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S-行列の超局所解析

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S-行列に対する不連続性公式を超局所化し、それにより、佐藤予想、即ち、S-行列が極大過剰決定系を満たす、という予想の証明を、かなり多くの物理領域の点において証明することを目標とする。最近詳しい報文(*)を王子セミナー報告集の為に準備したので、ここで詳細は略したい。ただ、その報告の中で触れた extended Landau variety について、その物理的背景を与える一文を、付録として掲載しておきたい。

この報文は、王子セミナーの際、柏原-河合-Stapp の報告の予稿(の一部)として配布されたものである。

(*) Discontinuity formula and Sato's conjecture, to appear in Proc. Oji Seminar (Publ. RIMS, Supplement)

§II Macrocausality at $u=0$ points

In ref. [ ] the singularity spectrum of the S-matrix at $u\neq 0$ points was derived from the macrocausality principle. This principle states that momentum-energy is carried over macroscopic distances only by stable systems. More specifically, it states that the probability of a transfer of momentum-energy that is not attributable to a network of stable particles (or objects) falls-off exponentially under space-time dilation. The aim of this section is to extract from this principle a general condition on the singularity spectrum of the S-matrix that covers both $u\neq 0$ points and $u=0$ points.

The mathematical formulation of macrocausality depends on the well-known close correspondence between claasical and quantum physics. The usual quantum mechanical expression for the scattering transition probability

$$(2.1)\quad \begin{aligned}&\mathcal{P}(\varphi_1,\cdots,\varphi_m;\ \varphi_{m+1},\cdots,\varphi_n)\\ &= |S(\varphi_1,\cdots,\varphi_m;\ \varphi_{m+1},\cdots,\varphi_n)|^2\\ &= |\langle\varphi_{m+1},\cdots,\varphi_n|S|\varphi_1,\cdots,\varphi_m\rangle|^2\end{aligned}$$

can be converted to the classical form

$$(2.2)\quad \begin{aligned}&\mathcal{P}(\varphi_1,\cdots,\varphi_m;\ \varphi_{m+1},\cdots,\varphi_n)\\ &= \int\prod_{i=1}^{n}\left[\frac{d^3p_i\,d^3x_i\,\rho_i(p_i,\ x_i)}{(2\pi)^3}\right]\\ &\qquad S(p_1,x_1,\cdots,p_m,x_m;\ p_{m+1},x_{m+1},\cdots,p_n,x_n)\end{aligned}$$

by introducing the expressions [ ]

(2.3) $$\rho_i(p_i, x_i)$$
$$= \int \varphi^*(M_i v_i - \tfrac{1}{2} q_i)\varphi(M_i v_i + \tfrac{1}{2} q_i) e^{-iq_i x_i} (\tfrac{M_i}{m_i})^{1/2}$$
$$\times 2\pi\, \delta(q_i \cdot v_i) \frac{d^4 q_i}{(2\pi)^4}$$

and

(2.4) $$S(p_1, x_1, ---, p_m, x_m; p_{m+1}, x_{m+1}, ---, p_n, x_n)$$
$$= \int \prod_{j=1}^{n} [\frac{d^4 q_j}{(2\pi)^4} 2\pi\, \delta(q_j \cdot v_j) e^{-iq_j x_j} (\tfrac{M_j}{m_j})^{1/2}]$$
$$S(\{M_j v_j \mp \tfrac{1}{2} q_j\}) S^*(\{M_j v_j \pm \tfrac{1}{2} q_j\}),$$

where the upper sign is for initial particles and the lower sign is for final particles, $p_j = m_j v_j$, and

