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Mathematical Programming Problems on an Infinite Network

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§ 1. Introduction and network definitions

In the present paper, we shall study the following mathematical programming problems on an infinite network: (1) Min-work problem, (2) Max-potential problem, (3) Max-flow problem, (4) Min-cut problem, (5) Extremal distance, (6) Extremal width. We shall discuss some duality relations of those problems. In relation to those problems, we shall classify the set of all infinite networks into parabolic networks and hyperbolic networks of order p and define a parabolic index of an infinite network.

Most of the results in this paper are extracted from [6], [7], [9] and [10].

We begin with some network definitions.

Let X and Y be countable (infinite) sets and K be a function on $X \times Y$ satisfying the following conditions:

(N. 1) The range of K is $\{-1, 0, 1\}$.

(N. 2) For each $y \in Y$, $e(y) = \{x \in X; K(x, y) \neq 0\}$ consists of exactly two points x_1, x_2 and $K(x_1, y)K(x_2, y) = -1$.

(N. 3) For each $x \in X$, $Y(x) = \{y \in Y; K(x, y) \neq 0\}$ is a nonempty finite set.

(N. 4) For any $x, x' \in X$, there are $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_{n+1} \in Y$ such that $e(y_j) = \{x_{j-1}, x_j\}$, $j = 1, \ldots, n + 1$ with $x_0 = x$ and $x_{n+1} = x'$.

Let r be a strictly positive function on Y. Then $N = \{X, Y, K, r\}$ is called an infinite network.

Let X' and Y' be subsets of X and Y respectively and let K' and r' be the restrictions of K and r onto $X' \times Y'$ and Y' respectively. Then $N' = \{X', Y', K', r'\}$ is called a subnetwork of the network N if conditions (N. 2) - (N. 4) are fulfilled replacing X, Y and K by X', Y' and K' respectively. Let us put for simplicity $\langle X', Y' \rangle = N'$. In case X' (or Y') is a finite set, $\langle X', Y' \rangle$ is a finite subnetwork.

A sequence $\{\langle X_n, Y_n \rangle\}$ of finite subnetworks of N is called an exhaustion of N if $X = \bigcup_{n=1}^{\infty} X_n$, $Y = \bigcup_{n=1}^{\infty} Y_n$ and $Y(x) \subset Y_{n+1}$ for all $x \in X_n$.

A path P from $x \in X$ to $x' \in X$ is the triple $(C_X(P), C_Y(P), p)$ of a finite ordered set $C_X(P) = \{x_0, x_1, \ldots, x_{n+1}\}$ of X, a finite ordered set $C_Y(P) = \{y_1, y_2, \ldots, y_{n+1}\}$ of Y and a function p on Y called the index of P such that

$$\text{(P)} \qquad \begin{aligned} &x_0 = x,\ x_{n+1} = x',\ x_i \neq x_k \text{ if } i \neq k,\ e(y_j) = \{x_{j-1}, x_j\}, \\ &p(y_j) = -K(x_{j-1}, y_j) \text{ and } p(y) = 0 \text{ if } y \notin C_Y(P). \end{aligned}$$

A path P from $x \in X$ to the ideal boundary ∞ of N is the triple $(C_X(P), C_Y(P), p)$ of an infinite ordered set $C_X(P) = \{x_0, x_1, \ldots\}$ of X, an infinite ordered set $C_Y(P) = \{y_1, y_2, \ldots\}$ of Y and a function p on Y which satisfies condition (P) except the terminal condition $x_{n+1} = x'$.

Denote by $P_{x,x'}$ (resp. $P_{x,\infty}$) the set of all paths from x to x' (resp. ∞). Note that condition (N. 4) means $P_{x,x'} \neq \emptyset$ for any x, x' $\in$ X. For mutually disjoint nonempty subsets A and B of X, denote by $P_{A,B}$ the set of all paths P such that $P \in P_{x,x'}$, $C_X(P) \cap A = \{x\}$ and $C_X(P) \cap B = \{x'\}$ for some $x \in A$ and $x' \in B$. Let A be a nonempty finite subset of X and let $P_{A,\infty}$ be the set of all paths P such that $P \in P_{x,\infty}$ and $C_X(P) \cap A = \{x\}$ for some $x \in A$.

