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# Higher Order Approximation in the Reductive Perturbation Method for the Weakly Dispersive System

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## §1. Introduction

The higher order approximations in the reductive perturbation method are studied for the weakly dispersive nonlinear system. It is shown that the secular terms appearing in the higher order terms are eliminated by adding to the Korteweg-de Vries equation the derivatives of the higher-order conserved densities, so that a general nonlinear dispersive system of equations can be approximated by the equation of the form,

$$u,_{\tau} - 6u\cdot u,_{\xi} + u,_{\xi\xi\xi} + \sum_{j\geq 2} \delta\lambda_j \mathcal{A}_{j+1}(u),_{\xi} = 0 , \qquad (1.1)$$

in which $\mathcal{A}_{j+1}$ is the j+1 th conserved density. It is shown further that the coefficients $\delta\lambda_j$ can be determined by the linear dispersion relation of the original system. The equation (1.1) is the so-called generalized KdV equation, which is completely integrable and physical effects of the conserved densities in this equation are given by the renormalization of the velocities of the KdV solitons. Also, eq.(1.1) is rewritten in terms of the conserved quantities, the form of which is more general in the sense that for the strongly dispersive system the nonlinear Schrödinger equation is not modified by the conserved densities but by the conserved quantities.

§2. Reductive Perturbation Method for Weakly Dispersive Systems

In this paper, we consider the following system of equations for a column vector U with n (real) components, $u_1, u_2, \cdots, u_n$,[1,2)]

$$U,_t + AU,_x + K_1[K_2(K_3U,_x),_x],_x = 0 . \qquad (2.1)$$

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Extension to a more general system was done in Ref.1. See also examples in §5.

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Here A is a $n \times n$ real matrix function of U, and its eigenvalues, $\lambda_i$ $(i=1,2,\cdots,n)$, are assumed real and distinct, so that the corresponding n eigenvectors $R_i$ $(i=1,2,\cdots,n)$ are linearly independent; $K_1$, $K_2$ and $K_3$ are also (real) $n \times n$ matrix functions of U. In what follows the neccessary analiticities of A, $\{\lambda_i\}$, $\{R_i\}$ and $K_\alpha(\alpha=1,2,3)$ with respect to U will be assumed in a domain of the U space, say $\Omega$. Let $U^{(0)}$ be a constant vector in $\Omega$, which is a trivial solution of eq.(2.1). Then a neighboring solution U is expanded as

$$U = U^{(0)} + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \cdots , \qquad (2.2)$$

where $\varepsilon$ is a small (positive) parameter. Consequently the matrices A and $K_\alpha$ are also expanded as

$$A = \sum_{n=0} \varepsilon^n A^{(n)} = A^{(0)} + \varepsilon U^{(1)} \cdot \nabla_u A^{(0)} + \varepsilon^2 \{U^{(2)} \cdot \nabla_u A^{(0)}$$

$$+ \frac{1}{2} U^{(1)} U^{(1)} : \nabla_u \nabla_u A^{(0)}\} + \cdots \qquad (2.3)$$

in which the notations are the same as are used in Ref.1) and 2) i.e. $A^{(0)}\equiv A(U^{(0)})$, $\nabla_u=(\partial/\partial u_1\ ,\cdots,\ \partial/\partial u_n)$, $\nabla_u A^{(0)}=(\nabla_u A)_{U=U^{(0)}}$ etc. The linear dispersion relation for the frequency $\omega$ and the wave-number $k$ is $\det|-\omega I+kA^{(0)}-k^3K_1^{(0)}K_2^{(0)}K_3^{(0)}|=0$, where $I$ is the unit matrix, and it yields the expansion in terms of small $k$

$$\omega = \lambda_i^{(0)}k - \mu k^3 + \cdots \tag{2.4}$$

where $\mu = (L_i^{(0)}K_1^{(0)}K_2^{(0)}K_3^{(0)}R_i^{(0)})/(L_i^{(0)}R_i^{(0)})$ ,[2] $(LA=\lambda A)$ .

We now assume that the i-th mode is genuinely weakly-dispersive as well as genuinely nonlinear,[2] that is, $\nabla_u\lambda_i^{(0)}\cdot R_i^{(0)}\neq 0,\ \mu\neq 0$, for which the Gardner-Morikawa transformation[2] is introduced by

$$\xi = \varepsilon^{1/2}\ (x-\lambda^{(0)}t)\ , \tag{2.5a}$$

$$\tau = \varepsilon^{3/2}\ t\ , \tag{2.5b}$$

in which and in what follows the subscript to specify the i-th mode is omitted. Under the boundary condition

$$U \to U^{(0)} \quad \text{for} \quad x\to\infty \quad \text{i.e.} \quad \text{for} \quad \xi\to\infty\ ,$$

following the standard procedure of the reductive perturbation method, we get in the order $\varepsilon^{3/2}$

$$W^{(0)}U^{(1)}_{,\xi} = 0$$

i.e. $$U^{(1)} = R^{(0)}u^{(1)}(\xi,\tau) \ .$$

Here $W^{(0)} \equiv -\lambda^{(0)}I + A^{(0)}$ and $u^{(1)}$ is a scalar function of $\xi$ and $\tau$ to be determined in the next order, $O(\varepsilon^{5/2})$, in which we have

$$W^{(0)}U^{(2)}_{,\xi} + U^{(1)}_{,\tau} + U^{(1)}\cdot\nabla_u A^{(0)}U^{(1)}_{,\xi} + K^{(0)}U^{(1)}_{,\xi\xi\xi} = 0 \ , \qquad (2.6)$$

where $K^{(0)} = K_1^{(0)}K_2^{(0)}K_3^{(3)}$. Multiplying this equation by $L^{(0)}$ from the left yields the KdV equation for $u^{(1)}$ as the compatibility condition,

$$\mathcal{K}(u^{(1)}) = u^{(1)}_{,\tau} - 6u^{(1)}u^{(1)}_{,\xi} + u^{(1)}_{,\xi\xi\xi} = 0 \ , \qquad (2.7)$$

in which $R^{(0)}$ is normalized by $\nabla_u\lambda^{(0)}\cdot R^{(0)} = -6$ and $\mu$(or eq.(2.4)) is assumed equal to unity. (If $\mu$ is $-1$, the transformation $\tau = -\varepsilon^{3/2}t$ and the normalization $\nabla_u\lambda^{(0)}\cdot R^{(0)} = 6$ gives (2.7).) On the other hand, solving eq.(2.6), we have

$$U^{(2)} = R^{(0)}u^{(2)} + \int V^{(2)}d\xi \qquad (2.8a)$$

where

$$V^{(2)} = \frac{\partial}{\partial\xi}\left[\frac{1}{2}\hat{R}^{(0)}(u^{(1)})^2 + \hat{K}^{(0)}R^{(0)}u^{(1)}_{,\xi\xi}\right] \ , \qquad (2.8b)$$

that is,

$$U^{(2)} = R^{(0)}u^{(2)} + \frac{1}{2}\hat{R}^{(0)}(u^{(1)})^2 + \hat{K}^{(0)}R^{(0)}u^{(1)}_{,\xi\xi} \ , \qquad (2.8c)$$