(2.5) $$M_j \equiv (m_j^2 - \tfrac{1}{4} q_j^2)^{1/2}.$$

The right hand side of (2.2) is identical to the expression for the scattering transition probability occurring in classical statistical mechanics. Classically the statistical weight $\rho_i(p_i, x_i)$ for an initial particle i is interpreted as the probability density that the associated statistical ensemble has a particle that carries momentum-energy p_i and moves on a space-time trajectory passing through x_i. For final particles the statistical weight $\rho_i(p_i, x_i)$ is interpreted as the efficiency for detecting a particle that carries momentum-energy p_i and moves on a trajectory passing through x_i. The function $S(p_1, x_1, ---, p_m, x_m; p_{m+1}, x_{m+1}, ---, p_n, x_n)$ represents probability that a system of m initial particles carrying momentum-energies $p_1, \text{---}, p_m$ and moving on trajectories passing through the space-time points $x_1, \text{---}, x_m$, respectively, will scatter into a system of $n-m$ final particles carrying momentum-energies $p_{m+1}, \text{---}, p_n$ and moving on trajectories passing through the space-time points $x_{m+1}, \quad , x_i$ respectively.

The quatum mechanical functions $\rho_i(p_i, x_i)$ and $S(p, x)$ are not necessarily positive and are subject to uncertainly principle limitations. But they are otherwise very similar to their classical counterparts. (See ref. [ ] for a detailed discussion) The formulas given above thus provide a very close correspondence between classical and quantum physics. Macrocausality furthers this correspondence by asserting that the classical idea that momentum-energy is transferred by physical particles becomes valid asymptotically. The asymptotic limit $\tau \longrightarrow \infty$ that we shall diccuss is essentially the same as the classical limit $\hbar \longrightarrow 0$, since $\hbar$ a parameter that fixes the space-time scale.

The macrocausality principle asserts that the probability of transfer of momentum-energy not attributable to a network of stable particles falls off exponentially under space-time dilation. This condition is made quantitative with the aid of a semi-classical model of the scattering process. In this model the momentum-energy of the initial particles is transferred to the final particles by some network of mechanisms, as indicated in Fig. 1.

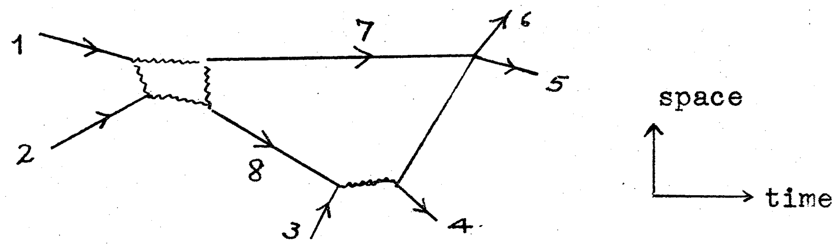

Fig. 1 A space-time diagram showing a typical network of mechanisms that transfers the energy-momentum carried by the initial particles 1, 2, and 3 a scattering process to the final particles 4, 5, 6. Momentum-energy is conserved at each vertex.

The straight lines in Fig. 1 represent transfers attributable to stable particles, whereas the wiggly lines represent transfers not attributable to physical particles. A transfer attributable to a stable particle of mass m is characterized by the classical condition $P = mv$, where P is the momentum-energy carried by the stable particle, m is its mass, and $v = dx/d\tau$ is its covariant velocity. A transfer represented by a wiggly line can occur only with a probability that falls off exponentially under space-time dilation.

The wiggly lines represent various mechanisms for momentum-energy transfers other than stable particles. It is possible, however, that momentum-energy can be conveyed also by stable particles traveling slightly off their mass shells. In this case macrocausality demands that the probability of such a transfer fall-off exponentially under space-time dilation.

The various possible networks are represented by points in a space of parameters ξ, and $\mathcal{P}(\xi)$ is the probability density that the momentum-energy of the initial particles of the network will be transferred by this network to the final particles. Thus $\mathcal{P}(\xi)$ is related to the function $S(p_1, x_1, ----, p_n, x_n) \equiv S(p, x)$ of (2.2) by the equation

$$(2.6) \quad S(p, x) = \int d\xi \mathcal{P}(\xi) \prod_{i=1}^{n} [\delta^3(p_i - p_i(\xi))\delta^3(x_i - x_i(\xi))].$$

