数理解析研究所講究録
第 332 巻 1978 年 171-230

CONCRETE PERIODIC INVERSE SPECTRAL TRANSFORM

by

W. E. Ferguson, Jr., H. Flaschka*, and D. W. McLaughlin*

Department of Mathematics and
Program in Applied Mathematics
University of Arizona
Tucson, Arizona 85721

*Supported in part by N.S.F. Grant MPS75-07530, A.R.O. Grant DAAG29-78-G-0059, and N.S.F. Exchange Travel Grant INT76-21516. This paper is an extended version of two lectures, one by H. Flaschka and one by D. W. McLaughlin, which were presented at a Conference on Solitons held at Kyoto University in January, 1978. Support by the N.S.F. of the United States and by the government of Japan is gratefully acknowledged.

INTRODUCTION

The inverse scattering method for evolution equations subject to periodic boundary conditions has recently been the subject of intensive investigation. At present, there is a rather complete theory of the periodic analog of N-soliton solutions. One has explicit formulas, and a beautiful mathematical structure underlying the whole theory. Unfortunately, these analytical results involve terribly transcendental operations: loop integrals, θ functions of many variables, etc. Little has been done to extract physically useful qualitative information from the general theory.

For several years, we have performed computer studies and analytical computations designed to understand, as simply and pictorially as possible, the solutions of some periodic integrable equations and the spectral theory of the associated Lax operator. We present here an outline of some of our results; more details will appear in papers currently in preparation.

In Part I, we try to explain the most important features of the periodic spectral theory in terms that should be understandable to a scientist who knows the inverse-scattering method, but has not studied the periodic problem. The point is that one can guess the characteristic features of a periodic problem without becoming involved with Riemann surfaces and θ functions. Our discussion even points to some interesting aspects of the periodic spectrum which have been completely ignored in current work on periodic problems, because this work has been concerned exclusively with the periodic N-soliton solutions (cf., for example, the question of spectral spikes in the sine-Gordon equation).

In Part II, we show how one can use the periodic spectral theory to predict the qualitative behavior of solutions of evolution equations. In principle, the method we describe could be applied to any of the equations integrable by inverse scattering. For partial differential equations, however, a lot of computer time would be required, and in fact, for most equations the periodic spectral theory is not far enough developed - or at least, not developed in the right direction for our needs. Therefore, we illustrate the method on an important finite-dimensional system, the periodic Toda lattice. We then describe some conjectures and ongoing work on an integrable discretization of the nonlinear Schrödinger equation to illustrate some of the difficulties that must be overcome in other applications of our ideas.

I. CONTINUOUS PROBLEMS

A. Periodic Problem is Posed

Consider the initial value problem for the Korteweg-deVries equation, with the initial data periodic in x (PKdV),

$$u_t = 6uu_x - u_{xxx}, \qquad -\infty < x < \infty,$$
$$u(x,t=0) = \overset{\circ}{u}(x), \qquad \overset{\circ}{u}(x+P) = \overset{\circ}{u}(x). \tag{PKdV}$$

This nonlinear evolution equation can be written in the Lax representation,

$$L_t = [B,L]$$

$$L(t=0) = \overset{\circ}{L}.$$

Here $L(t)$ denotes the Schrödinger operator,

$$L(t) \equiv -D^2 + u(\cdot,t),$$

and B is the third order differential operator introduced by Lax,

$$B \equiv -4D^3 + 3(uD + Du).$$

This Lax representation shows that (PKdV) defines an isospectral family of Schrödinger operators $\{L(t)\}$; that is, as $u(\cdot,t)$ flows in time t according to (PKdV), the spectrum of $L(t)$ remains independent of t. The spectrum of $L(t)$ is identical with that of $\overset{\circ}{L}$.

This situation should be compared with the initial value problem for the Korteweg-deVries equation, with initial data which vanishes as $|x| \to \infty$ (∞KdV),

$$u_t = 6uu_x - u_{xxx}, \qquad -\infty < x < \infty$$
$$u(x,t=0) = \overset{\circ}{u}(x), \quad \overset{\circ}{u}(x) \underset{\sim}{} 0 \quad \text{as} \quad |x| \to \infty. \qquad (\infty\text{KdV})$$

The Lax representation also applies under these ∞ boundary conditions; the family of Schrödinger operators is again isospectral. This isospectrality has led to a rather complete and thorough understanding of (∞KdV); however, (∞KdV) is considerably simpler than (PKdV). We describe a few of the reasons for this simplicity.