in which $\hat{R}^{(0)}$ and $\hat{K}^{(0)}$ are given by

$$\hat{R}^{(0)} = R^{(0)}\cdot\nabla_u R^{(0)} ,$$

$$W^{(0)}\hat{K}^{(0)} = I - K^{(0)} .$$

The terms of the order $\varepsilon^{7/2}$ are collected to give

$$W^{(0)}U^{(3)}_{,\xi} + u^{(2)}_{,\tau}R^{(0)} + R^{(0)}\cdot\nabla_u A^{(0)}R^{(0)}(u^{(1)}u^{(2)})_{,\xi} + K^{(0)}R^{(0)}u^{(2)}_{,\xi\xi\xi}$$

$$= S^{(2)}(u^{(1)}) , \qquad (2.9)$$

in which $S^{(2)}$ is the column vector dependent on $u^{(1)}$ and its $\xi$-derivatives only. Again, the compatibility condition of this equation gives the equation for $u^{(2)}$

$$\mathcal{L}(u^{(1)})u^{(2)} = s^{(2)}(u^{(1)}) . \qquad (2.10)$$

Here $\mathcal{L}(u^{(1)})$ is the linear operator defined by

$$\mathcal{L}(u^{(1)}) \equiv \frac{\partial}{\partial\tau} - 6\frac{\partial}{\partial\xi}u^{(1)} + \frac{\partial^3}{\partial\xi^3} \qquad (2.11)$$

and $s^{(2)}$ $(=L^{(0)}S^{(2)}/(L^{(0)}\cdot R^{(0)})$ is expressed by

$$s^{(2)} = s^{(2)}_1(u^{(1)})^2u^{(1)}_{,\xi} + s^{(2)}_2u^{(1)}u^{(1)}_{,\xi\xi\xi} + s^{(2)}_3u^{(1)}_{,\xi}u^{(1)}_{,\xi\xi}$$

$$+ s^{(2)}_4u^{(1)}_{,\xi\xi\xi\xi\xi} , \qquad (2.12)$$

in which $s^{(2)}_i$ (i=1,2,3,4) are constants. It should be noted that the homogeneous equation associated with eq.(2.10), $\mathcal{L}(u^{(1)})w=0$, is the linearized KdV equation.

It is straightforward to write down the higher order equations, namely in the order $\varepsilon^{n+3/2}$, assuming that $U^{(n)}=R^{(0)}u^{(n)}+\int V^{(n)}(u^{(1)},$

$\cdots, u^{(n-1)})d\xi$, we have

$$W^{(0)}U^{(n+1)}_{,\xi} + u^{(n)}_{,\tau}R^{(0)} + R^{(0)}\cdot\nabla_u A^{(0)}R^{(0)}(u^{(1)}u^{(n)})_{,\xi} + K^{(0)}R^{(0)}u^{(n)}_{,\xi\xi\xi}$$

$$= S^{(n)} , \qquad (2.13)$$

where $S^{(n)}$ depends on $u^{(1)},\cdots,u^{(n-1)}$ and independent of $u^{(n)}$. Hence the compatibility condition becomes the linear equation for $u^{(n)}$, $(n\geq 2)$

$$\mathcal{L}(u^{(1)})u^{(n)} = s^{(n)}(u^{(1)}, u^{(2)}, \cdots, u^{(n-1)}) , \qquad (2.14)$$

where $s^{(n)}$ $(\equiv L^{(0)}S^{(n)}/(L^{(0)}R^{(0)})$ is known function of $(\xi,\tau)$ whenever $u^{(1)},\cdots,u^{(n-1)}$ are known. On the other hand, eq.(2.13) gives

$$U^{(n+1)} = R^{(0)}u^{(n+1)} + \int V^{(n+1)}d\xi , \qquad (2.15a)$$

where $V^{(n+1)}$ and $\hat{S}^{(n)}$ are introduced by

$$V^{(n+1)} = [\hat{R}^{(0)}u^{(1)}u^{(n)} + \hat{K}^{(0)}R^{(0)}u^{(n)}_{,\xi\xi}]_{,\xi} + \hat{S}^{(n)} \qquad (2.15b)$$

$$W^{(0)}\hat{S}^{(n)} = S^{(n)} - s^{(n)}R^{(0)} . \qquad (2.15c)$$

It is to be noted that eq.(2.14) has the same homogeneous part for all n. We also remark that eq.(2.14) admits the secular solution, if $s^{(n)}$ contains a term $\bar{s}^{(n)}$ satisfying the homogeneous equation (the linearized KdV equation), e.g. $u^{(n)}=\tau\bar{s}^{(n)}+f^{(n)}$, where $f^{(n)}$ is given by $\mathcal{L}(u^{(1)})f^{(n)}=s^{(n)}-\bar{s}^{(n)}$. For example, if $u^{(1)}$ is the one-soliton solution of the KdV equation (2.7),

$$u^{(1)} = -2\kappa^2 \operatorname{sech}^2\eta , \quad \eta = \kappa(\xi-4\kappa^2\tau) + \theta , \qquad (2.16)$$

then $u^{(1)}_{,\eta} \propto \mathrm{sech}^2\eta\ \tanh\eta$ satisfies the linearized KdV equation; hence if $s^{(n)}$ contains a term proportional to $\mathrm{sech}^2\eta\ \tanh\eta$ we have the secular solution. This secular solution arises due to the self-resonance in the way closely analogous to that in the non-linear oscillation. Hence the term $\bar{s}^{(n)}$ in $s^{(n)}$ which gives rise to the secular term will be called the resonance term.

## §3. Solution to the Linearized KdV Equation

Since the resonance term is a solution to the linearized KdV equation, it is required to examine the properties of solutions of this equation. We first mention that the solution w to the linearized KdV equation

$$\mathcal{L}(u^{(1)})w = 0 \tag{3.1}$$

is expressed by

$$w = \frac{\partial}{\partial\xi}\Big[\sum_m^N c_m\psi_m^2(\xi,\tau)\exp(-8\kappa_\ell^3\tau) + \int_{-\infty}^{\infty} c(k)\psi^2(\xi,\tau;k)\exp(-8ik^3\tau)dk\Big] \ . \tag{3.2}$$

Here $c_m$ and $c(k)$ are arbitrary constants and $\psi$ is given by the inverse scattering scheme for the KdV equation (2.7):

$$-\psi_{,\xi\xi} + u^{(1)}\psi = k^2\psi \ , \tag{3.3a}$$

$$\psi_{,\tau} + 4\psi_{,\xi\xi\xi} - 6u^{(1)}\psi_{,\xi} - 3u^{(1)}_{,\xi}\psi = 4ik^3\psi \ , \tag{3.3b}$$

(c.f. Appendix A). Eq.(3.2) demonstrates that the solution to the linearized KdV equation (3.1) is given by the superposition of the derivative of the squared eigenfunction. Eq.(3.2) is proved by direct substitution: assuming $w=(\phi^2)_{,\xi}$ and substituting this in eq.(3.1) we find that the left hand side vanishes if $\phi$ satisfies eqs.(3.3), i.e. $\phi=\psi\exp(-4ik^3\tau)$. However, as will be shown in Appendix B, the set of functions $(\psi_m^2)_{,\xi}$ $(m=1,2,\cdots,N)$ and $(\psi^2(k))_{,\xi}$ $(-\infty<k<\infty)$ is not complete. Therefore the expression (3.2) does not give all the solutions of eq.(3.1), but it covers the set of the solutions of eq.(3.1) which are bounded for $\tau\to\infty$ (see Appendix B). On the other hand, the conserved densities of the KdV equation (2.7), which we denote by $\mathcal{A}_i(u^{(1)})$, are expressed similarly in terms of the squared eigenfunction as follows,

$$\mathcal{A}_{j+1} = (-1)^{j+1}\, 2\sum_m^N \kappa_m^{2j+1}\, C_m(\tau)\psi_m^2(\xi,\tau) + \frac{i}{\pi}\int_{-\infty}^{\infty} k^{2j+1}\, r(k)\psi^2(\xi,\tau;k)\exp(-8ik^3\tau)dk\ . \qquad (3.4)$$