Let A and B be mutually disjoint nonempty subsets of X. We say that a subset Q of Y is a cut between A and B if there exist mutually disjoint subsets Q(A) and Q(B) of X such that $A \subset Q(A)$, $B \subset Q(B)$, $X = Q(A) \cup Q(B)$ and the set

$$Q(A) \ominus Q(B) = \{y \in Y;\ e(y) \cap Q(A) \neq \emptyset \text{ and } e(y) \cap Q(B) \neq \emptyset\}$$

is equal to Q.

Let A be a nonempty finite subset of X. We say that a subset Q of Y is a cut between A and the ideal boundary ∞ of N if there exist mutually disjoint subsets Q(A) and Q(∞) of X such that $A \subset Q(A)$, $X = Q(A) \cup Q(\infty)$, Q(A) is a finite set and $Q = Q(A) \ominus Q(\infty)$. Denote by $Q_{A,B}$ (resp. $Q_{A,\infty}$) the set of all cuts between A and B (resp. ∞) and put $Q_{A,B}^{(f)} = \{Q \in Q_{A,B};\ Q \text{ is a finite set}\}$.

Let L(X) and L(Y) be the sets of all real functions on X and Y respectively, let $L_0(X)$ and $L_0(Y)$ be the subsets of L(X) and L(Y) respectively which consist of functions with finite support and let $L^+(Y)$ be the subset of L(Y) which consists of non-negative functions.

§ 2. Min-work problem and max-potential problem

Let $c \in L^+(Y)$ and let A and B be mutually disjoint nonempty subsets of X. Let us consider the following mathematical programming problems on N:

(MP. 1) (Min-work problem) Find

$$N(A, B; c) = \inf\{\sum_P c(y);\ P \in P_{A,B}\}.$$

(MP. 2) (Max-potential problem) Find

$$N^*(A, B; c) = \sup\{\inf_{x \in B} u(x) - \sup_{x \in A} u(x);\ u \in S^*\},$$

where $S^* = \{u \in L(X);\ |\sum_{x \in X} K(x, y)u(x)| \leqq c(y) \text{ on } Y\}$.

We proved in [9]

Theorem 1. $N(A, B; c) = N^*(A, B; c)$ holds and there exists an optimal solution u of (MP. 2) such that $u = 0$ on A.

Remark 1. There is no optimal solution of (MP. 1) in general.

§ 3. Max-flow problem and min-cut problem

Let A and B be mutually disjoint nonempty finite subsets of X. We say that $w \in L(Y)$ is a flow from A to B of strength I(w) if

$$\sum_{y \in Y} K(x, y)w(y) = 0 \quad \text{for } x \in X - A - B,$$

$$I(w) = -\sum_{x \in A} \sum_{y \in Y} K(x, y)w(y) = \sum_{x \in B} \sum_{y \in Y} K(x, y)w(y).$$

Denote by F(A, B) the set of all flows from A to B and put $G(A, B) = F(A, B) \cap L_0(Y)$. The spaces of flows on an infinite network have been analyzed by H. Flanders [2] and A. H. Zemanian [11].

Let $W \in L^+(Y)$ and consider the following mathematical programming problems:

(MP. 3) (Max-flow problem) Find

$$M(W; G(A, B)) = \sup\{I(w); w \in G(A, B) \text{ and } |w| \leqq W \text{ on } Y\}.$$

(MP. 4) (Min-cut problem) Find

$$M^*(W; Q_{A,B}) = \inf\{\sum_Q W(y); Q \in Q_{A,B}\}.$$

We can define $M(W; F(A, B))$ and $M^*(W; Q_{A,B}^{(f)})$ similarly.

We proved in [9]

Theorem 2. $M(W; G(A, B)) = M^*(W; Q_{A,B})$ holds and there exists an optimal solution of (MP. 4).

Proof. We only prove the inequality $M(W; G(A, B)) \geqq M^*(W; Q_{A,B})$. Let $\{<X_n, Y_n>\}$ be an exhaustion of N such that $A \cup B \subset X_1$ and define $W_n \in L(Y)$ by $W_n = W$ on Y_n and $W_n = 0$ on $Y - Y_n$. By a well-known result which states that max-flow equals min-cut in a finite network (cf. [3]), we have

$$M(W; G(A, B)) \geqq M(W_n; G(A, B)) = M^*(W_n; Q_{A,B}).$$

Since $\underline{\lim}_{n\to\infty} M^*(W_n; Q_{A,B}) \geqq M^*(W; Q_{A,B})$ (cf. [9]), we obtain the inequality.

