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FREE PRODUCTS WITH AMALGAMATION OF ORTHODOX SEMIGROUPS

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1. Introduction

A class of algebras $\mathcal{A}$ is said to have the strong amalgamation property if for any family of algebras $\{ A_i : i \in I \}$ from $\mathcal{A}$, each having an algebra $U \in \mathcal{A}$ as a subalgebra, there exist an algebra B in $\mathcal{A}$ and monomorphisms $\phi_i : A_i \to B$, $i \in I$, such that

(i) $\phi_i | U = \phi_j | U$ for all $i, j \in I$,

(ii) $A_i\phi_i \cap A_j\phi_j = U\phi_i$ for all $i, j \in I$ with $i \neq j$,

where $\phi_i | U$ denotes the restriction of ϕ_i to U. Omitting the condition (ii) gives us the definition of the weak amalgamation property. Adding the condition that $A_i = A_j$ for all $i, j \in I$, to the hypothesis of the definition of the strong amalgamation property gives us the definition of the special amalgamation property.

It is well-known (see [3]) that in a class of algebras closed under isomorphisms and the formation of the union of any ascending chain of algebras, each amalgamation property follows from the case $|I| = 2$. Hence we shall consider, in this paper, only the case $|I| = 2$.

The classes of algebras for which the strong or the weak amalgamation properties are known to hold are "groups, groups with a given operator domain, commutative groups, fields, defferential fields of characteristic 0, partially ordered sets, lattices, Boolean algebras, locally finite-dimensional cylindric algebras of a given infinite dimendion[6], pseudocomplemented distributive lattices $\mathcal{B}_n$, $n \leqq 2$ or $n = \omega$ [2], inverse semigroups, semilattices[3], commutative inverse semigroups[4]".

In section 2, we shall first study the free product of orthodox semigroups. Secondly, we shall give the free product of the variety $\mathcal{A}$ of bands defined by an identity. If $\mathcal{A}$ is the variety of [left, right] normal bands, we can describe it in a more useful form.

In section 3, we shall add the following classes of semigroups to the list above: "[left, right] normal bands, rectangular bands, left[right] zero semigroups, one element semigroups, M[L.N, R.N]-inversive semigroups". Moreover, the variety of bands defined by an identity $P = Q$ has the strong amalgamation property if and only if $P = Q$ is a permutation identity or a heterotypical identity.

In section 4, we shall show that the varieties of [left, right] regular bands, left[right] quasinormal bands and [left, right] generalized inverse semigroups have the special amalgamation property.

The notations and terminologies are those of [1] and [8], unless otherwise stated.

2. Free products

At first, we shall give the free product in the class of orthodox semigroups. Let S_i, $i \in I$, be a family of orthodox semigroups, and let S be the free product of the S_i in the class of semigroups. For $x = x_{i_1}x_{i_2}\dots x_{i_k}$, $x_{i_j} \in S_{i_j}$, in a reduced form in S, set

$$W(x) = \{ x'_{i_k}\dots x'_{i_2}x'_{i_1} : x_{i_j} \in V(x_{i_j}) \},$$

where $V(x_i)(= V_{S_i}(x_i))$ denotes the set of inverses of x_i in S_i. If $x \in S_i$, it is obvious $W(x) = V_{S_i}(x)$.

By an argument similar to the proof of [Theorem 1, 7], we have the following theorem.

THEOREM 2. 1. *Let* $\sim$ *denote the congruence on* S *generated by all pairs* $(xx'x, x)$ *and* $(x_1x_1'x_2x_2'\dots x_nx_n',$ $x_1x_1'x_2x_2'\dots x_nx_n'x_1x_1'x_2x_2'\dots x_nx_n')$ *for* $x, x_1, x_2, \dots, x_n$ *in* S *and for* $x' \in W(x)$ *and* $x_j' \in W(x_j)$, $j = 1, 2, \dots, n$. *Then* $S/\sim$ *is the free product of the* S_i *in the class of orthodox semigroups*.