The proof is given in Appendix C. Incidentally the expressions of $\mathcal{A}_j(u^{(1)})$ in terms of $u^{(1)}$ may be found by the recursion formulae

$$\mathcal{A}_{j+1,\xi} = -\frac{1}{4}(\mathcal{A}_{j,\xi\xi\xi}-4u^{(1)}\mathcal{A}_{j,\xi}-2u^{(1)}_{,\xi}\mathcal{A}_j)\ ,\quad \mathcal{A}_0=1\ , \qquad (3.5)$$

which enables us to immediately write down the expressions for the first three densities,

$$\mathcal{A}_1 = (1/2)u^{(1)}\ ,\quad \mathcal{A}_2 = (1/8)[3(u^{(1)})^2 - u^{(1)}_{,\xi\xi}]\ ,$$
$$\mathcal{A}_3 = -(1/32)[-10(u^{(1)})^3 + 5(u^{(1)}_{,\xi})^2 + 10u^{(1)}u^{(1)}_{,\xi\xi} - u^{(1)}_{,\xi\xi\xi\xi}]\ . \qquad (3.6)$$

Finally, we note that the solutions $(\psi_m^2)_{,\xi}$ correspond to the variation of the initial phases of the solitons $\delta u^{(1)}/\delta\theta_m$ which is seen easily by the one-soliton case, while $(\psi^2(\xi,\tau;k))_{,\xi}$

corresponds to that of the reflection coefficient $\delta u^{(1)}/\delta r_0(k)$. It is then understood that corresponding to the variation of the eigenvalue $\kappa_m$, $\delta u^{(1)}/\delta\kappa_m$ $(m=1,\cdots,N)$ also satisfy the linearized KdV equation, however as can be seen again by the one-soliton solution those are secular solutions growing proportional to $\tau$. It is shown in Appendix B that the set of functions given by the variations with respect to the full scattering data constitutes the complete set. Therefore the initial function can be expanded by the complete set of functions so that the resultant solution of the linearized KdV equation is obtained automatically. The existence of the unbounded (secular) solution of the linearized KdV equation does not contradict the stability of the soliton. Usually, the soliton is considered as stable, because the point spectrum of the Schrödinger operator does not change under initial perturbations. However for the carefully specified initial condition, which is given corresponding to $\delta u^{(1)}/\delta\kappa_m$ i.e. for the one soliton case by $2\operatorname{sech}^2(\kappa\xi+\theta)+\operatorname{sech}^2(\kappa\xi+\theta)\tanh(\kappa\xi+\theta)-2\kappa\xi\operatorname{sech}^2(\kappa\xi+\theta)\cdot\tanh(\kappa\xi+\theta)$, the point spectrum shifts by the order $\varepsilon$.

## §4. The Method of Renormalization

We first consider the purely one-soliton solution (2.16) for $u^{(1)}$. Introducing this equation into eq.(2.12) yields,

$$s^{(2)} = \kappa^7(b_1^{(2)}\operatorname{sech}^2\eta\,\tanh\eta+b_2^{(2)}\operatorname{sech}^4\eta\,\tanh\eta+b_3^{(2)}\operatorname{sech}^6\eta\,\tanh\eta)\ , \quad (4.1)$$

where $b_j^{(2)}$ $(j=1,2,3)$ are constants independent of $\kappa$. Here we note the following relations which will be used throughout in the subsequent computations;

$$\frac{d}{d\eta}\,\mathrm{sech}^{2n}\eta = -2n\ \mathrm{sech}^{2n}\eta\ \tanh\eta\ , \qquad (4.2a)$$

$$\frac{d}{d\eta}\,\mathrm{sech}^{2n}\eta\ \tanh\eta = -2n\ \mathrm{sech}^{2n}\eta + (1+2n)\,\mathrm{sech}^{2(n+1)}\eta\ . \qquad (4.2b)$$

The first term in the bracket of eq.(4.1) satisfies the linearized KdV equation (3.1), whilst the other terms do not. Hence the first term is the resonance term. It should be noted that the resonance term derives from the linear term in $s^{(2)}$, and this holds in any high order. We now attempt to eliminate this resonant term in all the $s^{(n)}$'s. For illustration, we first consider the set of eqs.(2.7) and (2.14) for $u^{(n)}$ $(n\geq 1)$ as the basic system to solve, which may be inclusively written as

$$\varepsilon\mathcal{K}(u^{(1)}) + \sum_{n\geq 2}\varepsilon^n\mathcal{L}(u^{(1)})u^{(n)} = \sum_{n\geq 2}\varepsilon^n s^{(n)}\ . \qquad (4.3)$$

Then, we add on both sides of eq.(4.3) the term $\sum_{n\geq 1}\varepsilon^n\delta\lambda u^{(n)}_{,\xi}$, where $\delta\lambda$ is given in a powerseries of $\varepsilon$, $\delta\lambda=\varepsilon\lambda^{(1)}+\varepsilon^2\lambda^{(2)}+\cdots$, with coefficients to be determined later. The crucial point in our procedure is that $\delta\lambda$ on the left hand side is not expanded while on the right hand side it is expanded so that $\lambda^{(n)}$'s are determined successively to cancel out the resonant term in $s^{(n)}$. Then the KdV equation is modified to

$$\tilde{u}^{(1)}_{,\tau} - 6\tilde{u}^{(1)}\tilde{u}^{(1)}_{,\xi} + \tilde{u}^{(1)}_{,\xi\xi\xi} + \delta\lambda\tilde{u}^{(1)}_{,\xi} = 0\ , \qquad (4.4)$$

whilst eq.(2.14) for $\tilde{u}^{(2)}$ becomes

$$\mathcal{L}(\tilde{u}^{(1)})\tilde{u}^{(2)} + \delta\lambda\tilde{u}^{(2)}_{,\xi} = \tilde{s}^{(2)} + \lambda^{(1)}\tilde{u}^{(1)}_{,\xi}\ , \qquad (4.5)$$

in which $\tilde{s}^{(2)}$ is defined by $\tilde{s}^{(2)} \equiv s^{(2)}(\tilde{u}^{(1)})$. From eq.(4.5) we obtain $\lambda^{(1)} = -\kappa^4 b_1^{(2)}/4$, hence up to this order $\delta\lambda$ is given by $-\varepsilon\kappa^4 b_1^{(2)}/4$. The one-soliton solution of eq.(4.4) derives by the Galilei-transformation $\tilde{\xi} = \xi - \delta\lambda\tau$, that is,

$$\tilde{u}^{(1)} = -2\kappa^2 \operatorname{sech}^2\tilde{\eta} , \quad \tilde{\eta} = \kappa\{\xi - (4\kappa^2 + \delta\lambda)\tau\} + \theta . \tag{4.6}$$

