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A system of differential equations with Kamke-type condition.

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1. Introduction. We shall consider the n-dimensional system

$$(1)\quad \frac{d}{dt} p_i = \lambda_i E_i(t, p) \qquad \text{for} \quad 1 \leqq i \leqq n ,$$

$$p = (p_1, p_2 \cdots, p_n) ,$$

where λ_i are positive constants and $E(t, p) = (E_i(t, p))$ is continuous on $-\infty < t < \infty$ and $p_i > 0$ $(1 \leqq i \leqq n)$. We set $P = \{p = (p_i) : p_i > 0 \text{ for } 1 \leqq i \leqq n\}$ and for $p \in P$, $|p| = \sum_{i=1}^{n} |p_i| = \sum_{i=1}^{n} p_i$. We assume that

(i) $E(t, p)$ is periodic in t, that is, there is a constant $\omega > 0$ such that $E(t+\omega, p) = E(t, p)$,

(ii) $E(t, p)$ satisfies Lipshitz condition in p such that for any compact set K in P, there is a constant $L = L(K) > 0$ such that

$$|E(t, p) - E(t, q)| \leqq L|p - q| \quad \text{for} \quad p, q \in K, \ t \in R .$$

(iii) $E(t, p)$ satisfies Kamke-type condition in p, that is, for each $i = 1, 2, \cdots, n$, we have

$$E_i(t, p) \leqq E_i(t, q) .$$

for any two points $p = (p_1, \cdots, p_n)$, $q = (q_1, \cdots, q_n)$ in P with $p_i = q_i$ and $p_j \leqq q_j$ $(j = 1, \cdots, n, i \neq j)$.

$$\text{(iv)} \quad \sum_{i=1}^{n} p_i E_i(t, p) = 0 .$$

For example we have

$$E_i(t, p) = \sum_{j=1}^{n} a_{ij}(t)\, p_j \,/\, p_i ,$$

where $a_{ij}(t) \geqq 0$ $(i \neq j)$, $\sum_{i=1}^{n} a_{ij}(t) = 0$ $(1 \leqq j \leqq n)$ and $a_{ij}(t+\omega) = a_{ij}(t)$ $(1 \leqq i, j \leqq n)$.

The system (1) is a genelalized form of dynamics in an economy. In the classical dynamics, the system (1) is autonomous, that is, $E(t, p) = E(p)$ and represents the law of supply and demand in an economy. Namely we assume that an economy is divided into n industries producing one good each. $p_i(t)$ denotes the price of i-th good at time t and is always assumed to be positive, that is,

$$p_i(t) > 0 \qquad \text{for} \qquad 1 \leqq i \leqq n .$$

$E_i(p)$ is the excess demand (demand - supply) for i-th good under price p. If $E_i(p)$ is positive (or negative), the price of i-th good is increased (or decreased).

Our concern in this topics is with the case where $E(t, p)$ depends on time t and is periodic in t. Such a genelalization seems a natural one considering the seasonal effects of an economy and moreover our method of the proof seems to improve classical results.

2. Theorems. First of all we have

Lemma 1. System (1) has invariant sets $\{p \in P : \sum_{i=1}^{n} \frac{p_i^2}{\lambda_i} = \text{const.}\}$. Consequently solutions are bounded as long as they are defined.

The proof is clear and will be omitted. Main results are the following

Theorem 1. Any compact solution is asymptotic periodic of period ω.

<u>Theorem 2</u>. The set of periodic points is connected. Here we shall note that

(i) a solution $p(t)$ is compact if $p(t)$ is define on $[t_0, \infty)$ for some $-\infty < t_0 < \infty$ and if there are constants $\alpha, \beta > 0$ such that

$$\alpha < p_i(t) < \beta \quad \text{for} \quad t \geqq t_0 , \quad 1 \leqq i < n ,$$

(ii) a solution $p(t)$ is asymptotic periodic of period ω if there is a periodic solution $q(t)$ of period ω such that $p(t) - q(t) \longrightarrow 0$ as $t \rightarrow \infty$,

(iii) a point $p \in P$ is called a periodic point if it is an initial value at $t = 0$ of periodic solutions of period ω.

We shall go back to the classical dynamics of system (1), where $E(t, p) =$ ~~$E(t, p)$~~, and see what information Theorem 1 provides us. $= E(p)$,

<u>Corollary</u>. Assume that $E(p)$ is homogeneous of order zero, that is, $E(\lambda p) = E(p)$ for $\lambda > 0$, $p \in P$. If there is at least one compact solution, then any solution is compact and asymptotically constant.

Remark. The conclusion of Corollary is called a global stability in an economy. To author's knowledge, sufficient conditions of the stability need the existence of not only a compact solution, but a critical point. So the Corollary seems to improve a classical result in an economy.

3. Proof of Theorem 1. We shall restrict the discussion to Theorem 1. Instead of treating (1) directly, it is more convenient to change the dependent variable of (1) by

$$x_i = p_i^{\,2} / \lambda_i$$

which reduces (1) to

$$(2) \quad \dot{x}_i = f_i(t, x) \qquad \text{for} \quad 1 \leqq i \leqq n ,$$

$$x = (x_1, x_2, \cdots, x_n) ,$$

where $f_i(t, x) = 2\sqrt{\lambda_i x_i}\, E_i\, (t, \sqrt{\lambda_1 x_1}, \cdots, \sqrt{\lambda_n x_n})$.
Here $f(t, x) = (f_i(t, x))$ satisfies the same conditions (i), (ii), (iii) of system (1) and

$$(\text{iv})' \quad \sum_{i=1}^{n} f_i(t, x) = 0 .$$

Moreover Theorem 1 is replaced by

<u>Theorem 1'</u>. Any compact solution of (2) is asymptotic periodic of period ω .

Therefore it is sufficient to prove Theorem 1'.

<u>Lemma 2</u>. Let $x(t)$ and $y(t)$ be any solution of (2) and set $|x(t) - y(t)| = \sum_{i=1}^{n} |x_i(t) - y_i(t)|$. Then $|x(t) - y(t)|$ has a right-hand derivative and

$$D^{+}|x(t) - y(t)| = \lim_{h \to 0+} \frac{1}{h}\{|x(t+h) - y(t+h)| - |x(t) - y(t)|\}$$

$$\leqq 0 .$$

Consequently any compact solution is uniformly stable.

This fact is known. For example see Coppel's book.

<u>Lemma 3.</u> In a periodic system, a compact and uniformly stable solution is asymptotically almost periodic.

This result was first proved by Deysach and Sell, and later improved by Yoshizawa.

References.

[1] W. A. Coppel, Stability and Asymptotic Behavior of Differential Equations, Heath Math. Mon. (1965), p. 59.

[2] F. Nikaido, Convex Structure and Economic Theory, Academic Press (1968), Chapter VI.

[3] T. Yoshizawa, Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions, Springer (1975), p. 185.