Let $\mathcal{A}$ be the variety of bands defined by an identity $P = Q$. If $P = Q$ is heterotypical, it follows from [8] that $\mathcal{A}$ is one of the varieties of rectangular bands, left zero semigroups, right zero semigroups and one element semigroups. Let $\mathcal{A}$ be the variety of rectangular bands. Let $\{ S_i : i \in I \}$ be a family of rectangular bands. For each S_i, there exist a left zero semigroup L_i, say, and a right zero semigroup R_i, say, such that $S_i \simeq L_i \times R_i$, $i \in I$. We can assume without loss of generality that $L_i \cap L_j = \square$ and $R_i \cap R_j = \square$ if $i \neq j$. Let $L = \cup \{ L_i : i \in I \}$, $R = \cup \{ R_i : i \in I \}$ and $B = L \times R$. Define a product $\circ$ on B as follows:

$$(a_1, b_1) \circ (a_2, b_2) = (a_1, b_2).$$

It is clear that $B(\circ)$ is the free product of the S_i in $\mathcal{A}$. Similarly, we can easily construct the free products in the other three varieties of bands.

So we now consider the free product in the variety of bands defined by a homotypical identity.

THEOREM 2. 2. *Let* $\mathcal{A}$ *be the variety of bands defined by a homotypical identity* $P(X_1, X_2, \ldots, X_n) = Q(X_1, X_2, \ldots, X_n)$. *Let* S_i, $i \in I$, *be a family of bands belonging to* $\mathcal{A}$, *and let* S *be the free product of the* S_i *in the class of semigroups. Let* $\sim$ *be the congruence on* S *generated by all pairs* (x^2, x) *and* $(P(x_1, x_2, \ldots, x_n), Q(x_1, x_2, \ldots, x_n))$ *for* $x, x_1, x_2, \ldots, x_n$ *in* S. *Then* $S/\sim$ *is the free product of the* S_i *in* $\mathcal{A}$.

Next, we shall consider the free product in the variety of [left, right] normal bands. In this case we can describe it in a more useful form. Let E_i, $i \in I$, be a family of normal bands. It follows from [9] that each E_i, $i \in I$, is isomorphic to the spined product $L_i \bowtie R_i(\Gamma_i)$ of a left normal band L_i and a right normal band R_i with respect to a semilattice Γ_i. Let $L_i \equiv \Sigma\{ L_{\alpha_i} : \alpha_i \in \Gamma_i \}$ and $R_i \equiv \Sigma\{ R_{\alpha_i} : \alpha_i \in \Gamma_i \}$ be the structure decompositions of L_i

and R_i, $i \in I$, respectively. Then the structure decomposition of each E_i is $E_i \equiv \Sigma\{ L_{\alpha_i} \times R_{\alpha_i} : \alpha_i \in \Gamma_i \}$. We identify each E_i with $\Sigma\{ L_{\alpha_i} \times R_{\alpha_i} : \alpha_i \in \Gamma_i \}$. To construct the free product of the E_i, we can assume without loss of generality that $E_i \cap E_j = \Box$ if $i \neq j$, and we assume $\cup \{ L_{\alpha_i} : \alpha_i \in \Gamma_i, i \in I \} = L$, say, and $\cup \{ R_{\alpha_i} : \alpha_i \in \Gamma_i, i \in I \} = R$, say, are both disjoint unions. Let $\Gamma = \{ (\alpha_i)_{i\in I} : \alpha_i \in \Gamma_i^{(1)}$, only finitely many but at least one α_i different from $1 \}$. For convenience, we write simply (α_i) instead of $(\alpha_i)_{i\in I}$. Let B be the subset of $L \times R \times \Gamma$ consisting of $(a,b,(\alpha_i))$ such that $a \in L_{\alpha_j}$ and $b \in R_{\alpha_k}$ for some $\alpha_j \neq 1$ and $\alpha_k \neq 1$ of (α_i). Hereafter, we sometimes denote such $(a,b,(\alpha_i))$ by $(a,b,(\alpha_i;j,k))$. By $e(\alpha_i)$ and $f(\beta_j)$, we mean elements e and f of L_{α_i} and R_{β_j}, respectively. Define a product $\circ$ on B as follows:

$$(a,b,(\alpha_i;j,k))\circ(c,d,(\beta_i;m,n))$$
$$= (a\cdot e(\alpha_j\beta_j),f(\alpha_n\beta_n)\cdot d,(\alpha_i\beta_i)).$$

