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On Levine pairings

by
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§1 Introduction

An n-knot will be a smooth oriented submanifold K of S^{n+2}, where K is homeomorphic to S^n. Since the complement $X = S^{n+2}-K$ is a homology circle, the universal abelian cover $\widetilde{X}$ of X has an infinite cyclic covering transformation group. This defines a unique module structure on $H_q(\widetilde{X};Z)$ over the ring $\Lambda = Z[t,t^{-1}]$. We will use the notation $A_q = H_q(\widetilde{X})$ and refer to it as the q-th Alexander module. Let T_q be the Z-torsion submodule of A_q. In case $n = 2q$, J.Levine has defined an obscure linking form $[\ ,\ ]:T_q \times T_q \longrightarrow Q/Z$, satisfying $(-1)^{q+1}$-symmetric property, [L]. In §2, we shall give an alternative description of it by making use of the τ-Seifert form, defined by M.A.Gutiérrez. This enables us to answer the classification problem. A simple 2q-knot K is odd finite if $\pi_q(X)$ is finite and 2-torsion free. Then the result is

Theorem ; Let K_0, K_1 be odd finite simple 2q-knots with isometric Levine pairings and $q > 3$. Then K_0 is isotopic to K_1.

A rough plan of the proof is described in §3. And some corollaries of this are stated.

§2 Levine pairings

2.1 We recall the definition of the Levine pairing. The property of duality for the Alexander module has been observed in many ways. The most suitable formulation for us is the following exact sequence for $0 \leq q \leq n$:

$$0 \longrightarrow \mathrm{Ext}^2_\Lambda(A_{n-q},\Lambda) \longrightarrow \overline{A}_q \longrightarrow \mathrm{Ext}^1_\Lambda(A_{n+1-q},\Lambda) \longrightarrow 0,$$

where $\overline{A}_q$ denotes the right Λ-module defined from the original left Λ-structure by usual means. This follows from the duality theorem of [M], the universal coefficient spectral sequence and the general property of A_q. As for a Alexander module A_i, $\mathrm{Ext}^2_\Lambda(A_i,\Lambda)$ is a Z-torsion module and depends on T_i, while $\mathrm{Ext}^1_\Lambda(A_i,\Lambda)$ is Z-torsion free and depends on A_i/T_i, therefore

(1) $\overline{T}_q \approx \mathrm{Ext}^2_\Lambda(T_{n-q},\Lambda)$ for $0<q<n$.

Now, for the finite Λ-module T_{n-q}, there is a canonical isomorphism as Λ-modules:

(2) $\mathrm{Ext}^2_\Lambda(T_{n-q},\Lambda) \approx \mathrm{Hom}_Z(T_{n-q},Q/Z)$.

So that we derive the Levine pairing by combining the isomorphisms (1), (2):

$$[\ ,\ ]: T_q \times T_{n-q} \longrightarrow Q/Z.$$

In case n = 2q, it satisfies the following four properties:

a) Z-linear: $[m\alpha,\beta] = [\alpha, m\beta] = m[\alpha,\beta]$ for $m \in Z$

b) conjugate self-adjoint: $[\lambda\alpha.\beta] = [\alpha,\overline{\lambda}\beta]$ for $\lambda \in \Lambda$

c) non-singular: the adjoint to $[\ ,\ ]$ is bijective as a homomorphism: $T_q \longrightarrow \mathrm{Hom}_Z(T_q;Q/Z)$

d) $(-1)^{q+1}$-symmetric: $[\alpha,\beta] = (-1)^{q+1}[\beta,\alpha]$.

Remark ; This is almost complete algebraic characterization of the Levine pairing. In fact, for any pairing $[\ ,\ ]$ on T_q

satisfying a), b), c) and d), there exists a 2q-knot with the Levine pairing [,] , provided $q \geq 2$, [L]. We don't know what kind of pairing can be realized as the Levine pairing of some 2-knot.

2.2 For the purpose of this section, some preliminaries are necessary. The Seifert manifold V of a 2q-knot K is a smooth oriented submanifold of S^{2q+2} which is bounded by K. Writing τ for the torsion subgroup, we define the τ-intersection pairing (classically called a linking number) $I: \tau H_q(V) \otimes \tau H_q(V) \longrightarrow Q/Z$ and the τ-linking pairing $L: \tau H_q(V) \otimes \tau H_q(S^{2q+2}-V) \longrightarrow Q/Z$ as follows. Let $\alpha \in \tau H_q(V)$ have order d. Represent α by a q-chain ξ, and let ζ be a (q+1)-chain such that $\partial \zeta = d \cdot \xi$. Then if $\beta \in \tau H_q(V)$ is represented by a q-chain η, we have $I(\alpha, \beta) \equiv \mathrm{Int}(\zeta, \eta)/d \pmod 1$, where Int(,) is the usual intersection number. And if $\gamma \in \tau H_q(S^{2q+2}-V)$ is represented by a q-chain μ with $\partial \rho = \mu$ for some (q+1)-chain ρ in S^{2q+2}, we have $L(\alpha, \gamma) \equiv \mathrm{Int}(\zeta, \rho)/d \pmod 1$. Then the τ-Seifert form of V

$$\theta: \tau H_q(V) \otimes \tau H_q(V) \longrightarrow Q/Z$$

is defined by letting $\theta(\alpha \otimes \beta)$ be the τ-linking number $L(\alpha, \beta_+)$, where β_+ is the translate in the positive normal direction off V of a cycle β.

