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# ON THE LOGARITHMIC POLE DIFFERENTIALS

by

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## §0. Introduction

In this note we treat a hypersurface which may have singularities and the complement of it in complex projective space $\mathbb{P}^n$. It is well known that the cohomology over $\mathbb{C}$ of $\mathbb{P}^n-D$, where D is a reduced divisor, can be represented by closed rational form on $\mathbb{P}^n$ with finite order pole along D. This is the algebraic de Rham theorem by Grothendieck [4].

On the other hand K. Saito defined the sheaf of logarithmic pole differentials along D, and he conjectured it will be related to the local topological property of the complement of D. But now our problem here is global one and is the following:

Problem: Can the cohomology of $\mathbb{P}^n-D$ be obtained by the complex of logarithmic pole differentials?

This problem is not solved yet, but is partially solved affirmatively. When D is a non-singular hypersurface, it was obtained with stronger results than us by Griffiths [3] and Deligne [2]. Here we treat D with singularities, and we know that the complex of logarithmic pole differentials have relation to the singularities of D.

## §1. Definition of logarithmic differentials and the results of Griffiths

Definition 1: Let X be a complex manifold and D a reduced divisor (i.e. all multiplicities of components are one). We define;

$$\Omega^q(kD)_x := \{\omega \in \text{germs of rational q-form; } Q^k\omega \text{ is holomorphic}\}$$

$$\Omega^q(\log kD)_x := \{\omega \in \text{germs of rational q-forms; } Q^k\omega \ \& \ Q^k d\omega \text{ are holomorphic}\}$$

, where Q is a local equation of D at x in X.

And we can define two sheaves on X;

$$\Omega^q(kD) = \bigcup_{x\in X}\Omega^q(kD)_x$$

$$\Omega^q(\log kD) = \bigcup_{x\in X}\Omega^q(\log kD)_x$$

which are called the sheaf of q-forms with k-th pole along D and the sheaf of q-forms with logarithmic k-th pole along D respectively.

K. Saito conjectured the following;

Saito's conjecture: If $\Omega^1(\log D)$ is locally free at x, then X-D is locally $K(\pi, 1)$ at x.

(In general a topological space X is called $K(\pi, 1)$ if $\pi_i(X)$ vanish for all $i \geq 2$.)

And he researched his conjecture in case of a discriminant with respect to a coxeter group. It is related to the lecture of

Yano-Sekiguchi in this symposium.

But now we treat only global problems, so we use the notations;

$$A_k^q = \Gamma(X, \Omega^q(kD))$$

$$B_k^q = \Gamma(X, \Omega^q(\log kD)).$$

For example, when X is $\mathbb{C}^n = (x_1, \ldots, x_n)$ and D is hyperplane defined by $x_1 = 0$, $\frac{dx_2}{x_1^2}$ is not an element of $B_2^2$ but that of $A_2^2$. In general it is clear that $A_k^q$ includes $B_k^q$ for every q, k. We call the element of $B_k^q$ logarithmic pole differentials.

Griffiths proved the following theorem [3];

Theorem 2: If V is non-singular hypersurface in complex projective space of dimension n,

$$A_k^n / dA_{k-1}^{n-1} \longrightarrow H^n(\mathbb{P}^n - V; \mathbb{C})$$

is injective for any k.

And we have the algebraic de Rham theorem by Grothendieck [4];

Theorem 3: Let X be a compact smooth complex manifold and D an ample reduced divisor in X.

Then

$$H^p\Gamma(X, \Omega^{\cdot}(*)) \cong H^p(X-D; \mathbb{C})$$

, where $\Omega^p(*) := \bigcup_k \Omega^p(k)$ and

$$\Gamma(X, \Omega^{\cdot}(*)) := \{ \cdots \xrightarrow{d} \Gamma(X, \Omega^{p-1}(*)) \xrightarrow{d} \Gamma(X, \Omega^p(*)) \xrightarrow{d} \cdots \}$$

In case $X = \mathbb{P}^n$, this theorem asserts that the cohomology class of $H^p(X-D;\mathbb{C})$ can be represented by a rational differential p-form with finite order pole along D.

In the light of Theorem 3, Theorem 2 asserts that if $\varphi$ is a closed n-form with k-th order pole along D, then $\varphi$ represents zero class in $H^n(\mathbb{P}^n-D;\mathbb{C})$ if and only if $\varphi = d\psi$ for some (n-1)-form $\psi$ with (k-1)-th order pole along D, under the assumption of smoothness of D.

§2. Formulation of problem by using spectral sequence

Let X and D satisfy the assumption of Theorem 3 throughout this paragraph.

From now on we use the terms of spectral sequence which will clarify the significance of logarithmic pole differentials.

