On the explicit formulae of characters for
discrete series representations

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Let us consider a simply connected complex simple Lie group $G_c$
and its connected real simple form $G$. We denote by $\mathfrak{g}_c$, $\mathfrak{g}$ the
Lie algebra of $G_c$, $G$ respectively.

According to a criterion of Harish-Chandra in [3], a square
integrable representation $\omega$ on $G$ (i.e., one of each matrix
coefficients of $\omega$ is square integrable with respect to a Haar
measure $dx$ on $G$) exists if and only if $G$ has a compact Cartan
subgroup $B$.

In this paper, we shall study the global characters of square
integrable representations which are called the discrete series.
Hence we shall assume that $G$ has a compact Cartan subgroup $B$.

For one of each irreducible representation $\omega$ of $G$ in the discrete
series, we define a distribution $\Theta$ on $G$ by

$$\Theta(f) = \text{Trace } \int_G f(x^{-1})\omega(x) \, dx$$

for all $C^\infty$-functions $f$ on $G$ with compact support.

Then $\Theta$ is a tempered invariant eigendistribution on $G$, and $\Theta$ is
real analytic on the set of all regular elements in $G$ (cf. [3]).
We select a maximal compact subgroup $K$ of $G$ containing $B$. Let

$\omega \mid K$ be the restriction of $\omega$ to $K$. Therefore we put, for each irreducible representation $\pi$ of $K$, by $\omega \mid K : \pi$ the multiplicity of $\pi$ occurring in $\omega \mid K$. Then the restriction of $\Theta$ to $B$ is expressed as

$$\Theta(b) = \sum_{\pi \in \mathcal{E}_K} |\omega \mid K : \pi| \text{Trace } \pi(b)$$

in the sense of the distributions on $B$, where $\mathcal{E}_K$ is the set of all inequivalent classes of irreducible representations on $K$.

We will summarize Harish-Chandra's parametrization in [11], [3] for the characters of representations in the discrete series.

Let us consider $\Sigma$ the root system of the pair $(\mathfrak{g}_c / \mathfrak{q}_c)$ where $\mathfrak{g}_c$ is the complexification of Lie algebra $\mathfrak{g}$ of $B$. By $P^-$, $P^+$, $P^-$, $L^+$, we shall denote the set of all positive roots, the set of all noncompact positive roots, the set of all compact positive roots, the set of all regular integral form on $\mathfrak{g}_c$ respectively.

Let $\Theta$ be the character of a fixed irreducible representation $\omega$ of $G$ in the discrete series. Then

$$\Theta = \xi(\Lambda)(-1)^{P^+} \sum_{s \in \mathcal{W}(G/B)} \xi(s) \exp sA \text{ on } B' \ldots \ldots (*)$$

for suitably chosen in $L'$

where $B' = \{ b = \exp H; b \in B, \alpha(H) \neq 0 \text{ for all } \alpha \in \mathfrak{P} \}$,
\[ W(G/B) = \text{the Weyl group of the pair } (G/B), \]
\[ \varepsilon(s) = \text{the signature of } s \text{ in } W(G/B), \]
and \[ \varepsilon(A) = \prod_{\alpha \in \Phi} \text{sgn}(A, \alpha). \]

Conversely, for each form \( \Lambda \) in \( L' \), there exists unique irreducible representation \( \omega \) of \( G \) in the discrete series such that

\[ \Theta = \text{Trace } \omega \text{ satisfies the above identity } (*). \]

Thus one of each discrete character \( \Theta \) is parametrized by \( \Theta = \Theta_{\Lambda}(\Lambda \in L') \) under the identity (*)

We now state our purpose of this paper. Let us consider a
regular dominant integral form \( \Lambda \) on \( G_C \) and finite dimensional irreducible representation \( \Pi_{\Lambda} \) of \( G_C \) with the highest weight
\[ \Lambda - \frac{1}{2} \sum_{\alpha \in \Phi} \alpha. \]
Therefore we define a distribution \( S_{\Lambda} \) on \( B \) as following;

\[ S_{\Lambda}(b) = \sum_{s \in W(G/B) \backslash W} \Theta_{s\Lambda} - (-1)^{|B'|} \text{Trace } \Pi_{\Lambda}(b), \quad b \in B \]

where \( W \) is the Weyl group of the pair \( (G_C, B_C) \).

By The identity (*), \( S_{\Lambda} \equiv 0 \) on \( B' \). Moreover, since the discrete representations are all infinite dimensional, \( S_{\Lambda} \neq 0 \) on \( B \).