For any network ξ there are others obtained from it by an overall space-time dilation. Let τ be a dilation parameter that increases linearly with the space-time size of the network. Let the network ξ dilated by the amount τ be represented by ξ^τ. Then macrocausality asserts that there are a pair of nonnegative continnous functions $C(\xi)$ and $\alpha(\xi)$ such that

$$(2.7) \quad |\mathcal{P}(\xi^\tau)| < C(\xi)\exp(-\alpha(\xi)\tau)$$

where $C(\xi)$ is integrable when restricted to compact sets in p space, and $\alpha(\xi)$ is strictly positive $(\alpha > 0)$ unless each line of the network satisfies the condition $p = mv$ associated with some stable physical particle. There is a positive contribution to $\alpha(\xi)$ from each wiggly line of ξ that has nonzero lingth, and the continuity of $\alpha(\xi)$ means that for any sequence of networks ξ_n the quantity

$\alpha(\xi_n)$ tends to zero only if the sum of the lengths of wiggly lines tends to zero, and all physical particle go to their masses shells:

(2.8a) $\alpha(\xi_n) \longrightarrow 0$

implies

(2.8b) $\sum_{i \in W} \|\Delta_{in}\| \longrightarrow 0,$

where Δ_{in} is the (Euclidean) length of line i of ξ_n, and the sum is over all wiggly lines of the network ξ_n, and also

(2.8c) $\sum_{i \in P} |p_i^2 - m_i^2| \longrightarrow 0$

where the sum is over all physical particle (solid) lines.

The set of variables $(p, x) \equiv (p_1, \text{---}, p_n; x_1, \text{---}, x_n)$ specifics the space-time trajectory lines of the set of external particles. In particular, $v_i = p_i/m_i$ defines the direction of the trajectory line of external particle i, and x_i is a point lying on this trajectory line. Define $u = x/\tau$. A set (p, u) is said to be <u>causal</u> if and only if the corresponding external trajectories can be joined by a nontrivial network of trajectories corresponding to stable particles. A trivial network is a network such that all of the vertices lie at a single point.

The ordinary Landau equations define the set of causal (p, u):

(2.9) $\{\text{causal } (p, u)\} = \{(p, u) : (p, u)$ is a solution of the Landau equations for some $D^+\}$

If for a given p there is a causal set $(p, u)=(p, 0)$ then p is called a $u=0$ point. For any such point one can easily construct a bounded sequence of causal sets (p_n, u_n) satisfying $p_n \longrightarrow p$ such that the corresponding sequence of causal space-time diagrams is unbounded in the sense that no bounded space-time region R_w contains all the vertices w_i of all the diagrams of the sequence. Conversely, if there is a bounded sequence of causal points (p_n, u_n) satisfying $p_n \longrightarrow p$ such that the corresponding sequence of causal space-time diagrams is unbounded (in this same sense) then p is a $u=0$ point. This follows from the fact that the sequence of growing diagrams can be scaled down by the minimum amount such that each vector lies inside the closure R_w of some neighborhood of the origin. The sequence of scale changes increases without bound. Hence the rescaled u_n, called u_n', satisfy $u_n' \longrightarrow 0$. Let $w_n \in \mathbb{R}^{4n'}$ be the collection of vectors that describes the positions of the n' vertices of the n-th rescaled diagram. The w_n lie in a compact subset of $\mathbb{R}^{4n'}$ and hence have accumulation point w, which defines a causal space-time diagrm. If this diagram is nontrivial then it defines a causal $(p, u)=(p, 0)$. If it is trivial then the common vertex must lie on the surface of R_w, and all of the external lines must pass through it, and also the origin. Placing another vertex at the origin one again gets a causal $(p, u)=(p, 0)$.

The conclusion is this:

(2.10a) $\{p;\ p$ is not a $u=0$ point$\}$

$= \{p;$ every bounded causal sequence (p^n, u^n) with $p^n \longrightarrow p$ corresponds to a bounded sequence of diagrams (i.e., to a sequence of diagrams whose vertices remain in a bounded $R_w)\}$.

