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On autonomous linear functional differential equations with a phase space of general type

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1. Introduction. We consider returded functional differential equations

$$(1.1)\qquad dx/dt = f(t,x_t),\quad x_t(\theta) = x(t+\theta),\ \theta\in[-r,0].$$

For a function ϕ defined on $[-r,0]$, we denote by $x(t,\sigma,\phi)$ the solution of (1.1) with the initial condition $x_\sigma(\sigma,\phi) = \phi$. When the retardation is finite, i.e., $r<\infty$, $x_t(\sigma,\phi)$ is a continuous function on $[-r,0]$ for $t \geqq \sigma+r$ whether ϕ is continuous or not. Hence, we choose for the space of initial data the space of continuous functions on $[-r,0]$. However, when the retardation is infinite, i.e., $r=\infty$, $x_t(\sigma,\phi)$ depends on ϕ in such a way that $x_t(\sigma,\phi)(\theta) = \phi(t-\sigma+\theta)$ for $\theta \leq -(t-\sigma)$. If ϕ is not continuous, $x_t(\sigma,\phi)$ is never a continuous function on $(-\infty,0]$ for any $t \geqq \sigma$. There are many possibilities of the choice for the space of initial functions.

Example 1.1 ([2]). Space of integrable functions. Let $\hat{\mathcal{B}} = \{\hat{\phi} : (-\infty, 0] \to R^n$ measurable and $|\hat{\phi}|_{\hat{\mathcal{B}}} < \infty\}$, where

$$|\hat{\phi}|_{\hat{\mathcal{B}}} = |\hat{\phi}(0)| + \int_{-\infty}^{0} g(\Theta) |\hat{\phi}(\Theta)| d\Theta$$

with $g:(-\infty, 0] \to R^+$ which satisfies some conditions.

Example 1.2 ([3],[8]). $\hat{\mathcal{B}} = \{\hat{\phi} : (-\infty, 0] \to R^n$ measurable on $(-\infty, 0]$, continuous on $[-r,0]$, $r \geqq 0$, and $|\hat{\phi}|_{\hat{\mathcal{B}}} < \infty\}$, where

$$|\hat{\phi}|_{\hat{\mathcal{B}}} = \{\sup_{-r \leqq \theta \leqq 0} |\hat{\phi}(\theta)|^p + \int_{-\infty}^{0} g(\Theta) |\hat{\phi}(\Theta)|^p d\Theta\}^{1/p}, \ p \geqq 1,$$

with $g:(-\infty, 0] \to R^+$ which satisfies some conditions.

Example 1.3 ([6]). Space of continuous functions. For any $\gamma \in R$, let $\hat{\mathcal{B}} = \{\hat{\phi} \in C((-\infty, 0], R^n) : e^{\gamma\theta}\phi(\theta) \to$ a limit as $\Theta \to -\infty\}$, and let

$$|\hat{\phi}|_{\hat{\mathcal{B}}} = \sup_{-\infty < \theta \leqq 0} e^{\gamma\Theta} |\hat{\phi}(\Theta)|$$

With the space of Example 1.1, Coleman and Mizel dicussed the existence, the uniqueness, the asymptotic behavior of solutions ([2]). The linear functional differential equations with phase spaces above are studied in [4], [7], [8], [9], and

[11]. Assuming that the phase space satisfies some conditions common to the above examples, Hale discussed the asymptotic behavior of solutions in [3] (see also [6]). Recently, in [5] Hale and Kato have presented new axioms and hypotheses for the phase space which seem to be natural. The all spaces listed above have these properties. In this lecture, we consider the solution operator of autonomous linear functional differential equations with this phase space. The theory given here is an extension of the one developed in [8].

