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On the Oscillation of Fourth Order Nonlinear Differential Equations with Deviating Argument

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Introduction

The purpose of this note is to report some results which we have recently obtained concerning the oscillation of solutions of the nonlinear differential equation

$$(*) \qquad [r(t)y''(t)]'' + f(y(g(t)), t) = 0,$$

where the following conditions are assumed to hold:

(a) $r(t)$ is continuous and positive on $[0,\infty)$;

(b) $g(t)$ is continuous on $[0,\infty)$ and $\lim_{t\to\infty} g(t) = \infty$;

(c) $f(y,t)$ is continuous on $(-\infty,\infty)\times[0,\infty)$ and $yf(y,t) > 0$ for $y \neq 0$.

It should be noticed that the deviating argument $g(t)$ is allowed to be retarded ($g(t) \leq t$) or advanced ($g(t) \geq t$) or otherwise.

Equation (*) is called superlinear if $f(y,t)/y$ is nondecreasing in y for $y > 0$ and nonincreasing in y for $y < 0$; (*) is called strongly superlinear if there is a number $\alpha > 1$ such that $f(y,t)/|y|^{\alpha}\operatorname{sgn} y$ is nondecreasing in $y > 0$ and nonincreasing in $y < 0$. Dually, equation (*) is called sublinear if $f(y,t)/y$ is nonincreasing in y for $y > 0$ and nondecreasing in y for $y < 0$; (*) is called strongly sublinear if there exists a positive number $\beta < 1$ such that $f(y,t)/|y|^{\beta}\operatorname{sgn} y$ is nonincreasing in $y > 0$ and nondecreasing in $y < 0$.

Our attention will be restricted to solutions $y(t)$ of (*) which exist on some ray $[T_y,\infty)$ and satisfy

$$\sup\{|y(t)| : t \geq T\} > 0 \quad \text{for any} \quad T \geq T_y.$$

Such a solution is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory otherwise.

We are interested in finding necessary and sufficient conditions for all solutions of (*) to be oscillatory. The results obtained extend considerably those of [1], [2] for ordinary differential equations of the form less general than (*). Some of the results of this note will appear in [3].

It is convenient to distinguish the two cases:

(A) $$\int_0^\infty \frac{t}{r(t)}\,dt = \infty,$$

(B) $$\int_0^\infty \frac{t}{r(t)}\,dt < \infty.$$

The following notation will be used throughout:

$$g^*(t) = \max\{g(t),t\}, \qquad g_*(t) = \min\{g(t),t\},$$

$$R(t) = \int_0^t \frac{(t-s)s}{r(s)}\,ds, \qquad \rho(t) = \int_t^\infty \frac{s-t}{r(s)}\,ds.$$

1. Nonoscillation Theorems

We first present sufficient conditions which guarantee the existence of nonoscillatory solutions of (*) with special asymptotic properties. From these results necessary conditions for the oscillation of all solutions of (*) will readily be derived.

THEOREM 1A. Let (*) be either superlinear or sublinear. Suppose that (A) holds.

(i) A necessary and sufficient condition for (*) to have a solution $y(t)$ such that $\lim_{t\to\infty} y(t)/R(t) = a \neq 0$ is that

$$\int^{\infty} |f(cR(g(t)),t)|\,dt < \infty \quad \text{for some} \quad c \neq 0.$$

(ii) A necessary and sufficient condition for (*) to have a solution $y(t)$ such that $\lim_{t\to\infty} y(t) = b \neq 0$ is that

$$\int^{\infty} R(t)|f(c,t)|\,dt < \infty \quad \text{for some} \quad c \neq 0.$$

THEOREM 1B. Let (*) be either superlinear or sublinear. Suppose that (B) holds.

(i) A necessary and sufficient condition for (*) to have a solution $y(t)$ such that $\lim_{t\to\infty} y(t)/t = a \neq 0$ is that

$$\int^{\infty} \rho(t)|f(cg(t),t)|\,dt < \infty \quad \text{for some} \quad c \neq 0.$$

(ii) A necessary and sufficient condition for (*) to have a solution $y(t)$ such that $\lim_{t\to\infty} y(t)/\rho(t) = b \neq 0$ is that

$$\int^{\infty} t|f(c\rho(g(t)),t)|\,dt < \infty \quad \text{for some} \quad c \neq 0.$$

We give an outline of the proof of Theorem 1B-(ii).

(Necessity) Let (*) have a solution $y(t)$ such that $\lim_{t\to\infty} y(t)/R(t) = a$. We may suppose that $a > 0$. There exist positive constants T, a_1, a_2 such that

$$a_1 g(t) \leq y(g(t)) \leq a_2 g(t) \quad \text{for} \quad t \geq T.$$

This implies that

(1.1) $f(y(g(t)),t) \geq f(a_1 g(t),t)$ if (*) is superlinear,

(1.2) $f(y(g(t)),t) \geq (a_1/a_2)f(a_2g(t),t)$ if (*) is sublinear.