Then it is clear that $B(\circ)$ is a normal band, and we have the following theorem.

THEOREM 2. 3. <u>Let</u> E_i, $i \in I$, <u>be a family of normal bands, and let</u> Γ_i, $i \in I$, <u>be the structure semilattice of</u> E_i. <u>Then</u> $B(\circ)$ <u>is the free product of the</u> E_i <u>in the variety of normal bands. Moreover, the structure semilattice of</u> $B(\circ)$ <u>is the free product of the</u> Γ_i, <u>in the variety of semilattices.</u>

COROLLARY 2. 4. <u>Let</u> L_i, $i \in I$, <u>be a family of left</u>

normal bands whose structure decompositions are $L_i \equiv \Sigma\{ L_{\alpha_i}:$ $\alpha_i \in \Gamma_i \}$. Let $L = \cup \{ L_{\alpha_i} : \alpha_i \in \Gamma_i, i \in I \}$, and let Γ be a semilattice defined above. Set

$$E = \{ (a,(\alpha_i;j)) \in L \times \Gamma: a \in L_{\alpha_j}, \alpha_j \neq 1 \}.$$

Then $E(o)$ is the free product of the L_i in the variety of left normal bands. Moreover, the structure semilattice of $E(o)$ is isomorphic to the free product of the Γ_i, in the variety of semilattices.

COROLLARY 2. 5. Let $\mathscr{A}$ be the variety of [left, right] normal bands. Let E_i, $i \in I$, be a family of [left, right] normal bands, and let B together with the monomorphisms ϕ_i be the free product of E_i, in the variety of [left, right] normal bands. If F_i is a subset of E_i, $i \in I$, then $< \cup \{ F_i\phi_i: i \in I \} >_{\mathscr{A}}$ is isomorphic to the free product of the F_i in the variety of [left, right] normal bands.

3. Strong amalgamation

We shall first show that the variety of left normal bands has the strong amalgamation property. Let L_1 and L_2 be left normal bands with a common subband U. Let the structure decompositions of L_1, L_2 and U be $L_1 \equiv \Sigma\{ L_1^{\alpha}: \alpha \in \Gamma_1 \}$, $L_2 \equiv \Sigma\{ L_2^{\alpha}: \alpha \in \Gamma_2 \}$ and $U \equiv \Sigma\{ U_{\alpha}: \alpha \in \Delta \}$, respectively. We can assume without loss of generality that $L_1 \cap L_2 = U$, $\Gamma_1 \cap \Gamma_2 = \Delta$ and $L_1^{\alpha} \cap L_2^{\alpha} = U_{\alpha}$ for all $\alpha \in \Delta$. Let $L = L_1 \cup L_2$ and $\Gamma = (\Gamma_1^{(1)} \times \Gamma_2^{(1)})\setminus\{(1, 1)\}$. It follows from Corollary 2.4 that

$$E = \{ (a, \alpha, \beta) \in L \times \Gamma : a \in L_1^{\alpha} \cup L_2^{\beta},\ \alpha \in \Gamma_1^{(1)},\ \beta \in \Gamma_2^{(1)} \}$$

is the free product of L_1 and L_2 in the variety of left normal bands, if its product is defined by

$$(a, \alpha, \beta)(b, \gamma, \delta) = \begin{cases} (a\cdot e(\alpha\gamma),\ \alpha\gamma,\ \beta\delta) & \text{if } a \in L_1^{\alpha}, \\ (a\cdot e(\beta\delta),\ \alpha\gamma,\ \beta\delta) & \text{if } a \in L_2^{\beta}, \end{cases}$$

where $e(\alpha)$ denotes an element of L_i^{α}, $i = 1, 2$. Hereafter, $e(1)$ means 1.