Thus the higher order effect is given by the renormalization of the soliton velocity. In this sence we call eq.(4.4) as the renormalized KdV equation. With the $\tilde{u}^{(1)}$ and $\delta\lambda$, eq.(4.5) becomes

$$\begin{aligned} &\tilde{u}^{(2)}_{,\tau} - 6(\tilde{u}^{(1)}(\tilde{\eta})\tilde{u}^{(2)})_{,\tilde{\xi}} + \tilde{u}^{(2)}_{,\tilde{\xi}\tilde{\xi}\tilde{\xi}} \\ &= \kappa^7 (b_2^{(2)} \operatorname{sech}^4\tilde{\eta} \tanh\tilde{\eta} + b_3^{(2)} \operatorname{sech}^6\tilde{\eta} \tanh\tilde{\eta}) . \end{aligned} \tag{4.5$'$}$$

By means of eqs.(4.2), a particular solution of this equation is obtained as

$$\tilde{u}_p^{(2)} = \kappa^4 (\beta_1^{(2)} \operatorname{sech}^2\tilde{\eta} + \beta_2^{(2)} \operatorname{sech}^4\tilde{\eta}) , \tag{4.6$'$}$$

where $\beta_1^{(2)} = -\frac{1}{96}(2b_2^{(2)} + b_3^{(2)}), \beta_2^{(2)} = b_3^{(2)}/48$. The general solution of eq.(4.5)' for $\tilde{u}^{(2)}$ is given by

$$\tilde{u}^{(2)} = \tilde{u}_0^{(2)} + \tilde{u}_p^{(2)} , \tag{4.7}$$

in which $u_0^{(2)}$ is a solution of the homogeneous equation for eq.(4.5)', the linearized KdV equation. However, as was noted already, the linearized KdV equation admits the secular solution, which is given by $\delta u^{(1)}/\delta\kappa$. Therefore, in general, $\tilde{u}_0^{(2)}$ should be expressed as

$$\tilde{u}_0^{(2)} = \tilde{w}^{(2)} + c_\kappa(\delta\tilde{u}^{(1)}/\delta\kappa) , \qquad (4.8)$$

where $\tilde{w}^{(2)}$ is given by eq.(3.2), being bounded for $\tau\to\infty$. The secular term in $\tilde{u}_0^{(2)}$ can be eliminated as follows. Solve the Schrödinger equation (3.3a) for the initial function $\tilde{u}^{(1)}(\xi,0) +\varepsilon c_\kappa(\delta\tilde{u}^{(1)}/\delta\kappa)|_{\tau=0}$. Then the eigenvalue $\kappa$ is shifted by which the evolution of $u^{(1)}$ is determined, while the special initial function $c_\kappa(\delta\tilde{u}^{(1)}/\delta\kappa)|_{\tau=0}$ must be subtracted from the initial function $\tilde{u}^{(2)}(\xi,0)$ so that $\tilde{u}_0^{(2)}$ becomes $\tilde{w}^{(2)}$. As the result, the second order effects on the KdV soliton are given by the renormalization of the amplitude and the width as well as the velocity. Since the velocity-shift can be replaced by the phase-shift, it may be stated that the second order effects are given by the renormalization of the scattering data. By means of the mathematical induction, it can be proved that $\lambda^{(n-1)}$, $\tilde{s}^{(n)}$ and the particular solution $\tilde{u}_p^{(n)}$ are given by

$$\lambda^{(n-1)} = \kappa^{2n} d^{(n-1)} , \qquad (4.9a)$$

$$\tilde{u}_p^{(n)} = \kappa^{2n} \sum_{j=1}^{n} \beta_j^{(n)} \operatorname{sech}^{2j}\tilde{\eta} , \qquad (4.9b)$$

$$\tilde{s}^{(n)} = \kappa^{2n+3} \sum_{j=2}^{n+1} b_j^{(n)} \operatorname{sech}^{2j}\tilde{\eta} \tanh\tilde{\eta} , \qquad (4.9c)$$

where $d^{(n)}$, $\beta_j^{(n)}$, $b_j^{(n)}$ are constants independent of $\kappa$, consequently $u^{(1)}$. Here it is to be noted that, for the one-soliton solution, the conserved density $\mathcal{A}_{j+1}(u^{(1)})$ becomes

$$\mathcal{A}_{j+1} = -\frac{1}{2}(-1)^{j+1} \kappa^{2j} u^{(1)} . \qquad (4.10)$$

Therefore, we have

$$\delta\lambda u^{(1)}_{,\xi} = \sum_{j=2} \varepsilon^{j-1} a^{(j-1)} \mathcal{A}_{j+1,\xi} \quad , \tag{4.11}$$

where $a_j$'s are constants independent of $\kappa$, that is, independent of $u^{(1)}$. Thus the renormalized KdV equation becomes well established one for $u^{(1)}$, when $\delta\lambda u^{(1)}_{,\xi}$ is expressed by (4.11).

We now show that for the N-soliton solution, the renormalization term can be represented samely by the higher order conserved densities. This may be seen by using the expression for the N-soliton wave function $u^{(1)}_N$

$$u^{(1)}_N = -\sum_{m=1}^{N} 4\kappa_m C_m \psi_m^{\,2} \quad , \tag{4.12}$$

where

$$\begin{aligned} C_m\psi_m^{\,2} &\to \frac{1}{2}\kappa_m \operatorname{sech}^2\eta_m \quad \text{for} \quad \tau \to \infty , \\ \eta_m &= \kappa_m(\xi - 4\kappa_m^{\,2}\tau) + \theta_m . \end{aligned} \tag{4.13}$$

Consequently, from eq.(3.4) we have

$$\mathcal{A}_{j+1} \to (-1)^{j+1} \sum_{m}^{N} \kappa_m^{\,2(j+1)} \operatorname{sech}^2\eta_m \ , \quad \text{for} \quad \tau \to \infty . \tag{4.14}$$

On the other hand from eq.(4.1) $s^{(2)}$ becomes

$$\begin{aligned} s^{(2)} \to &\sum_{m}^{N} \kappa_m^{\,7} (b_1^{(2)} \operatorname{sech}^2\eta_m \tanh\eta_m) \\ &+ \sum_{m}^{N} \kappa_m^{\,7} (b_2^{(2)} \operatorname{sech}^4\eta_m \tanh\eta_m + b_3^{(2)} \operatorname{sech}^6\eta_m \tanh\eta_m) \ , \end{aligned} \tag{4.15}$$

in which the coefficients $b_i^{(2)}$ $(i=1,2,3)$ are independent of $\kappa_m$ hence m. Consequently, the first term which is to be eliminated is equal to

$$-(b_1^{(2)}/2)\cdot(\partial \mathcal{A}_3/\partial\xi) \ . \tag{4.16}$$

Therefore, a generalization to the N-soliton case can be deduced as follows. Corresponding to $\sum_{n\geq 1} \varepsilon^n \delta\lambda u_{,\xi}^{(n)}$, add, on both sides of eq.(4.3) the term

$$\sum_{n\geq 1} \varepsilon^n \sum_{j\geq 2} \delta\lambda_j (\mathcal{A}_{j+1}^{(n)})_{,\xi}$$

where $\mathcal{A}_j^{(1)}=\mathcal{A}_j(u^{(1)})$, $\mathcal{A}_j^{(n)}\equiv(d/d\nu)\mathcal{A}_j(u^{(1)}+\nu u^{(n)})|_{\nu=0}=\mathcal{A}_j'u^{(n)}$ $(n\geq 2)$ and $\delta\lambda_j=\varepsilon^{j-1}a^{(j-1)}$ to which the same rule of expansion as that in the one-soliton case is applied. Then the renormalized KdV equation takes the form