2. The space $\mathcal{B}$. Let $\hat{\mathcal{B}}$ be a linear space of functions mapping $(-\infty,0]$ into C^n with elements designated by $\hat{\phi}$, $\hat{\psi}$, ... , and $\hat{\phi}=\hat{\psi}$ means $\hat{\phi}(t)=\hat{\psi}(t)$ for all $t\leq 0$. Assume that a semi-norm $|\cdot|_{\hat{\mathcal{B}}}$ is given in $\hat{\mathcal{B}}$, and assume that

$$\mathcal{B} = \hat{\mathcal{B}} / |\cdot|_{\hat{\mathcal{B}}}$$

is a Banach space with the norm $|\cdot|_{\mathcal{B}}$ which is naturally induced by $|\cdot|_{\hat{\mathcal{B}}}$. $\mathcal{B}$ consists of equivalence classes ϕ of $\hat{\phi}\in\hat{\mathcal{B}}$: $\phi = \{\hat{\psi}\in\hat{\mathcal{B}} : |\hat{\psi}-\hat{\phi}|_{\hat{\mathcal{B}}} = 0\}$. Given an $A > 0$ and a $\hat{\phi}\in\hat{\mathcal{B}}$, let $F_A(\hat{\phi})$ be the set of all functions $\hat{x}$ defined on $(-\infty,A]$ such that $\hat{x}_0 = \hat{\phi}$ and $\hat{x}(t)$ is continuous on $[0,A]$, and denote

$$F_A = \bigcup\{ F_A(\hat{\phi}) : \hat{\phi}\in\hat{\mathcal{B}} \} .$$

We assume that $\hat{\mathcal{B}}$ has the following properties, which are choosen from the axioms and hypotheses presented in [9].

(i) $\hat{x}_t \in \hat{\mathcal{B}}$ and x_t is continuous in t on [0,A] for all $\hat{x} \in F_A$.

(ii) $|\hat{\phi}(0)| \leqq K |\hat{\phi}|$ for any $\hat{\phi} \in \hat{\mathcal{B}}$ and some K.

For any $\beta > 0$, we define semi-norm $|\cdot|_\beta$, $|\cdot|_{(\beta)}$ of $\mathcal{B}$ as

$$|\phi|_\beta = \inf_{\hat{\phi} \in \phi} |\hat{\phi}|_\beta, \quad |\hat{\phi}|_\beta = \inf\{|\hat{\psi}|_{\hat{\mathcal{B}}} : \hat{\phi}(\theta) = \hat{\psi}(\theta), \theta \in (-\infty, -\beta]\}$$

$$|\phi|_{(\beta)} = \inf_{\hat{\phi} \in \phi} |\hat{\phi}|_{(\beta)}, \quad |\hat{\phi}|_{(\beta)} = \inf\{|\hat{\phi}|_{\hat{\mathcal{B}}} : \hat{\psi}(\theta) = \hat{\phi}(\theta), \theta \in [-\beta, 0]\} .$$

The following inequality is quite natural and useful.

(iii) $|\phi| \leqq |\phi|_{(\beta)} + |\phi|_\beta$ for any $\beta \geqq 0$.

Let $\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$. We assume that the operator S(t) is well defined on $\mathcal{B}_0$ in such a way that $S(t)\phi \ni \hat{\tau}^t \hat{\phi}$, where

$$\hat{\tau}^t \hat{\phi}(\theta) = \begin{cases} \hat{\phi}(t+\theta) & \text{for } \theta \leqq -t \\ 0 & \text{for } \theta \geqq -t . \end{cases}$$

The assumption on S(t) is

(iv) $\{S(t)\}_{t \geqq 0}$ is a strongly continuous semigroup of bounded linear operators on $\mathcal{B}_0$.

The next assumption will be used to obtain concrete representations of some elements in $\mathcal{B}$.

(v) If $\{\hat{\phi}^k\}$ converges to $\hat{\phi}$ uniformly on any compact set in

$(-\infty,0]$ and if $\{\hat{\phi}^k\}$ is a Cauchy sequence in $\hat{\mathcal{B}}$, then $\hat{\phi}\in\hat{\mathcal{B}}$ and $\phi^k\to\phi$ as $k\to\infty$.

Let $\hat{a}, a\in C^n$, be a function such that $\hat{a}(\theta)=a$ for $\theta\in(-\infty,0]$.