Definition: $K^{\cdot} := (\Gamma(X, \Omega^{\cdot}(*)), d)$

$$K^{\cdot}_{q-k} := ( \cdots \xrightarrow{d} A^{q-1}_{k-1} \xrightarrow{d} A^q_k \xrightarrow{d} A^{q+1}_{k+1} \xrightarrow{d} \cdots )$$

Thus we have a filtered complex

$$K^{\cdot} \supset \cdots \supset K^{\cdot}_i \supset K^{\cdot}_{i+1} \supset \cdots \supset K^{\cdot}_{n+1} = 0$$

We are going to research the spectral sequence of this filtered complex.

Then we can state Theorem 2 by using this spectral sequence;

Theorem 2': Under the assumption of Theorem 2, we have

$E_1^{p,q} = E_\infty^{p,q}$ $(p+q=n)$.

Explanation: Theorem 2 asserts that

$$A_k^n / dA_{k-1}^{n-1} = H^n(K^{\cdot}_{n-k}) \longrightarrow H^n(K^{\cdot})$$

is injective for any k. This means the natural map induced by the inclusion

$$H^n(K^{\cdot}_{q+1}) \longrightarrow H^n(K^{\cdot}_q)$$

is injective for any q, and we have

$$E_1^{p,q} = H^n(\mathrm{Gr}_p K^{\cdot}) \xrightarrow{\sim} \mathrm{Gr}_p H^n(K^{\cdot}) = E_\infty^{p,q}$$

for any p, q satisfying $p+q=n$.

This result is going to be generalized later.

Now the $E_2$-terms of this spectral sequence correspond to the logarithmic pole differentials!;

Theorem 4:

$$\cdots \to E_2^{q-k-1,k} \to H^q\Gamma(X, \Omega^{\cdot}(\log(k-1)D)) \to H^q\Gamma(X, \Omega^{\cdot}(\log kD))$$
$$\to E_2^{q-k,k} \to H^{q+1}\Gamma(X, \Omega^{\cdot}(\log(k-1)D)) \to \cdots$$

is an exact sequence.

proof: Notice $Z_1^{q-k,k} = \Gamma(X, \Omega^q(\log kD))$.

Thus we can state our fundamental conjecture about logarithmic pole differentials;

Conjecture: Our spectral sequence degenerate at $E_2$-terms for any D.

This conjecture means that the complex of logarithmic pole differentials determine the cohomology of X-D. In fact if this conjecture is true, we have;

$$H^q\Gamma(X, \Omega^{\cdot}(\log kD)) = \bigoplus_{i=0}^{k} E_2^{q-i,i} = \bigoplus_{i=0}^{k} E_\infty^{q-i,i} \subset H^q(X-D;\mathbb{C}).$$

§3. $E_1$-degeneracy and the codimension of singular locus of D

From now to the end we consider only complex projective space $\mathbb{P}^n$ as X.

The degeneracy of our spectral sequence is related to the codimension of singular locus of D, that is;

Theorem 5: $E_1^{p,q} = 0$ for $1<p+q<\operatorname{codim}_{\mathbb{P}^n}(\Sigma D)-1$, where $\Sigma D$ is singular locus of D.

Corollary 6: If D is non-singular, we have

$$E_1^{p,q} = E_\infty^{p,q} \quad (p+q=n)$$

and $E_1^{p,q} = 0 \quad (1<p+q<n)$.

<u>proof</u>: $H( E_1^{p-1,n-p} \longrightarrow E_1^{p,n-p} \longrightarrow 0 ) \cong E_2^{p,n-p}$ and we know from Theorem 5 that $E_1^{p,q} = 0$ for $p+q = n-1$.

So we get $E_1^{p,q} = E_\infty^{p,q}$ $(p+q=n)$ if $n>2$.

In case n is two, we can prove without using the result of Theorem 5.

This is a generalization of Theorem 2'.

<u>the idea of proof of Theorem 5</u>:

We define a filtered complex as follows;

$$L^{\cdot}_{i-k} = (\cdots \longrightarrow H_k^i \xrightarrow{d'} H_{k+1}^{i+1} \xrightarrow{d'} H_{k+2}^{i+2} \longrightarrow \cdots)$$

where $H_k^i: = \{$rational i-forms on $\mathbb{C}^{n+1}$ with at most k-th order pole along D with total degree zero (invariant under $\mathbb{C}^*$-action)$\}$ and $d'$: $H_k^i \longrightarrow H_{k+1}^{i+1}$ is defined by

$$d'\omega = \frac{dQ}{Q}\wedge\omega \text{ for } \omega \text{ belonging to } H_k^i.$$

(Q is a defining equation of D.)