Notice that $M(W; G(A, B)) \leqq M(W; F(A, B))$ and $M^*(W; Q_{A,B}) \leqq M^*(W; Q_{A,B}^{(f)})$ and the equalities do not hold in general. To give a sufficient condition for the equalities, we consider the value $M^*(W; Q_{F,\infty})$ of a mathematical programming problem similar to (MP. 4).

Definition 1. We say that $W \in L^+(Y)$ satisfies condition (∞)

if $M^*(W; Q_{F,\infty}) = 0$ for all nonempty finite subsets F of X.

The following two theorems were proved in [7].

Theorem 3. Let $W \in L^+(Y)$. Then W satisfies condition (∞) if and only if there exists an exhaustion $\{<X_n, Y_n>\}$ of N such that $\lim_{n\to\infty} \sum_{Z_n} W(y) = 0$ with $Z_n = Y_n - Y_{n-1}$ $(Y_0 = \emptyset)$.

Theorem 4. If $W \in L^+(Y)$ satisfies condition (∞), then $M^*(W; Q_{A,B}) = M^*(W; Q^{(f)}_{A,B}) = M(W; F(A, B))$.

Let A be a nonempty finite subset of X. We say that $w \in L(Y)$ is a flow from A to the ideal boundary ∞ of N of strength I(w) if

$$\sum_{y\in Y} K(x, y)w(y) = 0 \text{ for } x \in X - A,$$

$$I(w) = -\sum_{x\in A} \sum_{y\in Y} K(x, y)w(y).$$

Denote by $F(A, \infty)$ the set of all flows from A to ∞. We can define $M(W; F(A, \infty))$ similarly to (MP. 3).

We shall prove

Theorem 5. $M(W; F(A, \infty)) = M^*(W; Q_{A,\infty})$.

Proof. Let $w \in F(A, \infty)$ such that $|w| \leqq W$ on Y and let $Q = Q(A) \ominus Q(\infty) \in Q_{A,\infty}$. Define $u \in L(X)$ by $u = 1$ on Q(A) and $u = 0$ on $Q(\infty)$. Then $u \in L_0(X)$ and

$$I(w) = -\sum_{x\in X} u(x) \sum_{y\in Y} K(x, y)w(y) = -\sum_{y\in Y} w(y) \sum_{x\in X} K(x, y)u(x)$$

$$\leqq \sum_{y\in Y} |w(y)| |\sum_{x\in X} K(x, y)u(x)| \leqq \sum_{Q} W(y),$$

so that $M(W; F(A, \infty)) \leqq M^*(W; Q_{A,\infty})$. Let $\{<X_n, Y_n>\}$ be an exhaustion of N such that $A \subset X_1$. Then $M(W; G(A, X_{n+1} - X_n)) = M^*(W; Q_{A,X_{n+1}-X_n})$

$\geqq M^*(W; Q_{A,\infty})$ by Theorem 2. There exists $w_n \in F(A, X_{n+1} - X_n)$ such that $|w_n| \leqq W$ on Y and $I(w_n) = M(W; G(A, X_{n+1} - X_n))$. We may assume that $w_n(y)$ converges to $\hat{w}(y)$ as $n \to \infty$ for each $y \in Y$. It follows that $\hat{w} \in F(A, \infty)$, $|\hat{w}| \leqq W$ on Y and $M^*(W; Q_{A,\infty}) \leqq \lim_{n\to\infty} M(W; G(A, X_{n+1} - X_n))$ $= \lim_{n\to\infty} I(w_n) = I(\hat{w}) \leqq M(W; F(A, \infty))$, and hence $M(W; F(A, \infty)) = M^*(W; Q_{A,\infty})$.

§ 4. Path-cut inequalities

We shall improve the path-cut inequalities in [9]. Let $V, W \in L^+(Y)$ and let A and B be mutually disjoint nonempty finite subsets of X.