It turns out that the following three cases are possible:

(I) $y'(t) > 0$, $y''(t) > 0$ and $[r(t)y''(t)]' > 0$ for $t \geq T$;

(II) $y'(t) > 0$, $y''(t) < 0$ and $[r(t)y''(t)]' > 0$ for $t \geq T$;

(III) $y'(t) > 0$, $y''(t) < 0$ and $[r(t)y''(t)]' < 0$ for $t \geq T$.

Let Case (I) or (II) hold. Then, integrating (*), using (1.1), (1.2) and noting that $[r(t)y''(t)]' > 0$, we readily obtain

$$\int^{\infty} f(a_i g(t),t)dt < \infty,$$

where $i = 1$ if (*) is superlinear and $i = 2$ is (*) is sublinear. Let Case (III) hold. We multiply (*) by $\rho(t)$, integrate it from T to t, add $y'(t)$ to both sides of the resulting equation and let $t \to \infty$. Using the inequalities

$$y'(t) \geq -[r(t)y''(t)]'\rho(t),$$

$$y'(t) \geq -r(t)y''(t)\int_t^{\infty} \frac{ds}{r(s)},$$

we arrive at

$$\int^{\infty} \rho(t)f(a_i g(t),t)dt < \infty,$$

where $i = 1$ or 2 according as (*) is superlinear or sublinear.

(Sufficiency) Suppose $c > 0$. Put $a = c/2$ or c according as (*) is superlinear or sublinear. Take $T > 0$ so large that

$$\int_T^{\infty} \rho(t)f(cg(t),t)dt < \frac{a}{6} \quad \text{and} \quad T_0 = \inf_{t \geq T} g_*(t) > 0.$$

Let Y designate the linear space of all continuous functions $y(t)$ on $[T_0,\infty)$ such that

$$\|y\| = \sup_{t \geq T_0} [|y(t)|/t^2] < \infty$$

and put

$$X = \{y \in Y: at \leq y(t) \leq 2at \quad \text{for} \quad t \geq T_0\}.$$

Clearly, Y is a Banach space with norm $\|\cdot\|$ and X is a bounded, convex and closed subset of Y.

We define the operator Φ by

$$(\Phi y)(t) = at + \int_T^t \left(\int_s^t \frac{(\sigma - s)\sigma}{r(\sigma)} d\sigma \right) f(y(g(s)),s)ds$$

$$+ t\int_t^\infty \frac{ds}{r(s)} \cdot \int_T^t (t - s) f(y(g(s)),s)ds$$

$$+ t\int_t^\infty \frac{s - t}{r(s)} ds \cdot \int_T^t f(y(g(s)),s)ds$$

$$+ t\int_t^\infty \left(\int_s^\infty \frac{\sigma - s}{r(\sigma)} d\sigma \right) f(y(g(s)),s)ds \qquad \text{for } t \geq T,$$

$$(\Phi y)(t) = at + T\int_T^\infty \left(\int_s^\infty \frac{\sigma - s}{r(\sigma)} d\sigma \right) f(y(g(s)),s)ds \quad \text{for } T_0 \leq t \leq T.$$

It can be shown that Φ is a continuous operator mapping X into a compact subset of X. Thus the Schauder fixed point theorem is applicable, and Φ has a fixed point in X. This fixed point provides the required solution of (*).

2. Oscillation Theorems

Our next task is to give sufficient conditions for the oscillation of all solutions of (*) by limiting ourselves to the strongly superlinear and strongly sublinear cases.

THEOREM 2A. Let (*) be strongly superlinear. Suppose (A) holds. If there is a differentiable function $h(t)$ on $[0,\infty)$ such that

$$h(t) \leq g_*(t), \quad h'(t) \geq 0, \quad h(t) \to \infty \text{ as } t \to \infty,$$

and

$$\int^{\infty} R(h(t))\,|f(c,t)|\,dt = \infty \quad \text{for all} \quad c \neq 0,$$

then all solutions of (*) are oscillatory.

THEOREM 2B. Let (*) be strongly superlinear. Suppose (B) holds. If

$$\int^{\infty} g_*(t)\,|f(c\rho(g^*(t)),t)|\,dt = \infty \quad \text{for all} \quad c \neq 0,$$

then all solutions of (*) are oscillatory.

THEOREM 3A. Let (*) be strongly sublinear. Suppose (A) holds. If

$$\int^{\infty} \frac{R(g_*(t))}{R(g(t))}\,|f(cR(g(t)),t)|\,dt = \infty \quad \text{for all} \quad c \neq 0,$$

then all solutions of (*) are oscillatory.