We assume that

(vi) $\hat{a}\in\hat{\mathcal{B}}$ for any $a\in C^n$,

and denote by $\bar{a}$ the element of $\mathcal{B}$ determined by $\hat{a}\in\hat{\mathcal{B}}$.

The following assumption will be needed to prove the compactness of the operator $U(t)$ which will appear in Section 3.

(vii) There is a continuous function $K_1(\beta)$ of $\beta\geqq 0$ such that

$$|\phi|_{(\beta)} \leqq K_1(\beta)|\phi|_{[-\beta,0]} \quad , \quad \beta \geq 0 ,$$

where

$$|\phi|_{[-\beta,0]} = \inf \{\sup_{-\beta\leq\theta\leq 0} |\hat{\phi}(\theta)| : \hat{\phi}\in\phi\}.$$

3. The solution operator. Consider a linear functional differential equation

$$(3.1) \qquad dx/dt = L(x_t) ,$$

where L is a bounded linear operator on $\mathcal{B}$ into C^n.

Assume that the solution $\hat{x}(t,\phi)=\hat{x}(t,0,\phi)$ of (3.1) exists on $[0,\infty)$ uniquely for any ϕ in $\mathcal{B}$, and assume that the operators $T(t)$ defined as

$$T(t)\phi = x_t(\phi) \quad \text{for} \quad t \geqq 0 \ , \ \phi\in\mathcal{B}$$

make a strongly continuous semigroup of bounded linear operators on $\mathcal{B}$. Let U(t) be an operator defined by

$$T(t)\phi = U(t)\phi + S(t)P\phi \;, \quad t \geqq 0 \;, \quad \phi \in \mathcal{B},$$

where

$$P\phi = \phi^* \equiv \phi - \overline{\phi(0)} \;, \quad \phi \in \mathcal{B},$$

which is clearly a projection operator mapping $\mathcal{B}$ onto $\mathcal{B}_0$. Then $U(t)\phi$ is the equivalence class of $\hat{\xi}_t$ for $t \geqq 0$, where $\hat{\xi}$ is a function on $(-\infty,+\infty)$ such that

$$(3.2) \qquad \hat{\xi}(t) = \begin{cases} \hat{x}(t,\phi) = \phi(0) + \int_0^t L(x_s(\phi))ds & \text{for } t \geqq 0 \\ \phi(0) & \text{for } t \leqq 0 \;. \end{cases}$$

Using the assumptions (iii), (iv), (v), (vi) and (vii) and this representation of U(t), we can prove that U(t) is a compact operator (see [5]). From this fact, Hale obtained the following result (Theorem 3.1).

For any bounded set X of a Banach space, put $\alpha(X)=\inf\{ d>0 \;;\; X$ has a finite cover of diameter $<d\}$, which is called the measure of noncompactness of X. For example, $\alpha(X)=0$ if and only if X is relatively compact. For any continuous operator T on a Banach space into a Banach space which takes bounded sets into bounded sets, define $\alpha(T)$ by $\alpha(T) = \inf \{k: \alpha(TX) \leqq k\alpha(X)$

for all bounded sets X .

If T is a bounded linear operator, then $\alpha(T) \leqq |T|$.

T is a compact operator if and only if $\alpha(T)=0$. It is known that the raduis $r_e(T)$ of the smallest dosed disk in the complex plane with center zero which contains the essential spectrum of a bounded linear operator T is given by

$$(3.3) \qquad r_e(T) = \lim_{n\to\infty} \alpha(T^n)^{1/n}$$

(see [10]).