The field $u(x,t)$ under (∞KdV) consists of two very distinct components - solitons and radiation. The radiation component behaves very much like a linear-dispersive wave. As time increases, the radiation spreads wider and wider; of course, conservation of energy demands its amplitude decreases as it spreads. Thus, for reasonably large time, the support of the radiation is very wide and its amplitude very small. Physically, we think of the radiation as if it escapes any finite system to $x = \infty$. On the other hand, solitons are steady-localized pulses which translate at constant speeds without change of shape. These localized excitations remain in the field forever. This sharp, physical distinction between radiation and solitons could not be present under periodic boundary conditions, for if one views the periodic problem of period P as on a ring of circumference P, there is no point ∞ toward which the radiation can escape. All excitations must live forever on the finite ring, returning time and time again to any point x_0 on the ring. In spectral language, this sharp distinction between solitons and radiation takes the form of the distinction between point eigenvalues and continuous spectrum. As we shall see later, this distinction disappears under periodic boundary conditions.

Most scientists feel that the soliton component is the most important part of the (∞KdV) field. The real utility of the isospectrality of $\{L(t)\}$ for (∞KdV) is the clear identification of the number, speeds, and no-scattering property of solitons with properties of the discrete eigenvalues (bound states) whose invariance in t permits these physical characteristics to be extracted directly from the initial data. This utility is absent under (PKdV).

More technically, the Lax representation for (∞KdV) leads to the exact linearization of (∞KdV) through the scattering transform. The relative simplicity of this transform arises because, for all finite t, $u(\pm\infty,t) = 0$. The field at the points $x = \pm\infty$ is known to be zero for all time; this fact permits the scattering transform, which maps $u(x,t)$ to this point at infinity in a very precise manner, to linearize the time flow. Again, for the periodic problem, no distinguished point x_0 exists for which $u(x_0,t)$ is known for all time.

Given such difficulties, it was natural to pose the mathematical problem: "How can the isospectrality of the family $\{L(t)\}$ be used to integrate (PKdV)?" Several mathematicians [for example, H. P. McKean, Jr. and P. van Moerbecke, Inventions Math. 30 (1975), 217; B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, Uspekhi Mat. Nauk. 31 (1976), 55-136; E. Date and S. Tanaka, Prog. Theor. Phys. Suppl. 59 (1976), 107-125] throughout the world contributed to the solution of this problem, which is now rather completely solved from a mathematical perspective. Unfortunately, the mathematical methods employed are rather abstract, and there remains the problem of making the solution of (PKdV) concrete enough to be useful for applications. In this lecture, we will describe several tricks which are designed to provide qualitative insight into the periodic spectral transform and its connection to (PKdV).

B. Summary of the Mathematical Solution of the Periodic Problem

We begin with a concise summary of the mathematical solution of (PKdV). Fix t, and consider $L(t) = -D^2 + u(\cdot,t)$. (1) The first step is to transform variables from $u(x,\cdot)$ to appropriate spectral data for L, and to prove that this transformation is invertible. It turns out that sufficient spectral data consists of the eigenvalues $\{\lambda_j\}$ for L under periodic boundary conditions, eigenvalues $\{\mu_j\}$ for L under vanishing boundary conditions, and, for each j, an assignment of a $\pm$ sign which indicates on which side of a branch cut the eigenvalue μ_j resides. (Consider a solution of the Schrödinger equation at energy E, $\psi(x,E)$:

$$L\psi = (-D^2 + u)\psi = E\psi.$$

Then $\{\lambda_j\}$ are those values of E for which $\psi(x,\lambda_j) = \psi(x + P, \lambda_j)$ and $\{\mu_j\}$ are those values of E for which $\psi(0,\mu_j) = \psi(P,u_j) = 0.)$ The map from $\{u(x,\cdot)\} \to \{\lambda_j,(\mu_j,\pm)\}$ is invertible.

(2) Next, one asks how the spectral variables $\{\lambda_j,(\mu_j,\pm)\}$ evolve in t as $u(x,t)$ flows under (PKdV), which, for convenience of interpretation, we consider as an infinite dimensional Hamiltonian system

$$u_t = D\,\frac{\delta H}{\delta u}, \quad H(u) \equiv \int_0^P dx\left[\frac{u_x^2}{2} + u^3\right].$$

The existence of the Lax representation guarantees that the periodic eigenvalues $\{\lambda_j\}$ are constant in time t; thus, one half the degrees of freedom are constants of the motion and the (PKdV) is a completely integrable Hamiltonian system. On the other hand, the variables $\{\mu_j\}$ satisfy an infinite dimensional system of coupled, nonlinear, differential equations. These equations can be obtained from a canonical description of the spectral transform in the framework of Hamilton-Jacobi transformation theory. Formulas for the natural frequencies of oscillation exist; indeed, the entire transformation can be viewed in action-angle form [H. Flaschka and D. W. McLaughlin, Prog. Theor. Phys. 55 (1976), 438-456].