THEOREM 3B. Let (*) be strongly sublinear. Suppose (B) holds. If

$$\int^{\infty} \frac{g_*(t)}{g(t)}\,\rho(g^*(t))\,|f(cg(t),t)|\,dt = \infty \quad \text{for all} \quad c \neq 0,$$

then all solutions of (*) are oscillatory.

Here we sketch the proof of Theorem 3A. Let there exist a nonoscillatory solution $y(t)$. Without loss of generality we may suppose that $y(t)$ is eventually positive. It is easy to show that $y'(t) > 0$, $[r(t)y''(t)]' > 0$ for all large t, say $t \geq T$, that there are positive constants a_1, a_2 such

(2.1) $$a_1 \leq y(t) \leq a_2 R(t), \quad t \geq T,$$

and that the following inequality holds:

(2.2) $$y(t) \geq R_T(t)[r(t)y''(t)]', \quad t \geq T,$$

where

$$R_T(t) = \int_T^t \frac{(t-s)(s-T)}{r(s)}\, ds.$$

Let $T' > T$ be such that $g_*(t) \geq T$ for $t \geq T'$. An integration of (*) yields

(2.3) $$[r(t)y''(t)]' \geq \int_t^\infty f(y(g(s)),s)ds, \quad t \geq T'.$$

Combining (2.3) with the inequality

$$y(g(t)) \geq R_T(g_*(t))[r(t)y''(t)]', \quad t \geq T',$$

which follows from (2.2), we have

(2.4) $$y(g(t)) \geq R_T(g_*(t))\int_t^\infty f(y(g(s)),s)ds, \quad t \geq T'.$$

If we apply the lemma stated below to (2.4), observing from (2.1) that $y(g(t)) \leq aR_T(g(t))$, $t \geq T'$, for some constant $a > 0$, then we conclude that

$$\int^\infty \frac{R_T(g_*(t))}{R_T(g(t))} f(aR_T(g(t)),t)dt < \infty.$$

But this contradicts the assumption of Theorem 3A.

LEMMA. Let (*) be strongly sublinear. Let $u(t)$, $v(t)$, $w(t)$, $\mu(t)$ be positive continuous functions on (T,∞) such that $\mu(t) \leq 1$, $u(t) \leq aw(t)$, and

$$u(t) \geq \mu(t)w(t)\int_t^\infty v(s)f(u(s),s)ds \quad \text{for } t > T,$$

where a is a positive constant. Then,

It is of interest to observe that from the foregoing results necessary and sufficient conditions can be obtained for the oscillation of all solutions of certain classes of differential equations of the form (*).

THEOREM 4A. Let (*) be strongly superlinear and advanced. Suppose (A) holds. Then all solutions of (*) are oscillatory if and only if

$$\int^{\infty} R(t)|f(c,t)|dt = \infty \quad \text{for all} \quad c \neq 0.$$

THEOREM 4B. Let (*) be strongly superlinear and advanced. Suppose (B) holds. Then all solutions of (*) are oscillatory if and only if

$$\int^{\infty} t|f(c\rho(g(t)),t)|dt = \infty \quad \text{for all} \quad c \neq 0.$$

EXAMPLE 1. Consider the strongly superlinear retarded equation

$$[t^3y''(t)]'' + (3/16)t^{-1}|y(t^{1/2})|^2 \operatorname{sgn} y(t^{1/2}) = 0,$$

which has a nonoscillatory solution $y(t) = t^{-1/2}$. As easily verified, the integral condition of Theorem 4B is satisfied for this equation. This example shows that, as regards the oscillation of strongly superlinear equations of the form (*) with general deviating argument, there is a gap between the necessary condition and the sufficient condition that were given in the above theorems.

THEOREM 5A. Let (*) be strongly sublinear and retarded. Suppose (A) holds. Then all solutions of (*) are oscillatory if and only if

$$\int^{\infty} |f(cR(g(t)),t)|dt = \infty \quad \text{for all} \quad c \neq 0.$$

THEOREM 5B. Let (*) be strongly sublinear and retarded. Suppose (B) holds. Then all solutions of (*) are oscillatory if and only if

$$\int^{\infty} \rho(t)\,|f(cg(t),t)|\,dt = \quad \text{for all} \quad c \neq 0.$$

EXAMPLE 2. Consider the strongly sublinear advanced equation

$$[t^{1/2}y''(t)]'' + (1/2)t^{-7/2}|y(t^2)|^{1/2}\operatorname{sgn} y(t^2) = 0.$$

Although the integral condition of Theorem 5A is satisfied, this equation has a nonoscillatory solution $y(t) = t^2$. Thus it follows that there is a gap between the necessary condition and the sufficient condition for the oscillation of all solutions of the strongly sublinear equation (*) with general deviating argument.

REFERENCES

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