Hence \( S_{\Lambda} \) is a singular distribution on \( B \), because \( B' \) is open dense in \( B \). We shall, in this paper, give a characterization of

and obtain a global formula of the character \[ \sum_{s \in W(G/B) \backslash W} \Theta_{s\Lambda} \]
under the following assumptions.

(A2): All of noncompact roots in $\Sigma$ have the same length with each other.

For this purpose, we will state more precisely descriptions.

We define the generating function $\Phi_{s, \Lambda}$ ($s \in W$) on $B_C$ as followings;

$$\Phi_{s, \Lambda} = \prod_{\alpha \in P^-(s)} (1 - \exp(-\alpha)) \exp(s\Lambda + s\mathfrak{g}) / \prod_{\alpha \in P^+(s)} (1 - \exp(\alpha))$$

where $s\mathfrak{g} = \frac{1}{2} \sum_{\alpha \in P} \alpha$, $P^+(s) = \{ \alpha \in P^+ \cup P^- : s^{-1} \alpha > 0 \}$, and $P^-(s) = \{ \alpha \in P^- \cup P^- : s^{-1} \alpha > 0 \}$.

Then $\Phi_{s, \Lambda}$ is holomorphic on the complex domain

$$D_s = \{ b = \exp H \in B_C : |\exp \alpha (H)| < 1 \text{ for each } \alpha \text{ in } P^+(s) \}.$$

The functions $\Phi_{s, \Lambda}$ ($s \in W$) are concerned with Blattner's conjecture. This phenomenon is stated as followings. Let $L$ be the set of all integral form on $B_C$ and $Q_s(\mu)$ ($\mu \in L$) be the partition function on $L$, which is defined by

$$\frac{1}{\prod_{\alpha \in P^+(s)} (1 - \exp(\alpha))} = \sum_{\mu \in L} Q_s(\mu) \exp \mu.$$

Then $\Phi_{s, \Lambda}$ is expressed as

$$\Phi_{s, \Lambda} = \sum_{\mu \in L} Q_s(\mu) \prod_{\alpha \in P^-(s)} (1 - \exp(-\alpha)) \exp(s\Lambda + s\mathfrak{g} + \mu).$$

We now choose an element $u$ in $W$ satisfying $u^{-1} \alpha > 0$ for each $\alpha$ in $P^-$. 

Since $\Delta_K(b^g) = \mathcal{E}(s)\Delta_K(b)$ ($\Delta_K = \exp \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \cdot \pi(1 - \exp -\alpha)$) and $sP^+ \subseteq P^+ \cup -P^+$ for all $b \in B$, $s \in \mathcal{W}(G/B)$, we get

$$\sum_{s \in \mathcal{W}(G/B)} \Phi_{su,\lambda} = \sum_{\mu \in \mathcal{W}(G/B)} \Phi_{u,\lambda} \exp(su\Lambda + s\gamma^+(u) + \mu)$$

where $\gamma^+(u) = \frac{1}{2} \sum_{\alpha \in \Phi^+(u)} \alpha$. Moreover by $P^+(su) = sP^+(u)$, the above equation is rewritten as

$$\sum_{s \in \mathcal{W}(G/B)} \Phi_{su,\lambda} = \sum_{\mu \in \mathcal{W}(G/B)} \Phi_{u,\lambda} \exp(su\Lambda + s\gamma^+(u) + s\mu)$$

$$= \sum_{\mu \in \mathcal{W}(G/B)} \exp(s\mu - u\Lambda - \gamma^+(u)) \exp \mu.$$

Defining the Blattner's number $b_{u,\Lambda}(\mu)$ ($\mu \in \mathcal{W}(G/B)$) and a subset $L^+$ in $L$ by

$$b_{u,\Lambda}(\mu) = \sum_{s \in \mathcal{W}(G/B)} \mathcal{E}(s)\Phi_{u,\lambda} (s\mu - u\Lambda - \gamma^+(u)),$$

and $L_+ = \{ \mu \in \mathcal{W}(G/B) : (\mu, \alpha) > 0 \text{ for each compact positive root } \alpha \}$, then

$$\sum_{s \in \mathcal{W}(G/B)} \Phi_{su,\lambda} = \sum_{\mu \in \mathcal{W}(G/B)} b_{u,\Lambda}(\mu) \sum_{s \in \mathcal{W}(G/B)} \exp s\mu$$

$$= \sum_{\mu \in \mathcal{W}(G/B)} b_{u,\Lambda}(\mu) (\Delta_K)^2 \text{Trace } \pi \mu,$$

where $\pi \mu$ is the finite dimensional irreducible representation of $K$ with the highest weight $\mu - \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