We shall prove

Theorem 6. $N(A, B; V)M^*(W; Q_{A,B}) \leqq \sum_{y \in Y} V(y)W(y)$,

or equivalently

$$(\inf\{\sum_P V(y);\ P \in P_{A,B}\})(\inf\{\sum_Q W(y);\ Q \in Q_{A,B}\}) \leqq \sum_{y\in Y} V(y)W(y).$$

Proof. On account of Theorem 2, it suffices to show that $N(A, B; V)M(W; G(A, B)) \leqq \sum_{y\in Y} V(y)W(y)$. By means of Theorem 1, we can find $v \in L(X)$ such that $v = 0$ on A, $v = N(A, B; V)$ on B and $|\sum_{x\in X} K(x, y)v(x)| \leqq V(y)$ on Y. For any $w \in G(A, B)$ such that $|w(y)| \leqq W(y)$ on Y, we have

$$N(A, B; V)I(w) = \sum_{x\in B} v(x) \sum_{y\in Y} K(x, y)w(y)$$

$$= \sum_{y\in Y} w(y) \sum_{x\in X} K(x, y)v(x) \leqq \sum_{y\in Y} W(y)V(y),$$

which leads to the desired inequality.

By Theorem 4, we can similarly prove

Theorem 7. If W satisfies condition (∞), then

$$N(A, B; V)M^*(W; Q^{(f)}_{A,B}) \leqq \sum_{y \in Y} V(y)W(y).$$

Let A be a nonempty finite subset of X and let $\{\langle X_n, Y_n\rangle\}$ be an exhaustion of N such that $A \subset X_1$. Then $N(A, X_{n+1} - X_n; V) = N(A, X - X_n; V) \to N(A, \infty; V)$ as $n \to \infty$ (Lemma 2. 4 in [6]) and $M^*(W; Q_{A,\infty}) \leqq M^*(W; Q_{A,X_{n+1}-X_n})$ (cf. Lemma 5. 3 in [6]). We have by Theorem 6 with the understanding that $0 \cdot \infty = 0$

Theorem 8. $N(A, \infty; V)M^*(W; Q_{A,\infty}) \leqq \sum_{y \in Y} V(y)W(y)$.

§ 5. Extremal distance and extremal width

Hereafter let $1 < p \leqq \infty$ and $1/p + 1/q = 1$ $(1 < p < \infty)$. For $w \in L(Y)$, its energy $H_p(w)$ of order p is defined by

$$H_p(w) = \sum_{y \in Y} r(y)|w(y)|^p \quad (1 < p < \infty),$$

$$H_\infty(w) = \sup_{y \in Y} |w(y)|.$$

Denote by $L_p(Y; r)$ the set of all $w \in L(Y)$ such that $H_p(w) < \infty$ and by $L^+_p(Y; r)$ the subset of $L_p(Y; r)$ which consists of non-negative functions. For $u \in L(X)$, its Dirichlet integral $D_p(u)$ of order p is defined by

$$D_p(u) = H_p(r(y)^{-1} \sum_{x \in X} K(x, y)u(x)).$$

Denote by $D^{(p)}(N)$ the set of all $u \in L(X)$ such that $D_p(u) < \infty$.

Let A and B be mutually disjoint nonempty subsets of X and consider the following mathematical programming problems:

(MP. 5) Find $EL_p(A, B)^{-1} = \inf\{H_p(W); W \in E_p(P_{A,B})\}$,

where $E_p(P_{A,B}) = \{W \in L_p^+(Y; r); \sum_P r(y)W(y) \geqq 1$ for all $P \in P_{A,B}\}$.

(MP. 6) Find $EW_q(A, B)^{-1} = \inf\{H_q(W); W \in E_q^*(Q_{A,B})\}$,

where $E_q^*(Q_{A,B}) = \{W \in L_q^+(Y, r); \sum_Q W(y) \geqq 1$ for all $Q \in Q_{A,B}\}$.

Here we use the convention that the infimum of a real function on the empty set $\emptyset$ is equal to ∞.

Notice that $E_p(P_{A,B}) = \{W \in L_p^+(Y, r); N(A, B; W) \geqq 1\}$ and $E_q^*(Q_{A,B}) = \{W \in L_q^+(Y, r); M^*(W; Q_{A,B}) \geqq 1\}$.