We define a relation θ on E as follows:

(3. 1) For elements (a, α, β), (b, γ, δ) of E, define $(a, \alpha, \beta)\ \theta_0\ (b, \gamma, \delta)$ to mean that there exist $\sigma \in \Delta$ and $u \in U_{\sigma}$ such that

$$(a, \alpha, \beta) = (c_1, \xi_1, \eta_1)(u, \sigma, 1)(c_2, \xi_2, \eta_2),$$
$$(b, \gamma, \delta) = (c_1, \xi_1, \eta_1)(u, 1, \sigma)(c_2, \xi_2, \eta_2),$$

for some (c_1, ξ_1, η_1), $(c_2, \xi_2, \eta_2) \in E^1$.

Let $\theta_1 = \theta_0 \cup \theta_0^{-1} \cup \iota$ and let $\theta = \theta_1^t$.

Then of course θ is the congruence on E generated by $\{ ((u, \sigma, 1), (u, 1, \sigma)) : u \in U_{\sigma},\ \sigma \in \Delta \}$. Since any homomorphic image of a left normal band is also a left normal band, E/θ is a left normal bands.

LEMMA 3. 1. *If* $(a, \alpha, 1)\ \theta\ (b, \beta, \gamma)$, *then there exist* $\sigma \in \Delta^1$ *and* $u \in U_{\sigma}$ *such that*

$u\cdot e(\gamma) \in U$ *and* $a = b(u\cdot e(\gamma))$ (*in* L_1) *if* $b \in L_1^{\beta}$,

$bu \in U$ *and* $a = (bu)\cdot e(\beta)$ (*in* L_1) *if* $b \in L_2^{\gamma}$,

where $U_1 = \{1\}$.

Let $\phi_i : L_i \to E/\theta$, $i = 1, 2$, be mappings defined by

$$x\phi_1 = (x, \alpha, 1)\theta \quad \text{if} \quad x \in L_1^{\alpha},$$

$$y\phi_2 = (y, 1, \beta)\theta \quad \text{if} \quad y \in L_2^{\beta}.$$

It is clear that ϕ_1 and ϕ_2 are homomorphisms. Let x and y be elements of L_1^{α} and L_2^{β}, respectively, such that $x\phi_1 = y\phi_2$. Then $(x, \alpha, 1)\ \theta\ (y, \beta, 1)$. By the lemma above, there exist $\sigma, \tau \in \Delta^1$, $u \in U_\sigma$ and $v \in U_\tau$ such that $x = yu$ and $y = xv$. Therefore, $\alpha = \beta$ and $x = xy = (yu)y = y$. Hence ϕ_1 is a monomorphism. Similarly ϕ_2 is a monomorphism. By the definition (3.1), it is clear that $\phi_1|U = \phi_2|U$.

Next, we shall show that $L_1\phi_1 \cap L_2\phi_2 = U\phi_1$. Let x and y be elements of L_1^{α} and L_2^{β}, respectively, such that $x\phi_1 = y\phi_2$. Then $(x, \alpha, 1)\ \theta\ (y, 1, \beta)$. By the lemma above and its dual, there exist $\sigma, \tau \in \Delta^1$, $u \in U_\sigma$ and $v \in U_\tau$ such that

$$yu \in U \quad \text{and} \quad (yu)\cdot 1 = x \quad (\text{in } L_1),$$

$$xv \in U \quad \text{and} \quad (xv)\cdot 1 = y \quad (\text{in } L_2).$$

Then $x, y \in U$ and $x = y$. Thus we have $L_1\phi_1 \cap L_2\phi_2 \subseteq U\phi_1$. It is obvious $L_1\phi_1 \cap L_2\phi_2 \supseteq U\phi_1$. Hence $L_1\phi_1 \cap L_2\phi_2 = U\phi_1$.