There was conjectured that for the representation $\omega = \omega(u\Lambda)$ with its character $\overline{\mathfrak{S}}_{u\Lambda}$,

$$|\omega(u\Lambda)|_{K : \pi} = \begin{cases} 0 & \text{if } \pi = \pi_{\mu, \mu \Lambda}, \\ \sum_{\mu \in \mathcal{W}(G/B)} b_{u,\Lambda}(\mu) & \text{otherwise} \end{cases}$$
This conjecture implies

\[ \sum_{s \in \mathbb{W}} \mathfrak{F}_{s, \Lambda}(b) = (-1)^{|F| - |\Delta_s(b)|} \sum_{s \in \mathbb{W}(G/B) \setminus \mathbb{W}} \mathfrak{F}_{s, \Lambda}(b) \quad \ldots \ldots (**). \]

According to these observations, we arrive at the following situation; it will be suggested that a calculation of explicit relation between \( \sum_{s \in \mathbb{W}} \mathfrak{F}_{s, \Lambda}(b) \) and Trace \( \pi_{\Lambda}(b) \) \((b \in B)\) enable us to clear the gap \( S_{\Lambda} \).

As far as the author knows, the several results of multiplicity formulae of characters in the discrete series (Blattner's conjecture) have been obtained in the following papers; [14], [15] (W. Schmid), [16] (R. Hotta and K. R. Parthasarathy), [4] (H. Hecht and W. Schmid), [17] (N. Wallach).

In the amount of these papers, Blattner's conjecture was completely solved by [4], [15] and by [17] for all real semisimple matrix groups with compact Cartan subgroups. Especially, the explicit multiplicity formulae of characters of discrete series representations were obtained by [4] and [15].

However, we need not apply the multiplicity theorem in [4], [15] to our arguments. Our main results will be stated after the following preparations.

Definition; a subset \( F \) in \( \mathbb{P}^+ \) is strongly orthogonal if and only if \( F \) satisfies that for each of two distinct roots \( \alpha, \beta \) in \( F \),

\[ \alpha = -\beta \quad \text{and} \quad \alpha \pm \beta \notin \Sigma. \]
Definition: two strongly orthogonal system \( F_1, F_2 \) in \( \mathbb{P}^t \) are conjugate if and only if there exists \( t \) in \( \mathcal{W}(G/B) \) such that
\[
tF_1 - tF_1 = F_2 - F_2.
\]

By \( \Gamma_0 \), we denote a complete representative of all nonconjugate strongly orthogonal system in \( \mathbb{P}^t \) under \( \mathcal{W}(G/B) \). Choosing a system \( F^0 \) in \( \Gamma_0 \) satisfying \( |F^0| = \text{real rank of } G \), then we can assume that \( F \equiv F^0 \) for all \( F \) in \( \Gamma_0 \) under the condition \( A1, A2 \).

Let \( B(F) (F \in \Gamma_0) \) be the subgroup of \( B \), which is defined by
\[
B(F) = \{ b = \exp h \in B; \mathcal{A}(H) = 0 \text{ for each } \mathcal{A} \in F \}. \]

Let us consider \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) the Cartan decomposition of \( \mathfrak{g} \), here \( \mathfrak{k} \) is the Lie algebra of \( K \). Then for each \( \mathcal{A} \) in \( F \), there exist \( X_\mathcal{A}, X_{-\mathcal{A}} \) in \( \mathfrak{k} \) such that
\[
\text{ad}(H)X_{\pm \mathcal{A}} = \pm \mathcal{A}(H)X_{\pm \mathcal{A}} (H \in \mathfrak{k}), \Gamma_{-1}(X_\mathcal{A} + X_{-\mathcal{A}}), (X_\mathcal{A} - X_{-\mathcal{A}}) \in \mathfrak{p}.
\]
Therefore we put \( \mathfrak{g}_R(F), \mathfrak{g}_c(F), \mathfrak{g}_c(F), (\mathfrak{h}(F) (\equiv \mathfrak{g}_1(F)), \mathfrak{g}(F), \mathfrak{g}_c(F) \) by the followings;
\[
\mathfrak{g}_R(F) = \sum_{\mathcal{A} \in F} \Gamma_{-1}(X_\mathcal{A} + X_{-\mathcal{A}}), \mathfrak{g}_c(F) = \text{the centralizer of } \mathfrak{g}_R(F) \text{ in } \mathfrak{g}_c(F),
\]
\[
\mathfrak{g}_c(F) = \text{the orthogonal complement of } \mathfrak{g}_R(F) \text{ in } \mathfrak{g}_c(F) \text{ with respect to the Killing form on } \mathfrak{g}_c(F),
\]
\[
\mathfrak{h}(F) = \text{the Lie algebra of } B(F), \mathfrak{g}(F) = \mathfrak{g}_1(F) + \mathfrak{g}_R(F), \text{ and}
\]
\[
\mathfrak{g}_c(F) = \text{the complexification of } \mathfrak{g}(F) \text{ in } \mathfrak{g}_c(F).
\]