In case A is a nonempty finite subset of X, $EL_p(A, \infty)$ and $EW_q(A, \infty)$ are defined as above replacing $P_{A,B}$ and $Q_{A,B}$ by $P_{A,\infty}$ and $Q_{A,\infty}$ respectively.

We call $EL_p(A, B)$ (resp. $EL_p(A, \infty)$) the extremal distance of order p of N relative to A and B (resp. A and ∞) and $EW_q(A, B)$ (resp. $EW_q(A, \infty)$) the extremal width of order q of N relative to A and B (resp. A and ∞).

We proved in [8]

Theorem 9. Let A and B be mutually disjoint nonempty subsets of X and let $\{<X_n, Y_n>\}$ be an exhaustion of N such that $A \cap X_1 \neq \emptyset$ and $B \cap X_1 \neq \emptyset$ and put $A_n = A \cap X_n$ and $B_n = B \cap X_n$. Denote by $EL_p(A_n, B_n; N_n)$ and $EW_q(A_n, B_n; N_n)$ the values of (MP. 5) and (MP. 6) replacing A, B and N by A_n, B_n and $N_n = <X_n, Y_n>$. Then $EL_p(A_n, B_n; N_n) \to EL_p(A, B)$ and $EW_q(A_n, B_n; N_n) \to EW_q(A, B)$ as $n \to \infty$.

The following three theorems were proved in [6].

Theorem 10. Let A be a nonempty finite subset of X and let $\{<X_n, Y_n>\}$ be an exhaustion of N such that $A \subset X_1$. Then $EL_p(A, X - X_n) \to EL_p(A, \infty)$ and $EW_q(A, X - X_n) \to EW_q(A, \infty)$ as $n \to \infty$.

By the aid of Theorem 1, we have

Theorem 11. $EL_p(A, B)^{-1} = \inf\{D_p(u); u \in L(X), u = 0$ on A and $u = 1$ on $B\}$.

By the aid of Theorem 2, we have

Theorem 12. Let A and B be mutually disjoint nonempty finite subsets of X. Then

$$EW_q(A, B)^{-1} = \inf\{H_q(w); w \in G(A, B) \text{ and } I(w) = 1\}.$$

We have

Theorem 13. Let A and B be mutuallly disjoint nonempty subsets of X. Then $[EL_p(A, B)]^{1/p}[EW_q(A, B)]^{1/q} = 1$.

Proof. We proved this theorem in [6] in the case where A and B are finite sets. Our assertion follows from Theorem 9.

Remark 2. R. J. Duffin [1] proved Theorem 13 in the case where $p = 2$ and N is a finite network.

Remark 3. Even in the case where N is not locally finite, Theorem 13 also holds (cf. [8]). (MP. 5) and (MP. 6) can be defined even in the case where $p = 1$ and $q = 1, \infty$ respectively. We have $EL_1(A, B) = EW_\infty(A, B)^{-1}$ and $EL_\infty(A, B) = EW_1(A, B)^{-1}$ (cf. [8]).

By Theorems 10 and 13, we have

Theorem 14. Let A be a nonempty finite subset of X. Then

$EL_p(A, \infty) = [EW_q(A, \infty)]^{1-p}$.

Remark 4. In the case where N is not locally finite, Theorems 10 and 14 do not hold in general. We proved in [8] that they hold if N is p-almost locally finite, i.e., $\sum_{y \in Y(x)} r(y)^{1-p} < \infty$ for all $x \in X$ $(1 < p < \infty)$.

We shall prove

Theorem 15. $EW_q(A, \infty)^{-1} = \inf\{H_q(w);\ w \in F(A, \infty) \text{ and } I(w) = 1\}$.

Proof. Put $EW_q = EW_q(A, \infty)$ and $d_q^* = \inf\{H_q(w);\ w \in F(A, \infty)$ and $I(w) = 1\}$. To prove the inequality $EW_q^{-1} \leqq d_q^*$, we may assume that $d_q^* < \infty$, i.e., there is $w \in F(A, \infty)$ such that $I(w) = 1$ and $H_q(w) < \infty$. Then $W = |w| \in E_q^*(Q_{A,\infty})$ and $EW_q^{-1} \leqq H_q(W) = H_q(w)$, so that $EW_q^{-1} \leqq d_q^*$. To prove the converse inequality, we may suppose that $EW_q^{-1} < \infty$, i.e., there is $W \in L_q^+(Y; r)$ such that $M^*(W; Q_{A,\infty}) \geqq 1$. On account of Theorem 5, we can find $w \in F(A, \infty)$ such that $|w| \leqq W$ on Y and $I(w) \geqq 1$. Writing $w' = w/I(w)$, we have $w' \in F(A, \infty)$, $I(w') = 1$ and $d_q^* \leqq H_q(w') \leqq H_q(w) \leqq H_q(W)$. Thus $d_q^* \leqq EW_q^{-1}$.