It is clear that $< L_1\phi_1 \cup L_2\phi_2 > = E/\theta$ and that E/θ together with ϕ_1 and ϕ_2 is the colimit of L_1 and L_2 malgamating U, and we have the following theorem.

THEOREM 3. 2. <u>We use notations defined above. Then</u> E/θ <u>is the free product of</u> L_1 <u>and</u> L_2 <u>amalgamating</u> U, <u>in the variety of left normal bands. Moreover, the structure semilattice of</u> E/θ <u>is isomorphic to the free product of</u> Γ_1 <u>and</u> Γ_2 <u>amalgamating</u> Δ, <u>in the variety of semilattices.</u>

COROLLARY 3. 3. _The variety of [left, right] normal bands has the strong amalgamation property._

COROLLARY 3. 4. _The class of M[L.N, R.N]-inversive semigroups has the strong amalgamation property. Then the class of regular semigroups defined by a permutation identity has the strong amalgamation property._

The following example, due to T. E. Hall, shows that the variety of left regular bands does not have even the weak amalgamation property.

Example. Let $S = \{ e, f, g, h \}$, $T = \{ f, g, h, x, y \}$ and $U = \{ f, g, h \}$ be a left regular band, a left normal band and a left zero semigroup, respectively, whose multiplications are defined as follows:

	e	f	g	h
e	e	g	g	h
f	f	f	f	f
g	g	g	g	g
h	h	h	h	h

	f	g	h	x	y
f	f	f	f	x	x
g	g	g	g	y	y
h	h	h	h	x	x
x	x	x	x	x	x
y	y	y	y	y	y

Suppose that there exists a semigroup W such that $S \cup T$ can be embedded in W. Then, since W is associative,

$$ex = e(fx) = (ef)x = gx = y,$$

$$ex = e(hx) = (eh)x = hx = x.$$

Thus the elements x and y must coincide in W, a contradiction.

If $\mathcal{A}$ is one of the varieties of rectangular bands, left zero semigroups, right zero semigroups and one element semigroups, it is easy to see that $\mathcal{A}$ has the strong amalgamation property. Then the following theorem follows from [Corollary 1, 4], Corollary 3.3 and the example above.

THEOREM 3. 5. <u>Let $\mathcal{A}$ be the variety of bands defined by an identity</u> $P = Q$. <u>Then $\mathcal{A}$ has the strong amalgamation property if and only if</u> $P = Q$ <u>is a permutation identity or a heterotypical identity</u>.

4. Special amalgamation

We have seen that the variety of left regular bands does not have even the weak amalgamation property. However, we shall show that it has the special amalgamation property. Let $L \equiv \Sigma\{ L_\alpha : \alpha \in \Gamma \}$ be a left regular band and $U \equiv \Sigma\{ U_\alpha : \alpha \in \Delta \}$ a subband. We can assume without loss of generality that $\Gamma \supseteq \Delta$ and $L_\alpha \supseteq U_\alpha$ for all $\alpha \in \Delta$. Let L_1 and L_2 be left regular bands which are isomorphic to L such that $L_1 \cap L_2 = \Box$, and let $\nu_i : L \to L_i$, $i = 1, 2$, be isomorphisms. Let $U_i = U\nu_i$, $L_i^\alpha = L_\alpha \nu_i$ and $U_i^\beta = U_\beta \nu_i$ for all $\alpha \in \Gamma$, $\beta \in \Delta$ and $i = 1, 2$.