A finite abelian group G splits as the direct sum of its Sylow subgroups. Since these are clearly orthogonal with respect to any form on G, the whole problem splits accordingly. Thus without loss of generality, we may assume that $\tau H_q(V)$ is a p-group throughout this note.

Let $\{\alpha_i\}_{i=1}^{k}$ be a basis of $\tau H_q(V)$ having order p_i with elementary divisor $p_1 | p_2 | \cdots | p_k$, and $\{\beta_i\}_{i=1}^{k} \in \tau H_q(S^{2q+2}-V)$ be the Alexander dual basis of $\{\alpha_i\}$. The embeddings $\kappa, \iota : V \longrightarrow S^{2q+2}-V$ are defined by $\pm \varepsilon$-pushes. Then, for maps $\kappa_*, \iota_* : \tau H_q(V) \longrightarrow \tau H_q(S^{2q+2}-V)$ induced by κ, ι, we shall give the matrix presentations:

$$\kappa_*(\alpha_1, \alpha_2, \ldots, \alpha_k) = (\beta_1, \beta_2, \ldots, \beta_k)\, M_+$$
$$\iota_*(\alpha_1, \alpha_2, \ldots, \alpha_k) = (\beta_1, \beta_2, \ldots, \beta_k)\, M_-.$$

Indeed $M_\pm$ are not uniquely determined as integral matrices, but are uniquely determined modulo $\mathcal{P}$, where $\mathcal{P} = P \cdot J$,

$$P = \begin{pmatrix} p_1 & & 0 \\ & p_2 & \\ 0 & & \ddots & \\ & & & p_k \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1, & 1, \ldots, & 1 \\ 1, & \ddots & \vdots \\ 1, & \ldots\ldots, & 1 \end{pmatrix}.$$

And if $M_\theta = (p_i \cdot \theta(\alpha_i \otimes \alpha_j))$, then $M_+ = M_\theta \pmod{\mathcal{P}}$ and $M_- = M_\theta' \pmod{\mathcal{P}}$, where $M_\theta' = P \cdot {}^t M_\theta \cdot P^{-1}$. The result of [G] is

Proposition 1 (M.A.Gutiérrez) ; The presentation matrix of T_q is given by $(t \cdot M_\theta + (-1)^{q+1} M_\theta', P)$. More precisely, there is an exact sequence

$$\overset{2k}{\oplus}\Lambda \xrightarrow{\ d\ } \overset{k}{\oplus}\Lambda \longrightarrow T_q \longrightarrow 0$$

such that $d(u_1, \ldots, u_{2k}) = (v_1, \ldots, v_k)\,(t \cdot M_\theta + (-1)^{q+1} M_\theta', P)$, where $\{u_i\}$ and $\{v_i\}$ are bases of $\overset{2k}{\oplus}\Lambda$ and $\overset{k}{\oplus}\Lambda$.

In general, the τ-Seifert form of V is not non-singular. Therefore we demand some minimality of V so that it turns out to be non-singular. A Seifert manifold V is q-minimal if $\kappa_*, \iota_* : H_q(V) \longrightarrow H_q(S^{2q+2}-V)$ are injective. Now, if κ_* is isomorphic, then there is an inverse isomorphism κ_*^{-1}.

Let M_θ^{-1} be the representative matrix of κ_*^{-1} with respect to bases $\{\beta_i\}$ and $\{\alpha_i\}$, then $M_\theta \cdot M_\theta^{-1} = M_\theta^{-1} \cdot M_\theta = E \pmod{\mathcal{P}}$, where E is the identity matrix.

<u>2.3</u> Now, we shall give an alternative description of the Levine pairing. Let K be a 2q-knot and V be a q-minimal Seifert manifold of K. A square matrix $I_V = (I(\alpha_i, \alpha_j))$ with entries in Q/Z is the τ-intersection matrix of V. By Proposition 1, the element of T_q can be represented by a column vector of $\overset{k}{\oplus}\Lambda$. The <u>quotient form</u> $[\ ,\ ]_\theta$ on T_q determined by the τ-Seifert form θ of V with values in Q/Z, is defined by

$$[\alpha, \beta]_\theta = \text{the constant term of } {}^t\bar{u}\cdot\widetilde{U}_\theta\cdot v \pmod 1,$$

where u and v are representatives of α and β. Here, $\widetilde{U}_\theta$ is a square matrix with entries in $Q/Z[[t,t^{-1}]]$ defined as follows: set a Q/Z-matrix $U_\theta = {}^tM_\theta^* \cdot I_V \cdot M_\theta^*$, where $M_\theta^* = (M_\theta')^{-1}$ and a Z-matrix $A_\theta = (-1)^q M_\theta' M_\theta^{-1} \pmod{\mathcal{P}}$. Then $\widetilde{U}_\theta = \sum_{n=-\infty}^{\infty} t^n ({}^tA_\theta)^n U_\theta$.