$$\tilde{u}_{,\tau}^{(1)} - 6\tilde{u}^{(1)}\tilde{u}_{,\xi}^{(1)} + \tilde{u}_{,\xi\xi\xi}^{(1)} + \sum_{j\geq 2}\delta\lambda_j \mathcal{A}_{j+1}(\tilde{u}^{(1)})_{,\xi} = 0 \ , \tag{4.17}$$

which is called the generalized KdV equation and may be written as

$$\tilde{u}_{,\tau}^{(1)} + \tilde{\mathcal{A}}_{,\xi} = 0 \ , \tag{4.17$'$}$$

where $\tilde{\mathcal{A}}=\sum_{j\geq 1}\delta\lambda_j\tilde{\mathcal{A}}_{j+1}$ with $\delta\lambda_1\equiv -8$ and $\tilde{\mathcal{A}}_j\equiv\mathcal{A}_j(\tilde{u}^{(1)})$. The higher order equations are correspondingly modified, and corresponding to (4.5) we have

$$\tilde{\mathcal{L}}(\tilde{u}^{(1)})\tilde{u}^{(2)} \equiv \tilde{u}_{,\tau}^{(2)} + \mathcal{A}_{,\xi}^{(2)} = \tilde{s}^{(2)} + a^{(1)}\tilde{\mathcal{A}}_{3,\xi} \ . \tag{4.18}$$

The lefthand side of this equation is given by the linearization of the generalized KdV equation (4.18), while from eq.(4.16) it is anticipated that $a^{(1)}=(b_1^{(2)}/2)$.

The generalized KdV equation is completely integrable.[3)]

Namely it is solvable by means of the inverse scattering scheme, where the Schrödinger equation is valid samely so that the spectrum is not altered by the presence of the higher conserved densities, only the evolution equation for $\psi$ is modified as is given in Appendix D. Consequently, for $\tau\to\infty$, the N-soliton solutions are obtained by modifying the velocities of the solitons (see also Appendix D). Therefore the higher order effects are given by the renormalization of the velocities of the KdV soliton. The explicit form of the N-soliton solution (for $\tau\to\infty$) may be found by the following heuristic arguments. Assume that for $\tau\to\infty$

$$\tilde{u}_N^{(1)} \to \sum_{m=1}^{N} - 2\kappa_m^2 \operatorname{sech}^2\tilde{\eta}_m , \tag{4.19}$$

where $\tilde{\eta}_m=\kappa_m\{\xi-(4\kappa_m^2+\delta\lambda_m)\tau\}+\theta_m$ and $\delta\lambda_m$ is constant to be determined. Then $\tilde{\psi}_m$ (the m-th eigenfunction of the Schrödinger equation (3.3a) with potential $\tilde{u}_N^{(1)}$) takes the form $\sqrt{\tilde{c}_m}\ \tilde{\psi}_m \to \sqrt{\kappa_m/2}\ \operatorname{sech}\tilde{\eta}_m$. Hence by means of the definition of $\mathcal{A}_j$ we have

$$\mathcal{A}_{j+1} \to (-1)^{j+1} \sum_{m}^{N} \kappa_m^{2(j+1)} \operatorname{sech}^2\tilde{\eta}_m . \tag{4.20}$$

Introducing eqs.(4.19) and (4.20) in eq.(4.17) yields

$$\widetilde{\delta\lambda}_m = - \sum_{j\geq 2} \frac{1}{2}\, \delta\lambda_j (-1)^{j+1}\, \kappa_m^{2j} , \tag{4.21}$$

which is in agreement with that obtained in Appendix D by means of the inverse machienery.

It is also worthwhile to note that the generalized KdV equation (4.17) is given in terms of the conserved quantities, $\tilde{I}_j\equiv\int\tilde{A}_j d\xi$. That is, the last (renormalization) term on the left-hand side of eq.(4.17) is written as $\sum_{j\geq 2} \delta\nu_j \delta\tilde{I}_{j+2}/\delta\tilde{u}^{(1)}$ where

$\delta\nu_{j-1}$ is defined by $[2/(2j+1)]\delta\lambda_{j-1}$ (Appenfix C). Hence eq.(4.17) is expressed by the canonical form

$$\frac{\partial\tilde{u}^{(1)}}{\partial\tau} + \frac{\partial}{\partial\xi}\left[-\frac{16}{5}\frac{\delta\tilde{I}_3}{\delta\tilde{u}^{(1)}} + \sum_{j\geq 2}\delta\nu_j\frac{\delta\tilde{I}_{j+2}}{\partial\tilde{u}^{(1)}}\right] = 0\ . \qquad (4.22)$$

Here we note that in the strongly dispersive system reducible to the nonlinear Schrödinger equation, the renormalization term cannot be represented by the conserved densities but by the conserved quantities.[4] Therefore the canonical form (4.22) is more general than eq.(4.17).

With the lowest order solution $\tilde{u}^{(1)}$ given by eq.(4.17), we proceed to the next order equation (4.18), which has properties similar to those of eq.(2.10). In general, when $\tilde{u}^{(1)}$ satisfies eq.(4.17), $\tilde{\mathcal{A}}_{j,\xi}$ $(j=1,2,\cdots)$ satisfies the homogeneous equation for eq.(4.18). To prove this, using (4.17)' we first get $\tilde{\mathcal{A}}_{j,\tau}=-\tilde{\mathcal{A}}'_j\tilde{\mathcal{A}}_{,\xi}=-\tilde{\mathcal{A}}'_j\tilde{\mathcal{A}}'\tilde{u}^{(1)}_{,\xi}$, then by means of the relation

$$(\tilde{\mathcal{A}}'_j\tilde{\mathcal{A}}'-\tilde{\mathcal{A}}'\tilde{\mathcal{A}}'_j)\tilde{u}^{(1)}_{,\xi} = 0$$

which was first derived by Lax[5] and is proved in Appendix E, we obtain $\tilde{\mathcal{A}}_{j,\tau}+\tilde{\mathcal{A}}'\tilde{\mathcal{A}}_{j,\xi}= 0$, hence differentiating once with respect to $\xi$ gives

$$(\tilde{\mathcal{A}}_{j,\xi})_{,\tau} + (\tilde{\mathcal{A}}'\tilde{\mathcal{A}}_{j,\xi})_{,\xi} = 0.$$

Since the homogeneous part takes the same form in all order, this is valid in any higher order. Also, for the N-soliton wave function of $u^{(1)}$, in the limit $\tau\to\infty$, the extra term on the left-hand side of eq.(4.18) due to the renormalization $(\sum_{j\geq 2}\delta\lambda_j\tilde{\mathcal{A}}'_{j+1}\tilde{u}^{(2)})_{,\xi}$ can be eliminated locally by means of the Galilei-transformation

$\xi \to \tilde{\xi}_m = \xi - \delta\tilde{\lambda}_m \tau$ $(m=1,2,\cdots,N)$, where $\delta\tilde{\lambda}_m$ is given by eq.(4.21). In other words this term may be discarded provided $\xi$ is replaced by $\tilde{\xi}_m$. We thus find that for $\tau\to\infty$, discussions go parallel to that of the one-soliton case, because in the limit $\tau\to\infty$ the N-solitons are locally equivalent to the one soliton.