By definition we have a filtered complex;

$$\cdots\cdots \subset L^{\cdot}_k \subset L^{\cdot}_{k-1} \subset \cdots\cdots .$$

We can reduce Theorem 5 to the following two lemmas;

<u>Lemma 7</u>: (K. Saito [6])

$H^i(L^{\cdot}_k) = 0$ for any k and $i<\text{height}\,\mathfrak{a} = \text{codim}_{\mathbb{H}}\mathfrak{n}(\Sigma D)$, where $\mathfrak{a}$ is an ideal generated by $\left\{\frac{\partial Q}{\partial \xi_0},\ldots,\frac{\partial Q}{\partial \xi_n}\right\}$, and $(\xi_0,\ldots,\xi_n)$ is homogeneous coordinate of $\mathbb{P}^n$.

Remark: This lemma implies that if $\varphi \in H^i_k$ satisfying $\frac{dQ}{Q}\wedge\varphi = 0$ then there exists $\eta \in H^{i-1}_{k-1}$ such that $\frac{dQ}{Q}\wedge\eta = \varphi$ if $i < \mathrm{codim}_{\mathbb{P}^n}(\Sigma D)$.

So this is a generalization of de Rham's lemma in our case.

Lemma 8:

$$H^i(Gr^k L^\bullet) \xrightarrow{d} H^{i+1}(Gr^{k+1}L^\bullet) \xrightarrow{f} H^i(Gr^k K^\bullet) = E_1^{k,i-k}$$

$$H^i(Gr^k L^\bullet) \searrow H^{i+1}(L^\bullet_{k+1}) \nearrow H^{i+1}(Gr^{k+1}L^\bullet) \quad (\circlearrowleft)$$

In the diagrame above $f\circ d$ is surjective.

(The constructions of morphisms are omitted.)

Combining Lemma 7 and Lemma 8, we have Theorem 5.

We have the following corollaries easily;

Corollary 9: $H^q(\mathbb{P}^n - D;\mathbb{C}) = 0$ for $1 < q < \mathrm{codim}_{\mathbb{P}^n}(\Sigma D) - 1$.

This is known results in topology ([5]). It is an easy application of partial Poincaré duality.

Corollary 10: $\Gamma(\mathbb{P}^n, \Omega^i(\log D)) = B^i_1 = 0$ for $1 < i < \mathrm{codim}_{\mathbb{P}^n}(\Sigma D) - 1$.

Corollary 11: If $\varphi$ is a closed i-form with k-th order pole along D, then $\varphi$ represents zero class in $H^n(\mathbb{P}^n - D;\mathbb{C})$ if and only if $\varphi = d\psi$ for some (i-1)-form $\psi$ with (k-1)-th order pole along D, under the assumption of $1 < i < \mathrm{codim}_{\mathbb{P}^n}(\Sigma D) - 1$.

Remark: This result is in [2] when D is smooth divisor in any compact smooth complex manifold X instead of $\mathbb{P}^n$.

About the first cohomology, we have;

Theorem 12: $E_2^{0,1} = E_\infty^1$, that is,

$$H^1\Gamma(\mathbb{P}^n, \Omega^\cdot(\log D)) \cong H^1(\mathbb{P}^n - D;\mathbb{C})$$

proof: Take Poincaré residue of a closed 1-form with single pole along D, and we have constant function on each component of D. So we can choose basis of left handside $\{\omega_2, \ldots, \omega_k\}$, where

$$\omega_j = d_j \frac{dQ_1}{Q_1} - d_1 \frac{dQ_j}{Q_j}$$

$Q = Q_1 \cdots Q_k$; factorization into irreducible polynomials

$d_i = \deg Q_i \quad (i=1,\ldots,k)$

They are also the basis of $H^1(\mathbb{P}^n - D;\mathbb{C})$.

Thus we have known our spectral sequence degenerates at $E_1$-terms for lower cohomology than $\mathrm{codim}_{\mathbb{P}^n}(\Sigma D) - 1$. So the easiest conjecture is that our spectral sequence degenerates at $E_1$-terms for every dimensional cohomology, but it is false;

Example: Let D be three projective lines in two dimensional complex projective space, and D is defined by the equation $\{Q = xyz = 0\}$. If our spectral sequence degenerates at $E_1$-term, then $H^2(K_0^{\bullet}) \longrightarrow H^2(K_{-1}^{\bullet})$ is injective (see the explanation of Theorem 2').

This means;

$$A_2^2/dA_1^1 \longrightarrow A_3^2/dA_2^1$$

is injective and thus $A_2^2 \wedge dA_2^1 = dA_1^1$.

We have $A_2^2 \wedge dA_2^1 = dB_2^1$ by the very definition of logarithmic pole.

On the other hand

$$\varphi = \frac{x^4(ydz-zdy)}{Q^2}$$

is an element of $B_2^1$ and

$$d\varphi = \frac{-2x^3(xdy\wedge dz+ydz\wedge dx+zdx\wedge dy)}{Q^2}$$

But we can show that if

$$\psi = \frac{P}{Q^2}(xdy\wedge dz+ydz\wedge dx+zdx\wedge dy)$$

an element of $dA_1^1$, then P must belong to an ideal generated by $\left\{\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}\right\} = \{yz, zx, xy\}$.

