/w_1)$ is 2. Hence the transcendental degree of $C(w_2^2/w_1^2,\dots,w_7^2/w_1^2)$ is also 2. This yields the inequality

$$\dim W \geqq 3.$$

On the other hand, since w_j^2 is G-invariant, W can be regarded as a subspace of $H^0(S,\underline{O}(2K_S))$. But the Riemann-Roch theorem geves an equality $\dim H^0(S,\underline{O}(2K_S))$ = 3. This completes the proof.

Lemma 5. Let K be an extension of the rational function field $C(x_1,\dots,x_n)$ defined by

$$K_r = C(x_1,\dots,x_n,\sqrt{Q_1},\dots,\sqrt{Q_r}),$$

where Q_j is a quadric polynomial in x_i. Assume that $K_r : C(x_1,\dots,x_n) = 2^r$. Then the integral closure of $C[x_1,\dots,x_n]$ in K_r is $R_r = C[x_1,\dots,x_n, Q_1,\dots, Q_r]$.

Proof. Trivial.

Corollary. Let K be as above. Let Q_{r+1} be another quadric polynomial in x_i. Assume that $K_{r+1} = K_r$. Then $\sqrt{Q_{r+1}}$ is a linear combination of $x_1,\dots, x_n,\sqrt{Q_1},\dots,\sqrt{Q_r}$.

Let w_1^2, w_2^2, w_3^2 be a basis of W. From Lemma 4, we infer that there are quadric relations

$$w_j^2 = a_j w_1^2 + b_j w_2^2 + c_j w_3^2, \qquad j = 4, 5, 6, 7.$$

The above corollary asserts that, if the complete intersection defined by the above quadrics is reducible then its any irreducible component is contained in a hyperplane in P^6. Since the image V of $\bar{S}$ is contained in the complete intersection V' defined by the above 4 equations and V is not contained in any hyperplane, $V' = V$ is a irreducible surface. Thus we obtain the following

Corollary. V is a complete intersection of type $(2,2,2,2)$ in P^6.

As an immediate consequence of this corollary, we have

Theorem 8. The canonical homomorphism

$$\otimes^m H^0(\bar{S},\underline{O}(K_{\bar{S}})) \longrightarrow H^0(\bar{S},\underline{O}(mK_{\bar{S}}))$$

is surjective.

Proof. Let $\underline{O}_V(m)$ denote the sheaf of the hypersurface section of degree m. Since V is a complete intersection of type $(2,2,2,2)$, we have

$$\dim H^0(V,\underline{O}_V(m)) \geqq 8 + 8\,m(m-1) = \dim H^0(\bar{S},\ \underline{O}_{\bar{S}}(mK_{\bar{S}}))$$

Moreover $H^0(V,\underline{O}_V(1))$ generates $H^0(V,\underline{O}_V(m))$. This proves the theorem.

Now the following theorem is trivial.

Theorem 9. The canonical model $\bar{X}$ of $\bar{S}$ is isomorphic to a complete intersection of type $(2,2,2,2)$ in P^6. The canonical model X of S is the quotient of $\bar{X}$ by the action of following subgroup G of $PGL(6,C)$.

$$G = \left\langle \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & -1 & & \\ & 0 & & & & -1 & \\ & & & & & & -1 \end{pmatrix}, \begin{pmatrix} 1 & & & & & & \\ & -1 & & & & 0 & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & 0 & & & & -1 & \\ & & & & & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & & & & & \\ & -1 & & & & 0 & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & 0 & & & & 1 & \\ & & & & & & -1 \end{pmatrix} \right\rangle$$

The following theorem is a corollary of Theorem 9 and the forms of the defining equations.

Theorem 10. The moduli space of Campedelli surfaces is a ~~normal~~ unirational variety of dimension 6.

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