For a nonempty finite subset A of X, let us consider the following mathematical programming problem:

(MP. 7) Find $d_p(A, \infty) = \inf\{D_p(u);\ u \in L_0(X) \text{ and } u = 1 \text{ on } A\}$.

By Theorems 10 and 11, we have

Theorem 16. $d_p(A, \infty) = EL_p(A, \infty)^{-1}$.

§ 6. Parabolic and hyperbolic infinite networks

Let A and A' be nonempty finite subsets of X. Then $d_p(A, \infty) = 0$

if and only if $d_p(A', \infty) = 0$ (cf. [10]). Thus we can classify the set of all infinite networks as follows:

Definition 2. We say that an infinite network $N = \{X, Y, K, r\}$ is of parabolic type of order p if there exists a nonempty finite subset A of X such that $d_p(A, \infty) = 0$. We say that N is of hyperbolic type of order p if it is not of parabolic type of order p.

For a fixed $x_0 \in X$, we define $\|u\|_p$ by

$$\|u\|_p = [D_p(u) + |u(x_0)|^p]^{1/p} \qquad (1 < p < \infty),$$

$$\|u\|_\infty = D_\infty(u) + |u(x_0)|.$$

Lemma 1. For every finite subset F of X, there exists a constant M(F) such that

$$\sum_{x \in F} |u(x)| \leqq M(F)\|u\|_p$$

for all $u \in D^{(p)}(N)$.

We can prove by Lemma 1 and a standard argument that $D^{(p)}(N)$ is a Banach space with respect to the norm $\|u\|_p$. Denote by $D_0^{(p)}(N)$ the closure of $L_0(X)$ in $D^{(p)}(N)$ with respect to the norm. Notice that $D_0^{(p)}(N)$ does not depend on the choice of x_0.

By Theorems 14, 15 and 16 and Theorem 3.2 in [10], we have

Theorem 17. Let $1 < p < \infty$ and let A be a nonempty finite subset of X. An infinite network N is of parabolic type of order p if and only if any one of the following conditions is fulfilled:

(C. 1) $1 \in D_0^{(p)}(N)$.

(C. 2) $D_0^{(p)}(N) = D^{(p)}(N)$.

(C. 3) $EL_p(A, \infty) = \infty$.

(C. 4) $EW_q(A, \infty) = 0$, i.e., $E_q^*(Q_{A,\infty}) = \emptyset$.

(C. 5) There is no $w \in F(A, \infty)$ such that $I(w) = 1$ and $H_q(w) < \infty$.

Corollary 1. Let $1 < p < \infty$. If there exists an exhaustion $\{<X_n, Y_n>\}$ of N such that

$$\sum_{n=1}^{\infty} [\mu_n^{(p)}]^{1-q} = \infty \text{ with } \mu_n^{(p)} = \sum_{Y_n - Y_{n-1}} r(y)^{1-p},$$

then N is of parabolic type of order p.

Corollary 2. Assume that N is of parabolic type of order p. If $W \in L_q^+(Y; r)$, then W satisfies condition (∞).

Remark 5. For $u \in L(X)$, its p-Laplacian $\beta_p(u) \in L(X)$ is defined by

$$[\beta_p(u)](x) = - \sum_{y \in Y} K(x, y) g_p(r(y)^{-1} \sum_{z \in X} K(z, y) u(z)),$$

where $g_p(t) = |t|^{p-1}\text{sign}(t)$ $(t \in R)$. We say that $u \in L(X)$ is p-superharmonic on X if $\beta_p(u) \leqq 0$ on X. Denote by $SH^+(N)$ the set of all non-negative functions on X which are p-superharmonic on X. An infinite network N is of parabolic type of order p $(1 < p < \infty)$ if and only if $SH^+(N)$ consists only of constant functions (cf. [5]). In case $p = 2$, this is a discrete analogy of a well-known result in the classification theory of Riemann surfaces.