Then \( \mathfrak{g}_c(F) \) is a reductive subalgebra of \( \mathfrak{g}_c \) with Cartan subalgebra \( \mathfrak{h}(F) \). Moreover, the positive (noncompact positive) root system of \( (\mathfrak{g}_c(F), \mathfrak{g}_c(F)) \) can be identified with the set
\[ P(F) = \{ \alpha \in F; (\alpha, \beta) = 0, \beta \in F \} \] (resp. \( P^+(F) = \{ \alpha \in F^+; \alpha \not\in \beta \} \Sigma \) for all \( \beta \in F^\perp \). We now consider, for a fixed system \( F \) in \( \Gamma_0 \), a singular distribution \( \mathfrak{X}_{\alpha}(F; b) \) on \( B \), which is given by
\[
\langle \mathfrak{X}_{\alpha}(F; b), f(b) \rangle = \frac{1}{|W(G/B)|} \sum_{u \in W(G/B)} \int_{B(F)} \sum_{t \in W(\mathfrak{C}(F)/\mathfrak{B}_c(F))} \xi(ut)x
\]
\[
|\Delta_{\alpha}(F)(b)|^2 f(b) \quad \frac{\exp \text{uts}(\log b)}{\prod_{d \in P(F)} (1 - \exp -\delta(\log b))} db
\]
for all \( \mathfrak{C} \)-functions \( f \) on \( B \), where \( db \) is the Haar measure on \( B(F) \) normalized as \( \int_{B(F)} db = 1 \),
\[
\Delta_{\alpha}(F) = \exp \left( \frac{1}{2} \sum_{d \in P(\mathfrak{C})} \alpha_d \right) \prod_{d \in P(\mathfrak{C})} (1 - \exp -\delta), \quad P^{-}(F) = P(F) - P^+(F).
\]
and \( W(\mathfrak{C}(F)/\mathfrak{B}_c(F)) \) is the Weyl group of \( (\mathfrak{C}(F)/\mathfrak{B}_c(F)) \).

Our first main result is stated below.

Theorem I. Suppose \( G \) fulfills the conditions A1, A2. Then, for the functionals \( \mathfrak{F}_{s, \alpha}(s \in W), \mathfrak{X}_{s}(F; b) \) on \( B \), we have
\[
\sum_{s \in W(F)|W} (-1)^{|P^{-}(F)|} \mathfrak{F}_{s, \alpha}(b) = \sum_{s \in W(F)|W} \xi(s)(-1)^{|P^+(F)| + |F|} \mathfrak{X}_{s}(F; b)
\]
for all \( b \) in \( B \), where \( W(F) = \{ s \in W; sF \subseteq F - F \} \).

Let us consider, for a fixed system \( F \) in \( \Gamma_0 \), the Cartan subgroup \( A(F) \) of \( G \), which is corresponded to \( \mathfrak{C}(F) \). Let \( Z(F) \) be the centralizer of \( \mathfrak{C}(F) \) in \( G \). Then \( \mathfrak{C}_F \mathfrak{C}_c(F) \) is the Lie algebra of \( Z(F) \).
Moreover, there exists a unique parabolic subgroup \( Q(F) \) of \( G \) such that the reductive part coincides with \( Z(F) \). Therefore we define a representation \( \pi_{sA}^F \) of \( G \), which is induced from a finite dimensional irreducible representation of \( Q(F) \).

For the simplicity of our notations, we will identify \( \Sigma, W \) with the root system of \( (\mathfrak{q}_C/\mathfrak{a}_C(F)) \), the Weyl group of \( (\mathfrak{q}_C/\mathfrak{a}_C(F)) \) by using the Cayley transform. Putting

\[
T_{\Lambda} = \sum_{r \in \Gamma} \sum_{s \in W(F) \setminus \tilde{W}} \varepsilon(s) \text{ Trace } \pi_{sA}^F, \text{ then the Distribution } T_{\Lambda} \text{ on } G \text{ is an extention of } \sum_{s \in W} \Phi_{s, \Lambda} \text{ to } G. \text{ Speaking more precisely,}
\]

\[
\sum_{s \in W} \Phi_{s, \Lambda} = (-1)^{|F|} |\Delta_K|^2 T_{\Lambda} \text{ on } B \quad \ldots \ldots (\ast\ast\ast).
\]

We notice that the equation (\ast\ast\ast) is corresponded to (\ast\ast).