Let S be the free product of L_1 and L_2 in the variety of left regular bands. Hereafter, let a_i mean " a_i is an element of L_i ", where $i = 1$ or 2. Define a relation θ on S as follows:

$a_{i_1} a_{i_2} \ldots a_{i_r} \;\theta_0\; b_{j_1} b_{j_2} \ldots b_{j_s}$ if and only if there exist u in U and $c_{k_1} c_{k_2} \ldots c_{k_p}$, $d_{m_1} d_{m_2} \ldots d_{m_q}$ in S^1 such that

$$a_{i_1}a_{i_2}\cdots a_{i_r} = c_{k_1}c_{k_2}\cdots c_{k_p}(u\nu_1)d_{m_1}d_{m_2}\cdots d_{m_q},$$

$$b_{j_1}b_{j_2}\cdots b_{j_s} = c_{k_1}c_{k_2}\cdots c_{k_p}(u\nu_2)d_{m_1}d_{m_2}\cdots d_{m_p}.$$

Let $\theta_1 = \theta_0 \cup \theta_0^{-1} \cup \iota$, and let $\theta = \theta_1^t$.
It is clear that θ is a congruence on S. Then S/θ is a left regular band.

DEFINITION 4. 1. Let a be an element of L. A sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ of elements of $L_1 \cup L_2$ is said to <u>have property</u> $P_i(a)$, $i = 1, 2$, if there exist $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ in U^1 such that

(i) $u_1 = 1$,

(ii) $u_j(x_{i_j}\nu_{i_j}^{-1})v_j \in U$ if $i_j \neq i$,

(iii) $a = \prod_{j=1}^{n} u_j(x_{i_j}\nu_{i_j}^{-1})v_j$.

LEMMA 4. 2. <u>Let</u> a <u>be an element of</u> L_i, $i = 1, 2$. <u>If</u> $a \,\theta\, x_{j_1}x_{j_2}\cdots x_{j_r}$ <u>(in</u> S), <u>then</u> $(x_{j_1}, x_{j_2}, \ldots, x_{j_r})$ <u>has property</u> $P_i(a\nu_i^{-1})$.

Let $\phi_i\colon L_i \to S/\theta$, $i = 1, 2$, be mappings defined by

$$a\phi_i = a\theta \quad \text{for all } a \in L_i.$$

It is obvious that each ϕ_i, $i = 1, 2$, is a homomorphism. Let a and b be elements of L_i satisfying $a\phi_i = b\phi_i$. Then $a\theta = b\theta$. By the lemma above, there exist u and v in U^1 such that

$$a(u\nu_i) = b \quad\text{and}\quad b(v\nu_i) = a.$$

Since L_i is left regular,

$$a = b(v\nu_i) = bb(v\nu_i) = ba = bab$$
$$= a(u\nu_i)ab = a(u\nu_i)b = b,$$

and hence ϕ_i is a monomorphism.

By the definition of θ, it is obvious $\nu_1\phi_1|U = \nu_2\phi_2|U$. Let a and b be elements of L_1 and L_2, respectively, such that $a\phi_1 = b\phi_2$. Then we have $a\theta = b\theta$. By the lemma above, there exist u and v in U^1 such that $(a\nu_1^{-1})u \in U$, $(b\nu_2^{-1})v \in U$, $b\nu_2^{-1} = (a\nu_1^{-1})u$ and $a\nu_1^{-1} = (b\nu_2^{-1})v$. Then we have $a\nu_1^{-1} = b\nu_2^{-1} \in U$, and hence $L_1\phi_1 \cap L_2\phi_2 \subseteq U\nu_1\phi_1$. It is obvious that $L_1\phi_1 \cap L_2\phi_2 \supseteq U\nu_1\phi_1$. Therefore $L_1\phi_1 \cap L_2\phi_2 = U\nu_1\phi_1$, and hence we have the following theorem.