So far we have considered the particular solutions $u_p^{(n)}$ $(n\geq 2)$ assuming that $u_0^{(n)}=0$ $(n\geq 2)$. Since the equations for $u^{(n)}$ $(n\geq 2)$ are linear, the general solution can be obtained by adding to $u_p^{(n)}$ the homogeneous solutions $u_0^{(n)}$. In this case, when initial functions (3.2) is given by the point spectrum $\sum_m c_m \tilde{\psi}_m^{\,2}{}_{,\xi}$, $u_0^{(n)}$ thus specified will not give rise to the new secular term in $s^{(n+1)}$. This may be shown by the one-soliton case. Let $\tilde{u}_0^{(2)}$ be given by $-4\kappa C\psi^2{}_{,\xi}$ $\propto \mathrm{sech}^2\tilde{\eta}\, \tanh\tilde{\eta}$, then in $\tilde{s}^{(3)}$ it gives even functions, hence $\tilde{u}_0^{(2)}$ does not produce the secular term.

So far, we have considered the system of equations given by eqs.(2.7) and (2.14) as the basic set to be solved. However, the renormalization for the original system of equations, which may be taken as the set of eqs.(2.6), (2.9) and (2.13), is achieved similarly. Namely we first add on the lefthand side of eq.(2.6) the term $\varepsilon a^{(1)}\mathcal{A}_{3,\xi}R^{(0)}$. Then it yields the generalized KdV equation (4.22) while $\tilde{U}^{(2)}$ takes the same form as $U^{(2)}$, because $\tilde{u}^{(1)}_{,\tau}$ is eliminated by eq.(4.17). In the next-order equation (2.9) the terms $\varepsilon a^{(1)}\tilde{\mathcal{A}}'_3\tilde{u}^{(2)}_{,\xi}R^{(0)}$ and $\varepsilon a^{(1)}\tilde{\mathcal{A}}_{3,\xi}R^{(0)}$ are to be added on the left- and right-hand sides respectively. It is then obvious that eq.(4.18) for $\tilde{u}^{(2)}$ is reproduced, while eq.(2.9) so modified gives

$$\tilde{U}^{(3)} = R^{(0)}\tilde{u}^{(3)} + \int \tilde{V}^{(3)} d\xi \ ,$$

where $$\tilde{V}^{(3)} \equiv V^{(3)}(\tilde{u}^{(2)},\tilde{u}^{(1)}) \ .$$

The same applies to eq.(2.13). Thus in all order discussions go parallel to those done previously for the set of eqs.(2.6) and (2.14).

Finally we show that the coefficients $\delta\lambda_j$ in eq.(4.17) can be determined by the linear dispersion relation (2.4). Since $\delta\lambda_j$ are independent of $\tilde{u}^{(1)}$, they may be determined by the one-soliton solution of $\tilde{u}^{(1)}$. In this case, as can be seen frome eq.(4.9b) for $x\to\infty$ $\tilde{u}^{(n)}$ $(n\geq 1)$ damps at most as $\exp(-2\tilde{\eta})$. Hence from eq.(2.15) it follows that U is approximated by $U-U_0\propto\exp(-2\tilde{\eta})$ for $x\to\infty$. Substituting this expression for U in eq.(2.1) and linearizing, we get

$$\det|(\varepsilon^{1/2})\{-\lambda_0-\varepsilon(4\kappa^2+\delta\tilde{\lambda})\}I+A_0(\varepsilon^{1/2}\kappa)+4K_0(\varepsilon^{1/2}\kappa)^3|=0$$

that is

$$\det|[-\lambda_0-\varepsilon\{4\kappa^2-\sum_{j\geq 2}\frac{1}{2}(-1)^{j+1}\kappa^{2j}\delta\lambda_j\}]I+A_0+K_0(4\varepsilon\kappa^2)|=0 .$$

In comparison with the linear dispersion relation, one sees immediately

$$\lambda_0+\varepsilon\{4\kappa^2+\frac{1}{2}\kappa^4\ \delta\lambda_2-\frac{1}{2}\kappa^6\ \delta\lambda_3\pm\cdots\cdots\}$$

$$=[\omega/k]_{k=i\varepsilon^{1/2}(2\kappa)} . \tag{4.23}$$

## §5. Example (Ion Acoustic Wave[1])

For the system composed of warm electrons and cold ions, the basic equations can be reduced to

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\,v) - 2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(\frac{1}{n}\frac{\partial n}{\partial x}\right) = 0 \,, \qquad (5.1a)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{n}\frac{\partial n}{\partial x} = 0 \,, \qquad (5.1b)$$

in which x and t are stretched by the factor $\sqrt{2}$ in comparison to those in ref.1, n is the density of the electrons and v the flow velocity of the ions. This system can not be given in the matrix form (2.1), because of the existence of the time-derivative in the dispersive term; however, as was shown in ref.1, the reduction goes entirely parallel to that of eq.(2.1). That is, for $u_1=n$, $u_2=v$, we have $\lambda=v\pm 1$, hence for $n_1=1$, $v=0$ the linear dispersion relation is given by $\omega=\pm k(1+2k^2)^{-1/2}=\pm k(1-k^2+\frac{3}{2}k^4\pm\cdots)$. Consequently, we have the KdV eq.(2.7) for $u^{(1)}$ where $n^{(1)}=v^{(1)}=-6u^{(1)}$, while

$$n^{(2)} = -6u^{(2)} \,, \qquad v^{(2)} = -6u^{(2)} - 18(u^{(1)})^2 - 6u^{(1)}_{,\xi\xi} \,. \qquad (5.2)$$

The explicit form of $s^{(2)}$ is obtained as

$$s^{(2)} = 18(u^{(1)})^2 u^{(1)}_{,\xi} - 3u^{(1)}u^{(1)}_{,\xi\xi} - 6u^{(1)}u^{(1)}_{,\xi\xi\xi} - \frac{3}{2}u^{(1)}_{,\xi\xi\xi\xi\xi} \,.$$

Hence, after the renormalization we get

$$\tilde{u}^{(1)}_{,\tau} - 6\tilde{u}^{(1)}\tilde{u}^{(1)}_{,\xi} + \tilde{u}^{(1)}_{,\xi\xi\xi} + \varepsilon 48\tilde{\mathcal{A}}_3(\tilde{u}^{(1)})_{,\xi} + \cdots = 0 \,.$$

$$\tilde{\mathcal{L}}\tilde{u}^{(2)} = (63\tilde{u}^{(1)2}\tilde{u}^{(1)}_{,\xi} - 33\tilde{u}^{(1)}_{,\xi}\tilde{u}^{(1)}_{,\xi\xi} - 21\tilde{u}^{(1)}\tilde{u}^{(1)}_{,\xi\xi\xi}) \,.$$

The N-soliton solution becomes

$$\tilde{u}^{(1)} \to \sum_{m}^{N} - 2\kappa_m^{\ 2} \ \mathrm{sech}^2 \tilde{\eta}_m \ ,$$

where $\tilde{\eta}_m \equiv \kappa_m\{\xi - (4\kappa_m^{\ 2} - \varepsilon 24\kappa_m^{\ 2} + O(\varepsilon^2))\tau\} + \theta_m$ , and the dressed part is expressed by

$$\tilde{u}^{(2)} \to \sum_{m} [-9\kappa_m^{\ 4}\{\mathrm{sech}^2\tilde{\eta}_m + 6\mathrm{sech}^4\tilde{\eta}_m\}] ;$$

$\tilde{v}^{(1)} = \tilde{n}^{(1)} = -6\tilde{u}^{(1)}$ and the equations for $\tilde{n}^{(2)}$ and $\tilde{v}^{(2)}$ take the same form as eq.(5.2). (Note that $\delta\lambda_2 = 48\varepsilon$ derives directly from eq.(4.23)).