It is contradiction because $x^3 \notin \{yz, zx, xy\}$, so we know $E_1$-degeneracy is false in this case.

Then is $E_2$-degeneracy true in this case?

The answer is yes.

## §4. $E_2$-degeneracy of restricted polynomials

Definition 13: If Q is a homogeneous polynomial, then Q is called separated type if Q is a sum of monomials without common variables.

Example: $Q = xyz$, $Q = x^3+y^2z$, and $Q = x^3+y^3+z^3$ are polynomials of separated type and $Q = x^3+y^3+xyz$ is not.

Theorem 14: If Q is a polynomial of separated type, we have;

$$E_2^{p,q} = E_\infty^{p,q} \text{ for any } p,\ q \text{ satisfying } p+q=n.$$

the idea of proof: We defined a filtered complex $L^\bullet$ in the idea of proof of Theorem 5.

One can define $\tilde{Z}_k^i$ and $\tilde{B}_k^i$ as usual;

$$\tilde{Z}_k^i: = \{\omega\in H_k^i \ : \ d'\omega = 0\}$$

$$\tilde{B}_k^i: = d'H_{k-1}^{i-1}\subset H_k^i$$

Using these, the $E_2$-degeneracy at top term (n-th cohomology) reduces to the following lemma;

Lemma 15: If $d\tilde{Z}_k^n \cap H_{k-1}^{n+1} = d\tilde{Z}_{k-1}^n$ for any k, then we have

$$E_2^{p,q} = E_\infty^{p,q}$$

for any p, q satisfying $p+q=n$.

Remark: In Lemma 7, we stated $\tilde{Z}_k^i = \tilde{B}_k^i$ for any k and $i<\operatorname{codim}_{\mathbb{P}^n}(\Sigma D)$.

K. Saito proved in [6] that there exists positive integer $l$ such that $\tilde{Z}_k^i$ is included in $\tilde{B}_{k+l}^i$ for any i and k.

But in fact one can prove that $\tilde{Z}_k^i$ is included in $\tilde{B}_{k+1}^i$ for any i and k in this case. Moreover we know

$$d\tilde{Z}_{k-1}^n = d\tilde{B}_k^n \wedge H_{k-1}^{n+1}.$$

So the assumption of Lemma 15 is equivalent to

$$d\tilde{Z}_k^n \wedge H_{k-1}^{n+1} \subset d\tilde{B}_k^n \text{ when } i=n.$$

Lemma 16: If Q is a polynomial of separated type, then there exists the complementary space $\tilde{D}_k^n$ satisfying the following three conditions;

(i) $\tilde{Z}_k^n = \tilde{B}_k^n + \tilde{D}_k^n$

(ii) $d\tilde{Z}_k^n \wedge H_{k-1}^{n+1} = d\tilde{B}_k^n \wedge H_{k-1}^{n+1} + d\tilde{D}_k^n \wedge H_{k-1}^{n+1}$

(iii) $d\tilde{D}_k^n \wedge H_{k-1}^{n+1} = d\tilde{D}_{k-1}^n$

The proof of this lemma is complicated, so it is omitted in this note.

By the remark above and Lemma 16, we can show;

$$\begin{aligned} d\tilde{Z}_k^n \wedge H_{k-1}^{n+1} &= d\tilde{B}_k^n \wedge H_{k-1}^{n+1} + d\tilde{D}_{k-1}^n \\ &= d\tilde{Z}_{k-1}^n + d\tilde{D}_{k-1}^n \\ &= d\tilde{Z}_{k-1}^n \end{aligned}$$

Thus we have proved Theorem 14.

Corollary 17: If Q is a polynomial of separated type and $D = \{Q = 0\}$ is a hypersurface with isolated singularities in $\mathbb{P}^n$, then $E_2^{p,q} = E_\infty^{p,q}$ for any p, q.

proof: By using Theorem 5, one knows $E_2^{p,q} = 0$ if $p+q \leq n-2$. Combining Theorem 14, we have the result above.

Remark: The following two conditions are equivalent;

(i) $E_2^{p,q} = E_\infty^{p,q}$ for p, q satisfying $p+q=n$

(ii) Let $\tilde{D}$ be a cone in $\mathbb{C}^{n+1}$ defined by $Q = 0$ and X be a homogeneous algebraic vector field which is tangential to the each level surfaces of $\tilde{D}$. Assume that divX vanishes on $\tilde{D}$. There exists a homogeneous algebraic vector field Y which tangents to the each level surfaces of $\tilde{D}$ satisfying divX = QdivY.

So this is also a problem of differential equations.

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