(3) Finally, the actual integration of the system, constrained to admit only N degrees of freedom, can be accomplished using θ functions of many variables.

We take the point of view that the most important physical information is contained in the amplitudes and frequencies of vibration, both of which are fixed by the constants of motion. Indeed, the beauty of the Hamilton-

Jacobi method of integrating (PKdV) is that this formalism concentrates upon these physical characteristics. Using standard constructions from classical mechanics together with the spectral transformation, one obtains concrete formulas for the action variables and the vibrational frequencies. The action variables admit a rather explicit pictorial representation through the Floquet discriminant, a representation which we will use rather extensively in Part II. Thus, our general goal is to understand, as concretely as possible, qualitative relations between the constants $\{\lambda_j\}$ (or equivalently the action variables), the initial data $\overset{\circ}{u}$, and the physical characteristics of the wave $u(x,t)$.

C. Insight into Periodic Problems from Whole-Line Problems

Consider a solution $u(x,t)$ of (PKdV) which evolves from initial data $\overset{\circ}{u}(x)$. Isospectrality guarantees that the spectrum of $L(t) \equiv -D^2 + u(\cdot,t)$ is identical with that of $L(0) = -D^2 + \overset{\circ}{u}(\cdot)$. Denote this spectrum by $\sigma(L)$. Quite generally for any periodic potential, the spectrum $\sigma(L)$ is displayed through the Floquet discriminant $\Delta(E)$, which we now define. Consider two solutions $\phi_\pm(x,E)$ of the Schrödinger equation which are defined by the initial value problem

$$[-D^2 + \overset{\circ}{u}(x)]\phi_\pm = E\phi_\pm$$

$$\phi_\pm(x=0,E) = 1$$

$$\partial_x\phi_\pm(x=0,E) = \pm i\sqrt{E}.$$

This pair $\{\phi_+,\phi_-\}$ is a basis of solutions at energy E. Since the potential $\overset{\circ}{u}(x)$ is periodic of period P, $\phi_{\pm}(x + P, E)$ are also solutions and the pair $\{\phi_{+}(x + P, E), \phi_{-}(x + P, E)\}$ is also a basis. These two bases are related by

$$\begin{pmatrix} \phi_{+}(x + P, E) \\ \\ \phi_{-}(x + P, E) \end{pmatrix} = T(E) \begin{pmatrix} \phi_{+}(x,E) \\ \\ \phi_{-}(x,E) \end{pmatrix}$$

where $T(E)$ is a 2×2 matrix which transfers the function $\phi_{\pm}(x,E)$ across one period of the potential. More compactly, we have

$$\vec{\phi}(x + P, E) = T(E)\vec{\phi}(x,E).$$

If we wish to transfer across N periods, we have

$$\vec{\phi}(x + NP, E) = T^{N}(E)\vec{\phi}(x,E).$$

Now E belongs to the spectrum $\sigma(L)$ if and only if $\vec{\phi}(x,E)$ is bounded for all x; equivalently, if and only if $\vec{\phi}(x + NP, E)$ is bounded for all N. But since $\vec{\phi}(x + NP, E) = T^{N}(E)\vec{\phi}(x,E)$, $\vec{\phi}(x + NP, E)$ will be bounded for all N iff the eigenvalues of the transfer matrix $T(E)$ have unit modulus. If we denote the eigenvalues of this 2×2 matrix by $\rho_{\pm}(E)$,

$$\det\,[T(E) - \rho_{\pm}(E)I] = 0,$$

we have

$$E \in \sigma(L) \quad \text{iff} \quad |\rho_{\pm}(E)| = 1.$$

Computing the 2×2 determinant, we find the eigenvalues

$$\rho_{\pm}(E) = \frac{\text{tr } T(E) \pm \sqrt{(\text{tr } T(E))^2 - 4}}{2} ,$$

where tr T(E) denotes the trace of the transfer matrix. Clearly, $|\rho_{\pm}(E)| = 1$ demands tr T(E) is real and $|\text{tr } T(E)| \leq 2$. Thus, we find $E \in \sigma(L)$ iff $\Delta(E)$ is real and $|\Delta(E)| \leq 2$, where $\Delta(E) \equiv \text{tr } T(E)$ is called the Floquet discriminant.