Theorem II. Under the same assumptions as in Theorem I,

\[
\varepsilon_R(F;h)\Delta(F;h) T_{\Lambda}(h)
\]

\[
= \sum_{s \in W} \sum_{u \in \mathcal{V}_0(F) \setminus \tilde{W}(G/A(F))} \prod_{F} \varepsilon(s^{-1}A) \exp sA(\log h_1) \times
\]

\[
\prod_{u \in F} \exp -|\mathfrak{a}(\log(h_R)^u)| |(sA_u^\infty)|/|W|^2
\]

for all regular elements \( h = h_1 h_R (h_1 \in \mathfrak{a}(\mathcal{A}(F) \cap K), h_R \in \mathcal{A}(F) \cap \exp p) \)

where \( \mathcal{W}(G/A(F)) \) = the Weyl group of \( (G/A(F)) \),

\[
\mathcal{V}_0(F) = \{ s \in \mathcal{W}(G/A(F)); sF \subseteq F \cup {^{-1}F}\}
\]
\[ \Delta(F; h) = \exp\left(\frac{1}{2} \sum_{\alpha \in \Phi} \alpha' \right) \log h \prod_{\alpha \neq 0 \text{ on } \mathbb{R}^n} (1 - \exp -\alpha' \log h), \]

\[ \mathcal{E}_R(F; h) = \prod_{\alpha \neq 0 \text{ on } \mathbb{R}^n} (1 - \exp -\alpha' \log h), \]

and \( \varepsilon(\alpha') = \) the signature of root \( \alpha' \).

Theorem III. We keep the same conditions as in Theorem I.

Then \( T_{\Lambda} = \sum_{\sigma \in \mathbb{Z}(G/B) \backslash \mathbb{W}} \mathbb{G}_{s\Lambda} \). Consequently, the right hand side in the equation of Theorem I obtains a global formula for the character \( \sum_{\sigma \in \mathbb{Z}(G/B) \backslash \mathbb{W}} \mathbb{G}_{s\Lambda} \).

The formulae in Theorem I, II will be calculated out by using the certain properties for the groups \( \mathbb{W}(F), V_0(F) \) \( (F \subseteq \Gamma_0) \).

The relation in Theorem III can be proved by using Theorem II, and by using the uniqueness of tempered invariant eigen-distributions in [1], the explicit formulae of \( \mathbb{G}_{s\Lambda} \) (\( \sigma \in \mathbb{W} \)) on \( B' \) (cf. [3]).

For the global character formulae of discrete series representations, there were known the following cases: real rank one groups in [2], pp.120-122 (Harish-Chandra), indefinite unitary groups, \( S_p(n,\mathbb{R}) \) in [8],[9] (T. Hirai) respectively. On the other hand, [14] (W. Schmid) has given the explicit formulae of characters for discrete series representations on split real Cartan subgroup
of $S_p(n, R)$. This calculation is based on the relation of $c$ in Theorem (4.15),[15] (W. Schmid).

The same relation as in Theorem III have been observed by [12] for real rank one case, and by [18](G. Zuckerman) for real rank one case, indefinite unitary groups. In essence, our direction of this paper is similar to the one in [12],[18]: Harish-Chandra has given general principle for these relations. However, the explicit relations are not completely known in general.

References

[1] Harish-Chandra, Discrete series for semisimple Lie groups I,

[2] ————, Two theorems on semisimple Lie groups,

[3] ————, Discrete series for semisimple Lie groups II,

[4] H. Hecht and W. Schmid, A proof of Blattner's conjecture,
preprint.


[6] S. Helgason, Differential geometry and symmetric spaces,
114


[7] T. Hirai, The character of some induced representation of
semisimple Lie groups, J. Math., Kyoto Univ. 8, 313-363
(1962)

8 ———, Invariant eigen-distributions of Laplace operators
on a semisimple Lie groups I: Case of SU(p,q), Japan J. of

9 ———, Invariant eigen-distributions of Laplace operators
on real simple Lie groups. IV: Explicit forms of the
characters of discrete series representations, preprint.

10 R. Hotta and K. R. Parthasarathy, Multiplicity formulae for

11 S. Martnennes. The characters of holomorphic discrete series,

12 H. Midorikawa. On a relation between characters of discrete

13 W. Schmid. Homogeneous complex manifolds and representations
of semisimple Lie groups. Thesis at Univ. of California-
(1968).

14 ————, Some remarks about the discrete series characters
of $S_p(n, \mathbb{R})$, preprint.