This result entails that

(2.10b) $\{p;\ p$ is not a $u=0$ point$\}$

$= \{p;$ for any bounded set U of vectors u of p and some R_w such that for every causal (p', u) with $p' \in N_p$ and u in U the corresponding causal diagram has its vertices in $R_w\}$

A similar result is this:

(2.11) $\{(p, u);\ (p, u)$ is not causal, p is not $u=0$ point$\}$

$\supset \{(p, u);$ there is a neighborhood N_p of p and a neighborhood N_u of u such that every network ξ with its set (p', u') in (N_p, N_u) has $\alpha(\xi) > a(N_p, N_u) > 0\}$

Here the dilation parameter τ is set to unity: $u=x$.

To prove (2.11) assume that the condition on the right-hand side is false. Then there must be a sequence of networks ξ_n such that the $(p_n, u_n) \longrightarrow (p, u)$ and $\alpha(\xi_n) \longrightarrow 0$. The condition $\alpha(\xi_n) \longrightarrow 0$ implies that the sum of the lengths of the wiggly lines goes to zero. Thus if all the end points of all the stable lines of all the networks ξ_n are confined to a bounded region R_w and if one has to consider only networks with a finite number of stable-particle lines, then the compactness of the space of variables describing the end points of these stable particle lines implies that an accumulation point in this space must exist. At this accumulation point the wiggly lines all have zero length. Thus the limit point defines a causal space-time diagram having a set (p', u') that equals (p, u). But then (p, u) is causal, contrary to the first assumption on the left-hand side of (2.11).

The remaining possibities are either that in the sequence of networks ξ_n the end points of the stable-particle lines do not remain in any compact region R_w or that networks with an unbounded number of stable-particle lines must be considered.

We shall not consider the possibility that an infinite number of stable particles conspire together to give a point in the singularity spectrum. We simply assume that the singularity spectrum of the S-matrix union over all finite N of the singularity spectrums obtained by considering networks with only N stable particles. This assumption disposed of one of the two remaining cases.

The final case is that in which the end points of the (finite set of) stable particle lines do not remain in any compact region R_w. However, the sum of the lengths of the wiggly lines tend to zero. Consequently the construction that was used to prove (2.10) works also in this case and shows that p must be a u=0 point.

The bound (2.11) on $\alpha(\xi')$, inserted into the bound (2.7), gives, with the aid of (2.6), a bound on $S(p, u\tau)$: for any noncausal set (p, u) such that p is not a u=0 point there are neighborhoods N_p of p and N_u of u, and numbers $C > 0$ and $\alpha > 0$ such that for all $p' \in N_p$, all $u' \in N_u$, and all

(2.12) $S(p', u'\tau) < C\exp(-\alpha\tau)$

This bound on $S(p, x)$ is a quantitive expression of macrocausality in the semi classical framework. The quantitative expression of macrocausality in quantum theorey is the set of bounds on transition probabilities obtained by inserting the semi-classical bounds on $S(p', u'\tau)$ into (2.2).

To derive conditions on the singularity spectrum of S from this macrocausality property one can use in (2.2) wave functions of the form

$$(2.12)\quad \varphi_i(p_i) = \chi_i(p_i)\exp[-(\vec{P}_i - \vec{p}_i)^2\gamma\tau + ip_iu_i\tau]$$
$$= \varphi_i(p_i;\ P_i,\ u_i,\ \gamma,\ \tau).$$

Here $\vec{p}_i$ and $\vec{P}_i$ are the vector parts of two mass-shell variables p_i and P_i, and $\chi_i(p_i)$ is an infinitely differentiable function of compact support that satisfies $|\chi_i(p_i)| \leqq 1$ and is analytic at $p_i = P_i$. The product of functions $|\rho_i(p_i, x_i)|$ correspondens to these functions φ_i enjoys a strong exponential fall off property: Let N_P and N_u be any open neighborhoods of the points P and u, respectively, and let $(N_P \times N_u^\tau)'$ be the complement of $N_P \times N_u^\tau$, where $N_u^\tau \equiv \{x;\ x = u'\tau,\ u' \in N_u\}$. Then there are strictly positive numbers $c > 0$, $\alpha > 0$, and $\gamma_o > 0$, which depend only on N_P and N_u, such that for all $0 \leqq \gamma \leqq \gamma_o$ and all (p_i, x_i) in $(N_P \times N_u^\tau)'$