THEOREM 4. 3. *Let U be a subband of a left regular band L. We use the notations defined above. Then S/θ is the free product of L_1 and L_2 amalgamating U in the variety of left regular bands. Thus the variety of left regular bands has the special amalgamation property. Moreover, the structure semilattice of S/θ is isomorphic to the free product of Γ amalgamating Δ in the variety of semilattices.*

COROLLARY 4. 4. *The varieties of [right] regular bands and left [right] quasinormal bands have the special amalgamation property.*

We can not answer whether or not the variety of bands has the special amalgamation property. There is a counterexample in [5] that the variety does not have the special amalgamation property. However, we remark the example is not true. So we raise the following problem.

Probem 1. Does the variety of bands have the special amalgamation property ?

An orthodox semigroup S is called a [<u>left</u>, <u>right</u>] <u>generalized inverse semigroup</u> if $E(S)$ forms a [left, right] normal band. Let S be a left generalized inverse semigroup and U a left generalized inverse subsemigroup. Then S and U are isomorphic to the left quasi-direct products $Q(L \otimes \Gamma;\ \Delta)$ and $Q(V \otimes \Omega;\ \Lambda)$, respectively, for some left normal bands L and V and for some inverse semigroups $\Gamma(\Delta)$ and $\Omega(\Lambda)$. We can assume without loss of generality that $L \supseteq V$, $\Gamma \supseteq \Omega$ and $\Delta \supseteq \Lambda$. Let L_1 and L_2 be left normal bands which are isomorphic to L such that $L_1 \cap L_2 = \square$, and let $\nu_i\colon L \to L_i$, $i = 1, 2$, be isomorphisms. Let Γ_1 and Γ_2 be inverse semigroups which are isomorphic to Γ such that $\Gamma_1 \cap \Gamma_2 = \square$, and let $\pi_i\colon \Gamma \to \Gamma_i$, $i = 1, 2$, be isomorphisms. Let $V_i = V\nu_i$, $\Delta_i = \Delta\pi_i$, $\Omega_i = \Omega\pi_i$ and $\Lambda_i = \Lambda\pi_i$, $i = 1, 2$. Let $L \equiv \Sigma\{\ L(\alpha)\colon \alpha \in \Delta\ \}$ be the structure decomposition of L. Then the structure decomposition of each L_i, $i = 1, 2$, is $L_i \equiv \Sigma\{\ L(\alpha)\nu_i\colon \alpha \in \Delta\ \} \simeq \Sigma\{\ L(\alpha_i)\colon \alpha_i \in \Delta_i\ \}$, where $L(\alpha_i) = L(\alpha_i\pi_i^{-1})\nu_i$. We identify each L_i with $\Sigma\{\ L(\alpha_i)\colon \alpha_i \in \Delta_i\ \}$.

Let Γ^* be the free product of Γ_1 and Γ_2 in the class of semigroups, and let $L^* = L_1 \cup L_2$. For any element $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n}$ in Γ^*, $\alpha_{i_j} \in \Gamma_{i_j}$, let us denote $(\alpha_{i_1}\pi_{i_1}^{-1})(\alpha_{i_2}\pi_{i_2}^{-1})\cdots(\alpha_{i_n}\pi_{i_n}^{-1})$ by $\overline{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n}}$. Let $T = \{\ (a,\ \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n}) \in L^* \times \Gamma^*\colon a \in L_{i_1}$ and $a\nu_{i_1}^{-1} \in L(\overline{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n}\alpha_{i_n}^{-1}\cdots\alpha_{i_2}^{-1}\alpha_{i_1}^{-1}})\ \}$. Define a product on T as follows:

$$(a,\ \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n})(b,\ \beta_{j_1}\beta_{j_2}\cdots\beta_{j_m})$$
$$= (ac,\ \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n}\beta_{j_1}\beta_{j_2}\cdots\beta_{j_m}),$$

where c is an element of L_{i_1} such that $c\nu_{i_1}^{-1}$ is contained in $L(\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n}\beta_{j_1}\beta_{j_2}\cdots\beta_{j_m}\beta_{j_m}^{-1}\cdots\beta_{j_2}^{-1}\beta_{j_1}^{-1}\alpha_{i_n}^{-1}\cdots\alpha_{i_2}^{-1}\alpha_{i_1}^{-1})$.