<u>Lemma 2.1</u> ; <u>The quotient form</u> $[\ ,\ ]_\theta$ <u>is well defined and satisfies four properties</u> a), b), c) <u>and</u> d).

We recall the formula [G]:

$$\theta(\alpha\otimes\beta) + (-1)^{q+1}\theta(\beta\otimes\alpha) = -I(\alpha,\beta) \pmod 1,$$

namely $M_\theta + (-1)^{q+1}M_\theta' = -P\, I_V \pmod{\mathcal{P}}$. This enables us to prove Lemma by routine calculus.

<u>Lemma 2.2</u> ; <u>The quotient form</u> $[\ ,\ ]_\theta$ <u>coincides with the Levine pairing</u> $[\ ,\ ]$ <u>of</u> K.

The proof is supplied by Levine's observation. Proposition 7.1 in [L] is valid in more general situation that $H_q(V) \approx H_q(\tilde{X})$.

2.4 We shall expose some examples.

Example 1 ; 2-twist-spun torus knot of type (2,p).
Here, p is an odd integer. This is fibered with the fiber punctured L(p,1). Then

$$M_\theta = (p-1)$$
$$A_1 \approx \Lambda/(t+1,p)$$
$$U = \sum_{n=-\infty}^{\infty} (-t)^n/p.$$

Example 2 ; 3-twist-spun torefoil.
In this case, the fiber V has the homology $H_1(V) \approx Z_2 \oplus Z_2$.

$$M_\theta = \begin{pmatrix} 1, & 0 \\ 1, & 1 \end{pmatrix}$$
$$A_1 \approx \Lambda\oplus\Lambda \Big/ \begin{pmatrix} t+1, & 1, & 2, & 0 \\ t, & t+1, & 0, & 2 \end{pmatrix}$$
$$U = \sum_{n=-\infty}^{\infty} \left\{ t^{3n}\begin{pmatrix} 0, & 1/2 \\ 1/2, & 0 \end{pmatrix} + t^{3n+1}\begin{pmatrix} 1/2, & 0 \\ 1/2, & 1/2 \end{pmatrix} + t^{3n+2}\begin{pmatrix} 1/2, & 1/2 \\ 0, & 1/2 \end{pmatrix} \right\}.$$

§3 Summary of the proof and Corollaries

3.1 Our new description of the Levine pairing demands minimality of the Seifert manifold. For this reason, the following result, which is due to the surgery, is necessary.

Proposition 2 (J.Levine) ; Let K be a simple 2q-knot. Then there exists a q-minimal Seifert manifold of K, provided $q \geq 2$.

Now, by virtue of this, we can describe the situation for the proof of Theorem as follows. Let K_0, K_1 be odd finite simple 2q-knots with isometric Levine pairings. And V, W are their q-minimal Seifert manifolds. The τ-Seifert form of V (W) is denoted by θ (η). Our process is divided into the following Lemmas.

Lemma 3.1 ; If the quotient form $[\ ,\ ]_\theta$ is isometric to $[\ ,\ ]_\eta$, then the τ-Seifert form θ is isometric to η.

Lemma 3.2 ; In case q is odd and ≥ 5, if θ is isometric to η, then the Seifert manifold V is isotopic to W in S^{2q+2}.

Lemma 3.3 ; If q is even and ≥ 4, then the fact in the above Lemma is valid.

The hypothesis "odd" of knots is essentially used in Lemma 3.2, 3.3. The proof is so long that we shall refer to [K2] for details.

3.2 Here, we shall give some corollaries of Theorem. Notations and their definitions are followed in the previous article [K1]. The first result is established by summing up Theorem and [K1].

Corollary 1 ; Let K be an odd simple fibered 2q-knot and $q > 3$. Then the isotopy class of K is completely determined by the 1st, the 2nd and the τ- Seifert forms.

Now, the followings are immediately obtainable from this.

Corollary 2 ; *If a 2q-knot K is as above, then there is a unique splitting* $K = K_F \# K_T$ *such that* $\pi_q(X_F)$ *is torsion free and* $\pi_q(X_T)$ *is finite, where* # *denotes the knot connected sum and* X_F (X_T) *is the complement of a simple knot* K_F (K_T).

Corollary 3 ; *Let* K *be as above and* $p, p' : X \longrightarrow S^1$ *be fibrations of it. Then there is a diffeomorphism* f *of* S^{2q+2} *such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ f|X\ } & X \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle p'} & \\ & S^1 & \end{array}$$

commutes.

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