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## Appendix A. Inverse Scattering Scheme

For eq.(3.3a) we introduce the Jost functions $\psi$ and $\phi$ with the boundary conditions[6)] for the right scattering

$$\psi(\xi,\tau;k) \to e^{-ik\xi} \qquad \xi \to -\infty , \tag{A.1a}$$

$$\phi(\xi,\tau;k) \to e^{ik\xi} \qquad \xi \to \infty . \tag{A.1b}$$

For real k, it holds

$$\begin{gathered} \psi(\xi,\tau;k) = a(k,\tau)\phi(\xi,\tau;-k) + b(k,\tau)\phi(\xi,\tau;k) \\ |a(k,\tau)|^2 - |b(k,\tau)|^2 = 1 \end{gathered} \tag{A.2}$$

and the bound states are given by

$$a(i\kappa_\ell , \tau) = 0 \quad (k=i\kappa_\ell , \ \ell=1,2,\cdots,N) \tag{A.3a}$$

$$\psi(\xi,\tau;i\kappa_\ell) = b_\ell(\tau)\phi(\xi,\tau;i\kappa_\ell) . \tag{A.3b}$$

From eq.(3.3b) we get

$$\begin{aligned} a(k,\tau) &= a_0(k) \\ b(k,\tau) &= b_0(k)\exp(8ik^3\tau) \\ B_\ell(\tau) &\equiv b_\ell(\tau)/(i\mathring{a}(i\kappa_\ell)) = B_{\ell 0} \exp(8\kappa_\ell^3 \tau) . \end{aligned} \tag{A.4}$$

Hence the right scattering data $S(\tau)$ are given by

$$S(\tau) = [R(k,\tau)\equiv b(k,\tau)/a(k)=R_0(k)e^{8ik^3\tau}, \{B_{\ell 0}e^{8\kappa_\ell^3\tau}, \kappa_\ell\}_{\ell=1}^{N}] \tag{A.5a}$$

and the left scattering data are given by

$$s(\tau) = [r(k,\tau) \equiv b(-k,\tau)/a(k) = r_0(k)e^{-8ik^3\tau}, \{C_{\ell 0}e^{-8\kappa_\ell^3\tau}, \kappa_\ell\}_{\ell=1}^{N}] \,. \tag{A.5b}$$

Appendix B. Squared-Eigenfunctions

We introduce the squared-eigenfunction and its spatial derivative by

$$\Phi(\xi,\tau;k) \equiv \phi^2(\xi,\tau;k) \equiv \langle\xi|k\rangle \,, \tag{B.1a}$$

$$\Psi(\xi,\tau;k) \equiv \frac{\partial}{\partial\xi}\psi^2(\xi,\tau;k) \equiv \langle k|\xi\rangle \,. \tag{B.1b}$$

By means of eqs.(A.1a) and (A.2) $\Psi$ and $\Phi$ satisfy the equations

$$L_S\Phi = k^2\Phi \,, \qquad L_S^A\Psi = k^2\Psi - \frac{1}{2}u^{(1)}_{,\xi}\psi^2(-R,\tau;k) \tag{B.2}$$

where $$L_S = -\frac{1}{4}\frac{\partial^2}{\partial\xi^2} + u^{(1)} + \frac{1}{2}\int_\xi^\infty d\xi' u^{(1)}_{,\xi'}$$

and $L_S^A$ is adjoint to $L_S$, i.e.

$$L_S^A = -\frac{1}{4}\frac{\partial^2}{\partial\xi^2} + u^{(1)} + \frac{1}{2}u^{(1)}_{,\xi}\int_{-R}^{\xi} d\xi' \,.$$

From eqs.(B.2) we have

$$\Psi L_S\Phi - \Phi[L_S^A\Psi + \frac{1}{2}u^{(1)}_{,\xi}\psi^2(-R,\tau;k)] = \partial_\xi F \tag{B.3}$$

where $$F \equiv -\frac{1}{4}[\Psi\frac{\partial}{\partial\xi}\Phi - \Phi\frac{\partial}{\partial\xi}\Psi]$$

$$+ \frac{1}{2}[\int_{-R}^{\xi}\Psi d\xi' + \psi^2(-R,\tau;k)]\cdot\int_\xi^\infty u^{(1)}_{,\xi'}\Phi d\xi' \,,$$

while the evolutions of $\Psi$ and $\Phi$ are governed by the associated KdV equation[7] and the linearized KdV equation, respectively.
Let the innerproduct be defined by

$$\int \Psi(\xi,k)\Phi(\xi,k')d\xi \equiv \langle k|k'\rangle \,. \tag{B.4}$$

Then in view of eqs.(B.2) it becomes

$$\langle k|k'\rangle = \frac{1}{k^2-k'^2}\,\frac{1}{4}[\Psi(\xi,k)\frac{\partial}{\partial\xi}\Phi(\xi,k') - \Phi(\xi,k')\frac{\partial}{\partial\xi}\Psi(\xi,k)]\Big|_{\xi=-\infty}^{\xi=\infty} \,.$$

Hence for real k, k' we have

$$\langle k|k'\rangle = -2i\pi ka^2(k)\delta(k-k') \;; \tag{B.5}$$

for complex k, k' with positive imaginary parts, at the zero points of a(k) we get

$$\frac{\partial}{\partial k}\langle k|i\kappa_m\rangle|_{k=i\kappa_\ell} = i\kappa_\ell \mathring{a}_\ell^2 \delta_{\ell m} \,. \tag{B.6}$$

We also obtain

$$\langle i\kappa_\ell|i\kappa_m\rangle = \langle k|i\kappa_m\rangle = 0 \tag{B.7}$$

and
$$\frac{\partial}{\partial k}\cdot\frac{\partial}{\partial k'}\langle k|k'\rangle|_{\substack{k=i\kappa_\ell\\ k'=i\kappa_m}} = (\mathring{a}_\ell^2 + i\kappa_\ell \mathring{a}_\ell \ddot{a}_\ell)\delta_{\ell m} \,. \tag{B.8}$$

The completeness relation can be shown in the same way as was done by Kaup[8] by considering the integral

$$J \equiv i\oint \frac{dk}{2\pi}\,|k\rangle\,\frac{1}{ka^2(k)}\,\langle k|$$

where the contour C passes through the upper half plane above all the zero points of $a(k)$, namely

$$J = i\int_{-\infty}^{\infty} \frac{dk}{2\pi}|k\rangle \frac{1}{ka^2(k)} \langle k| + \sum_{\ell=1}^{N} \frac{1}{i\kappa_\ell \dot{a}_\ell^2}\left|\frac{\partial}{\partial k}(|k\rangle\langle k|)\right|_{k=i\kappa_\ell}$$

$$+ \sum_{\ell=1}^{N} \left(\frac{1}{\kappa_\ell^2 \dot{a}_\ell^2} + i\frac{\ddot{a}_\ell}{\kappa_\ell \dot{a}_\ell^3}\right)|i\kappa_\ell\rangle\langle i\kappa_\ell| \ . \tag{B.8}$$