Now, we will compute $r_e(T(t))$. Since U(t) is a compact operator, it is easy to see that

$$\alpha(T(t)) = \alpha(S(t)P) \quad \text{for} \quad t \geq 0 .$$

Since $\{S(t)\}$ is a semigroup on the range of the projection P, we have $S(t)PS(s)P=S(t+s)P$ for $t,s\geqq 0$. Using this and the formula (3.3), we obtain

$$r_e(T(t)) = r_e(S(t)P) .$$

Browder proved that the point λ_0 of the spectrum of a closed linear operator T does not lie in the essential spectrum if and only if $\bigcup_{n\geqq 0} N((\lambda_0 I-T)^n)$ is of finite dimension and the resolvent $(\lambda I-T)^{-1}$ is holomorphic in the neiborhood of λ_0 and has a pole at λ_0 ([1]). Using this result, we can easily prove that the essential spectrum of S(t)P coincides with that of S(t).

Hence we have

$$r_e(S(t)P) = r_e(S(t)) .$$

Let β be the number defined as

(3.4) $$\beta = \lim_{t\to\infty} \frac{\log \alpha(S(t))}{t} = \inf_{t>0} \frac{\log \alpha(S(t))}{t} < \infty .$$

Then, it follows that

$$\lim_{n\to\infty} \alpha(S(t)^n)^{1/n} = e^{t\beta} .$$

Finally, from the relations above we obtain

$$r_e(T(t)) = e^{t\beta}$$

This implies the following theorem.

Theorem 3.1.

$$\{ \lambda \in C : \mathrm{Re}\lambda > \beta \} \subset P_\sigma(A) \cup \rho(A) .$$

4. The point spectrum and the resolvents of A. Let $V(\lambda)$ be the linear subspace of C^n consisting of vectors $b \in C^n$ such that $\omega(\lambda)b \in \mathcal{B}$, where $\omega(\lambda)b$ is the equivalence class of the function

$e^{\lambda\theta}b$, $\theta \in (-\infty,0]$. Define a linear operator $\Delta(\lambda):V(\lambda)\to C^n$ as

$$\Delta(\lambda)b = \lambda b - L(\omega(\lambda)b) \quad \text{for} \quad b\in V(\lambda) .$$

Theorem 4.1 ([5]).

$$P_\sigma(A) = \{\lambda\in C;\ \dim \ker\Delta(\lambda) \neq 0\} .$$

Proof. Suppose that λ lies in $P_\sigma(A)$. Then, there exists a $\phi \in \mathcal{B}$, $\phi \neq 0$, such that $A\phi = \lambda\phi$, which implies

$$(4.1) \qquad T(t)\phi = e^{\lambda t}\phi \quad \text{for} \quad t \geqq 0 .$$

For a $\hat{\phi}\in \phi$, define the function $\hat{x}(t)$ by

$$(4.2) \qquad \hat{x}(t) = \begin{cases} (T(t)\phi)(0) = e^{\lambda t}\phi(0) & t \geqq 0 \\ \hat{\phi}(t) & t \leqq 0 . \end{cases}$$

Since $\hat{x}_t\in T(t)\phi$, relation (4.1) implies that $\hat{\phi}^t \equiv e^{-\lambda t}\hat{x}_t$ is in ϕ for $t \geqq 0$. From definition (4.2) of $\hat{x}$, we have

$$\hat{\phi}^t(\theta) = e^{-\lambda t}\hat{x}(t+\theta) = e^{\lambda\theta}\phi(0) \quad \text{for } \theta \geqq -t .$$

Hence, the sequence $\{\hat{\phi}^k\}_{k=1}^{\infty}$ of $\hat{\mathcal{B}}$ converges to $e^{\lambda\theta}\phi(0)$ uniformly on any compact set of $(-\infty,0]$. Since $\hat{\phi}^k\in \phi$ for $k=1,2,\ldots$, $\{\phi^k\}$ is a Cauchy sequence of $\mathcal{B}$. Therefore, hypothesis (v) implies

$\omega(\lambda)\phi(0) = \phi$, and also we have that $\phi(0) \neq 0$. Since $\hat{x}(t)$ is a solution of (3.1), it holds that

$$\lambda e^{\lambda t}\phi(0) = d\hat{x}/dt = L(T(t)\phi) = L(e^{\lambda t}\phi) = L(e^{\lambda t}\omega(\lambda)\phi(0)) ,$$

which implies that

$$\Delta(\lambda)\phi(0) \equiv \lambda\phi(0) - L(\omega(\lambda)\phi(0)) = 0 .$$

Since $\phi(0) \neq 0$, it follows that dim ker$\Delta(\lambda) \neq 0$. It is easy to prove that if dim ker$\Delta(\lambda) \neq 0$, then λ lies in $P_\sigma(A)$. Q.E.D.