Let θ be a relation on T defined by

$(a, \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n})\ \theta\ (b, \beta_{j_1}\beta_{j_2}\cdots\beta_{j_m})$ if and only if $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n} \sim \beta_{j_1}\beta_{j_2}\cdots\beta_{j_m}$ and one of the following conditions:

(i) $a = b$ if $i_1 = j_1$,

(ii) there exists $v \in V$ such that

$a\nu_{i_1}^{-1} = b\nu_{j_1}^{-1} = v(\nu_{i_1}^{-1})$ if $i_1 \neq j_1$,

where $\sim$ is the congruence on Γ^* such that $\Gamma^*/\sim$ is the free product of Γ_1 and Γ_2 amalgamating Δ in the class of inverse semigroups.

We can easily see that θ is a congruence on T, and hence T/θ is a left generalized inverse semigroups. Let $(a, \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n})\theta$ denote the θ-class containing $(a, \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n})$.

LEMMA 4. 5. _Let_ $\phi_i: S_i \to T/\theta$, $i = 1, 2$, _be mappings defined by_

$(a, \alpha_i)\phi_i = (a, \alpha_i)\theta$ _where_ $\alpha_i \in \Gamma_i$, $a \in L(\alpha_i\alpha_i^{-1})$.

Then ϕ_i, $i = 1, 2$, _are monomorphisms satisfying the following two conditions_:

$(\nu_1, \pi_1)\phi_1|U = (\nu_2, \pi_2)\phi_2|U$,

$S_1\phi_1 \cap S_2\phi_2 = U(\nu_1, \pi_1)\phi_1$ $(= U(\nu_2, \pi_2)\phi_2)$,

where (ν_i, π_i), $i = 1, 2$, _are isomorphisms of_ S _onto_ S_i _defined by_ $(v, \alpha)(\nu_i, \pi_i) = (v\nu_i, \alpha\pi_i)$ _for_ $(v, \alpha) \in S$.

Moreover, the structure inverse semigroup of T/θ *is the free product* $\Gamma^*/\sim$ *of* Γ_1 *and* Γ_2 *amalgamating* Δ *in the class of inverse semigroups.*

THEOREM 4. 6. *The class of [left, right] generalized inverse semigroups has the special amalgamation property.*

It is natural to raise the following problem.

Problem 2. Does the class of [left, right] generalized inverse semigroups have the strong amalgamation property ?

REFERENCES

[1] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys, No. 7, Amer. Math. Soc., Providence, R. I., Vol. I (1961), Vol. II (1967).

[2] G. Gratzer and H. Lakster, *The structure of pseudocomplemented distributive lattices II: congruence extension and amalgamation*, Trans. Amer. Math. Soc. 156(1971), 343 - 357.

[3] T. E. Hall, *Free products with amalgamation of inverse semigroups*, J. Algebra 34(1975), 375 - 385.

[4] T. Imaoka, *Free products with amalgamation of commutative inverse semigroups*, J. Austral. Math. Soc. 22(1976), 246 - 251.

[5] J. R. Isbell, *Epimorphisms and dominions*, Proceedings of the conference on categolical algebras, La Jolla 1965, Springer, 1966, 232 - 246.

[6] Bjarni Jonsson, Extensions of relational structures, Theory of models, Proceedings of 1963 international sympodium at Berkely, edited by J. W. Addision, L. Henkin and A. Tarski, North Holland, Amsterdam, 1965, 146 - 157.

[7] G. B. Preston, Inverse semigroups: some open questions, Proceedings of a symposium on inverse semigroups and their generalizations, Northern Illinois University, 1973, 122 - 139.

[8] M. Yamada, The structure of separative bands, Doctoral dissertation, University of Utah, 1962.

[9] M. Yamada and N. Kimura, Note on idempotent semigroups II, Proc. Japan Acad. 34(1958), 110 - 112.

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