The secular solution for one-soliton solution of $u^{(1)}$, $\frac{\delta u^{(1)}}{\delta\kappa}$ corresponds to

$$e^{-8\kappa^3\tau} \frac{\partial}{\partial k}\Psi(\xi,\tau;k)|_{k=i\kappa} = -\frac{i\kappa}{C_0}(2+\tanh\eta)\ \mathrm{sech}^2\eta$$

$$+ 2\frac{i\kappa^2}{C_0}\ \xi\ \mathrm{sech}^2\eta\ \tanh\eta$$

where $\quad \eta = \kappa(\xi-4\kappa^2\tau) + \theta.$

Appendix C. The Conserved Density and the Conserved Quantity

For the conserved quantity, $I_j=\int \mathcal{A}_j\, d\xi$, we have[9)]

$$I_{j+1} = \frac{2j+1}{2\pi}\int_{-\infty}^{\infty} \ln|a(k)|^2 \cdot k^{2j}\, dk$$

$$+ (-1)^{j+1}\, 2\sum_{m}^{N} \kappa_m^{2j+1} \tag{C.1}$$

Hence $\frac{\delta I_{j+1}}{\delta u^{(1)}}$ is calculated as

$$\frac{\delta I_{j+1}}{\delta u^{(1)}} = \frac{2j+1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{a(k)}\frac{\delta a(k)}{\delta u^{(1)}} + \frac{1}{a^*(k)}\frac{\delta a^*(k)}{\delta u^{(1)}}\right)\cdot k^{2j}\, dk$$

$$+ (-1)^{j+1}\, 2\sum_{m}^{N}(2j+1)\kappa_m^{2j}\frac{\delta\kappa_m}{\delta u^{(1)}} \ . \tag{C.2}$$

Using the relations proved by Kodama and Wadati,[10)]

$$\frac{1}{a(k)}\frac{\delta a(k)}{\delta u^{(1)}} + \frac{1}{a^*(k)}\frac{\delta a^*(k)}{\delta u^{(1)}} = \frac{1}{2ik}[r(k)\psi^2(\xi,k) - r(-k)\psi^2(\xi,-k)] \tag{C.3a}$$

$$\frac{\delta\kappa_m}{\delta u^{(1)}} = -\frac{C_m}{2\kappa_m}\psi^2(\xi,\, i\kappa_m) \;, \tag{C.3b}$$

we obtain

$$\begin{aligned}\frac{2}{2j+1}\frac{\delta I_{j+1}}{\delta u^{(1)}} &= -\frac{i}{\pi}\int_{-\infty}^{\infty} k^{2j-1}\cdot r(k)\psi^2(\xi,k)\,dk \\ &\quad + (-1)^j\, 2\sum_m^N \kappa_m^{\,2j-1} C_m\psi^2(\xi,i\kappa_m) \\ &= \mathcal{A}_j(u^{(1)}) \;. \end{aligned} \tag{C.4}$$

## Appendix D. The Generalized KdV Equation

In the canonical form, the generalized KdV equation is given by

$$\frac{\partial \tilde{u}^{(1)}}{\partial \tau} = \frac{\partial}{\partial \xi} \cdot \frac{\delta H[\tilde{u}^{(1)}]}{\delta \tilde{u}^{(1)}} \tag{D.1}$$

where $H[\tilde{u}^{(1)}] \equiv \frac{16}{5} I_3[\tilde{u}^{(1)}] - \sum_{j\geq 2} \delta\nu_j I_{j+2}[\tilde{u}^{(1)}]$

while the canonical variables are represented by the scattering data

$$P(k) = \frac{k}{\pi} \ln|a(k)|^2 , \quad Q(k) = \arg b(k)$$

$$p_m = \kappa_m^2 , \quad q_m = -2 \ln b_m \quad (m=1,2,\cdots,N) \tag{D.2}$$

consequently

$$I_{j+1} = \frac{2j+1}{2} \int_{-\infty}^{\infty} k^{2j-1} P(k)dk + (-1)^{j+1} 2 \sum_m^N p_m^{(2j+1)/2} . \tag{D.3}$$

Hence we have

$$H = 8 \int_{-\infty}^{\infty} k^3 P(k)dk - \sum_{j\geq 2} \frac{2j+3}{2} \delta\nu_j \int_{-\infty}^{\infty} k^{2j+1} P(k)dk$$

$$- \frac{32}{5} \sum_m^N p_m^{5/2} + \sum_{j\geq 2} 2(-1)^{j+1} \delta\nu_j \sum_m^N p_m^{(2j+3)/2} . \tag{D.4}$$

The canonical equations (Hamilton's equation) are integrated as follows.

For the continuous spectrum,

$$\frac{dP(k)}{d\tau} = -\frac{\delta H}{\delta Q(k)} = 0 \quad \text{i.e.} \quad \frac{d}{d\tau} a(k,\tau) = 0 \ , \tag{D.5a}$$

$$\frac{dQ(k)}{d\tau} = \frac{\delta H}{\delta P(k)} = 8k^3 - \sum_{j\geq 2} \frac{2j+3}{2} \delta\nu_j k^{2j+1} \ , \tag{D.5b}$$

which yields

$$b(k,\tau) = b_0(k)\exp(i8k^3\tau - i \sum_{j\geq 2} \frac{2j+3}{2} \delta\nu_j k^{2j+1} \tau) \ .$$

For the point spectrum,

$$\frac{dP_m}{d\tau} = -\frac{\delta H}{\delta q_m} = 0 \quad \text{i.e.} \quad \frac{d}{d\tau} \kappa_m = 0 \ , \tag{D.5c}$$

$$\frac{dq_m}{d\tau} = \frac{\delta H}{\delta p_m} = -16p_m^{3/2} + \sum_{j\geq 2} (-1)^{j+1} (2j+3)\delta\nu_j p_m^{(2j+1)/2} \ , \tag{D.5d}$$

which gives

$$b_m(\tau) = b_{m0}\exp(8\kappa_m^3\tau - \sum_{j\geq 2} (-1)^{j+1} \frac{2j+3}{2} \delta\nu_j \kappa_m^{2j+1} \tau) \ .$$

Then the asymptotic solution (pure solitons, $r(k)=0$) reduces to

$$u^{(1)} \rightarrow \sum_m -2\kappa_m^2 \operatorname{sech}^2[\kappa_m\{\xi - (4\kappa_m^2 - \sum_{j\geq 2} (-1)^{j+1} \frac{2j+3}{4} \delta\nu_j \kappa_m^{2j})\tau\} + \theta_m] \tag{D.6}$$

Appendix E. Lax's Theorem[5)]

Consider the linearized equation of eq.(4.17),

$$\frac{\partial}{\partial \tau} v = - \frac{\partial}{\partial \xi}(\mathcal{A}'(u^{(1)})v) \quad . \tag{E.1}$$

Let the solutions of (4.17) of one-parameter family be given by $u_{\varepsilon}^{(1)} = u^{(1)} + \varepsilon v$. Then

$$\frac{d}{d\varepsilon} I_{j+1}[u_{\varepsilon}^{(1)}]\Big|_{\varepsilon=0} = \frac{2j+1}{2}(\mathcal{A}_j, v) \tag{E.2}$$

is time invariance, where $(f,g) = \int f \cdot g d\xi$ $(f, g \in L_2(R))$, consequently,

$$\frac{\partial}{\partial \tau}(\mathcal{A}_j, v) = (\mathcal{A}_j' \frac{\partial u^{(1)}}{\partial \tau}, v) + (\mathcal{A}_j, \frac{\partial v}{\partial \tau})$$

$$= -(\mathcal{A}_j' \frac{\partial}{\partial \xi}\mathcal{A} - \mathcal{A}'\frac{\partial}{\partial \xi}\mathcal{A}_j, v) = 0 \ .$$

Namely $\quad \mathcal{A}_j' \frac{\partial}{\partial \xi}\mathcal{A}_j - \mathcal{A}'\frac{\partial}{\partial \xi}\mathcal{A}_j = 0 \ .$

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