We proved in [10]

Theorem 18. An infinite network N is of parabolic type of order ∞ if and only if there exists a nonempty finite subset A of X such that $\sum_P r(y) = \infty$ for all $P \in P_{A,\infty}$.

Corollary. If N is of hyperbolic type of order ∞, then it is of hyperbolic type of order p for all $p > 1$.

§ 7. Parabolic index of an infinite network

We proved in [10]

Theorem 19. Let $1 < p_1 < p_2$. If N is of hyperbolic type of order p_2, then N is of hyperbolic type of order p_1.

By this fact, we can define a parabolic index ind N of an infinite network N which is of parabolic type of order ∞:

$$\text{ind } N = \inf\{p > 1;\ N \text{ is of parabolic type of order } p\}.$$

A geometric meaning of ind N may be seen by the following examples:

Example 1. Let $\{t_n\}$ be a sequence of positive integers and denote by J the set of all positive integers. Let us take

$$X = \{x_n;\ n \in J\},\ Y = \{y_1^{(n)}, y_2^{(n)}, \ldots, y_{t_n}^{(n)};\ n \in J\},$$

$$K(x_n, y_1^{(n)}) = \ldots = K(x_n, y_{t_n}^{(n)}) = -1 \quad \text{for } n \in J,$$

$$K(x_{n+1}, y_1^{(n)}) = \ldots = K(x_{n+1}, y_{t_n}^{(n)}) = 1 \quad \text{for } n \in J,$$

$$K(x, y) = 0 \quad \text{for any other pair } (x, y).$$

Let $r = 1$ on Y. Then $N = \{X, Y, K, r\}$ is an infinite network. Let α be a non-negative number and let t_n be the greatest integer less than or equal to n^α. Then we have ind $N = \alpha + 1$. In case $t_n = 2^n$, ind $N = \infty$.

Example 2. Let $X = \bigcup_{n=0}^{\infty} C_n$ and $Y = \bigcup_{n=1}^{\infty} Z_n$, where $C_n =$

$\{x_i^{(n)};\ i = 1, 2, \ldots, 2^n\}$ and $Z_n = \{y_i^{(n)};\ i = 1, 2, \ldots, 2^n\}$.
For each $n \in J$, we define

$$K(x_i^{(n)}, y_i^{(n)}) = 1 \text{ for } i = 1, 2, \ldots, 2^n,$$

$$K(x_i^{(n-1)}, y_i^{(n)}) = K(x_i^{(n-1)}, y_{2^{n-1}+i}^{(n)}) = -1 \text{ for } i = 1, 2, \ldots, 2^{n-1}.$$

For any other pair (x, y), we set $K(x, y) = 0$. Let $\{r_n;\ n \in J\}$ be a set of positive numbers and define $r \in L(Y)$ by $r(y) = r_n$ on Z_n for each $n \in J$. Then $N = \{X, Y, K, r\}$ is an infinite network which may be called a binary tree stemmed from $x_1^{(0)}$. It is shown that N is of parabolic type of order p $(1 < p < \infty)$ if and only if $\sum_{n=1}^{\infty} 2^{n(1-q)} r_n = \infty$. Thus we can calculate ind N for several choices of $\{r_n;\ n \in J\}$. In case $r_n = 1$ for $n \in J$, ind $N = \infty$. In case $r_n = 2^{n/\alpha}$ $(\alpha > 0)$ for $n \in J$, ind $N = \alpha + 1$ and N is of parabolic type of order ind N. In case $r_n = n^{-2} 2^{n/\alpha}$ $(\alpha > 0)$ for $n \in J$, ind $N = \alpha + 1$ and N is of hyperbolic type of order ind N. In case $r_n = 2^{n^2}$ for $n \in J$, ind $N = 1$.

Remark 6. F-Y. Maeda [4] proved that the infinite network formed by the lattice points and the segments parallel to coordinate axes in the d-dimensional euclidean space has parabolic index equal to the dimension. This result implies that the dimension of a general infinite network may be defined by its parabolic index.

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