The following proposition is proved from hypothesis (v).

Proposition 4.2. Let $\hat{\xi}(t)$ be the function defined in (3.2). Then, the integration $\int_0^t e^{-\lambda s}U(s)\phi ds$ is the equivalence class of the function of θ given by

$$\int_0^t e^{-\lambda s}\hat{\xi}(s+\theta)ds, \quad \theta \in (-\infty,0] .$$

Let γ be a number such that

$$\gamma = \lim_{t\to\infty} \frac{\log|S(t)|}{t} = \inf_{t>0} \frac{\log|S(t)|}{t} .$$

It is trivial that $\beta \leqq \gamma$ (see (3.4)). From the definition of semi-norm $|\cdot|_{(t)}$, $t \geqq 0$, we have the following proposition.

Proposition 4.3.

$$|\int_t^\infty e^{-\lambda s}S(s)\phi^* ds|_{(t)} \leq \int_t^\infty |e^{-\lambda s}| \, ||S(s)\phi^*|_{(t)} ds = 0 \text{ for } \phi\in\mathcal{B},$$

where Re $\lambda>\gamma$.

Preparing these propositions, we can prove the following theorem.

Theorem 4.4. Let $\lambda\in C$ be a number such that Re $\lambda>\gamma$ and $\lambda \notin P_\sigma(A)$, which implies $\lambda\in\rho(A)$ (Theorem 4.1). Then, the resolvent $R(\lambda;A) = (\lambda I-A)^{-1}$ is given as

$$R(\lambda;A)\phi = \omega(\lambda)b + \lambda^{-1}\overline{\phi(0)} + \int_0^\infty e^{-\lambda s}S(s)\phi^* ds \quad \text{for} \quad \phi\in\mathcal{B} ,$$

where

$$\Delta(\lambda)b = L(\lambda^{-1}\overline{\phi(0)} + \int_0^\infty e^{-\lambda s}S(s)\phi^* ds) .$$

Let B be the infinitesimal generator of $\{S(t)\}_{t\geq 0}$.

It is known that λ is in $\rho(B)$ for sufficiently large $\lambda > 0$ with the resolvent

$$(4.3) \qquad R(\lambda;B)\phi = \int_0^\infty e^{-\lambda s}S(s)\phi ds \quad \text{for} \quad \phi\in\mathcal{B}_0 .$$

Theorem 4.5

$$A\phi = \lim_{\lambda\to\infty} \{\lambda BR(\lambda;B)\phi^* + \omega(\lambda)\lambda\Delta(\lambda)^{-1}L(\overline{\phi(0)}+\lambda R(\lambda;B)\phi^*)\} ,$$

if and only if this limit exists.

Proof. For sufficiently large $\lambda > 0$, we have from Theorem 4.4 and relation (4.3) that

$$R(\lambda;A)\phi - R(\lambda;B)\phi^*$$
$$= \omega(\lambda)\Delta(\lambda)^{-1}L(\lambda^{-1}\overline{\phi(0)} + R(\lambda;B)\phi^*) + \lambda^{-1}\overline{\phi(0)} \quad .$$

Note that $\lambda R(\lambda;T)\phi = TR(\lambda;T)\phi + \phi$ for the resolvent of a closed linear operator. Hence, it follows that

$$AR(\lambda;A)\phi - BR(\lambda;B)\phi^*$$
$$= \omega(\lambda)\Delta(\lambda)^{-1}L(\overline{\phi(0)} + \lambda R(\lambda;B)\phi^*) \quad .$$

Since $\lambda R(\lambda;A)\phi$ approaches ϕ as $\lambda \to \infty$, we obtain Theorem 4.4.

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