$$(2.13) \qquad |\prod_i \rho_i(p_i, x_i)| < C\exp(-\alpha\gamma\tau)$$

Moreover, this function $|\Pi\rho_i|$ has compact support in p space. Thus the integrability property of the function $C(\xi)$ of (2.7) entails that a bound of the form (2.13) holds also for the part of the integral (2.2) coming from the region $(N_P \times N_u^\tau)'$. If (P, u) is a noncausal set, and P is not a u=0 point, then the condition (2.12) ensures that the contribution to (2.2) from the remaining set $N_P \times N_u'$ is also exponentially bounded. In particular, there are three strictly positive numbers $c > 0$, $\alpha > 0$, and $\gamma_o > 0$ such that the function

$$(2.14) \qquad \mathcal{P}(\varphi_1(p_1;\ P_1, u_1, \gamma, \tau), \cdots, \varphi_n(p_n;\ P_n, u_n, \gamma, \tau))$$
$$\equiv \mathcal{P}(P, u, \gamma, \tau)$$

as defined in (2.1) and calculated by (2.2), satisfies

(2.15) $\mathcal{P}(P, u, \gamma, \tau) < C\exp(-\alpha\gamma\tau)$

The continuity properties of all the functions involved in the derivation of (2.15) entail that this bound hold uniformly in some neighborhood of the original point (P, u).

The condition (2.15), holding uniformly in a neighborhood of (P, u), is, by definition, the statement that (P, u) lies outside the essential support of S. But the concepts of essential support and singularity spectrum have been shown to be equivalent [ ], at least for distributions, and hence for S. Thus macrocausality implies that all noncausal (p, u) with p not a u=0 point lie outside the singularity spectrum of S'.

Consider now the u=0 points. The new feature at these points is that the condition that the u remain in a bounded region R_u does not entail that the vertices remain in a bounded region R_w. Thus there may, for $u \in R_u$, be sequences of networks ξ_n such that the sum of the lengths of the wiggly lines tend to zero and all physical particle momenta tend to their mass shells but no causal diagram exists.

Because of this fact the macrocausality condition fails to yield at u=0 points the conclusion that the singularity spectrum of S is confined to the solutions of the positive-α Landau equations. It leads rather to the conclution that the singularity spectrum of S is confined to the set of points (p, u) for which there is a sequence of networks $\xi^{(m)}$ satisfying

$$(2.16) \qquad (p^{(m)}, u^{(m)}) \longrightarrow (p, u)$$

and

$$\alpha(\xi^{(m)}) \longrightarrow 0$$

If such a sequence exists then the proof that (p, u) lies outside the singularity spectrum fails. On the other hand, if no such sequence exists then there must be some neighborhoods N_p and N_u of p and u and an associated number $a(N_p, n_u) > 0$ such that $\alpha(\xi) > a(N_p, N_u)$ for all ξ such that $(p(\xi), u(\xi))$ lies in $N_p \times N_u$. Over this information the proof proceeds exactly as before, and one can conclude that (p, u) is not in the singularity spectrum of S.

In view of (2.8) the final conclusion is this:

$$(2.17) \qquad \text{S.S. } S(p) \subset \{(p, u);\ \exists\ \xi^{(m)} \text{ such that } (p(\xi^{(m)}), u(\xi^{(m)}) \longrightarrow (p, u), \sum_{i \in W} \|\Delta_i(\xi^{(m)})\| \longrightarrow 0, \text{ and } \sum_{i \in P} |p_i^2(\xi^{(m)}) - m_i^2| \longrightarrow 0\}.$$

In S-matrix theory the external particles of one scattering process are internal particles of some larger process. It is thus unnatural to treat them differently, and doing so would be expected to lead to inconsistencies. Thus the natural, and conservative, course is to allow in (2.17) both the internal and external solid lines of the networks ξ_n to be off-mass shell.