This characterization of the spectrum of $L = -D^2 + \overset{\circ}{u}$ for periodic $\overset{\circ}{u}(x)$ is quite general; however, if we restrict ourselves to $\overset{\circ}{u}(x)$ which have compact support within the period P (Figure I.1), we can obtain rather concrete information about $\sigma(L)$. In this case,

$$\phi_{\pm}(x,E) = e^{\pm i\sqrt{E}x} \quad \text{near} \quad x \simeq 0.$$

To compute $\phi_{\pm}(x,E)$ near $x \simeq P$, we need consider only one period (Figure I.2). Since $\overset{\circ}{u}(x) = 0$ near $x \simeq P$, $\phi_{\pm}(x,E)$ must be linear combinations of $\{e^{i\sqrt{E}x}, e^{-i\sqrt{E}x}\}$ near $x \simeq P$:

$$\phi_{+}(x,E) = -b(\sqrt{E})e^{-i\sqrt{E}x} + \bar{a}(\sqrt{E})e^{i\sqrt{E}x}$$

$$\phi_{-}(x,E) = -\bar{b}(\sqrt{E})e^{i\sqrt{E}x} + a(\sqrt{E})e^{-i\sqrt{E}x} \quad \text{near} \quad x \simeq P.$$

Here the coefficients $\{a,b,\bar{a},\bar{b}\}$ are the familiar scattering coefficients from whole line scattering theory. They are the scattering coefficients for one period of $\overset{\circ}{u}(x)$ with the rest of $\overset{\circ}{u}(x)$ set at zero (as in Figure I.2). They possess all the familiar properties, in particular

$$b(\sqrt{E})/a(\sqrt{E}) = \text{reflection coefficient}$$

$$1/a(\sqrt{E}) = \text{transmission coefficient}$$

$$|a^2(\sqrt{E})| = 1 + |b^2(\sqrt{E})|, \quad E \text{ real and positive}$$

$$a(\sqrt{E}) \to 1 \quad \text{as} \quad |E| \to +\infty$$

$$a(\sqrt{E_j}) = 0 \quad \text{at the bound states} \quad E_N < E_{N-1} < \ldots < E_1 < 0.$$

Using this representation of $\phi_\pm(x,E)$ yields

$$\begin{pmatrix} \phi_+(x + P, E) \\ \phi_-(x + P, E) \end{pmatrix} = \begin{bmatrix} \bar{a}(\sqrt{E})e^{i\sqrt{E}P} & -b(\sqrt{E})e^{-i\sqrt{E}P} \\ -b(\sqrt{E})e^{i\sqrt{E}P} & a(\sqrt{E})e^{-i\sqrt{E}P} \end{bmatrix} \begin{pmatrix} e^{-i\sqrt{E}x} \\ e^{i\sqrt{E}x} \end{pmatrix}, \quad x \simeq 0,$$

that is,

$$\vec{\phi}(x + P, E) = T(E)\vec{\phi}(x,E), \quad x \simeq 0,$$

where the transfer matrix $T(E)$ is given explicitly in terms of the scattering coefficients by

$$T(E) = \begin{bmatrix} \bar{a}(\sqrt{E})e^{i\sqrt{E}P} & -b(\sqrt{E})e^{-i\sqrt{E}P} \\ -\bar{b}(\sqrt{E})e^{i\sqrt{E}P} & a(\sqrt{E})e^{-i\sqrt{E}P} \end{bmatrix}.$$

From this formula, we obtain an extremely useful representation of the Floquet discriminant $\Delta(E) = \operatorname{tr} T(E)$,

$$\Delta(E) = a(\sqrt{E})e^{-i\sqrt{E}P} + \bar{a}(\sqrt{E})e^{i\sqrt{E}P}. \tag{I.1}$$

This formula for the discriminant $\Delta(E)$ is valid as long as the potential $\overset{\circ}{u}$ has compact support within a period; it can be used to display rather detailed information about the spectrum $\sigma(L)$. Since L is self-adjoint, we know that $\sigma(L)$ lies on the real axis. Fix E real and positive. Then $\bar{a}(\sqrt{E}) = [a(\sqrt{E})]^*$, and

$$\Delta(E) = 2|a(\sqrt{E})| \cos(\sqrt{E}P - \text{ph } a(\sqrt{E})),$$

where ph (a) denotes the phase of a. Since $|a(\sqrt{E})|^2 = 1 + |b(\sqrt{E})|^2$, $|a(\sqrt{E})| > 1$ and $|\Delta(E)|$ can exceed 2. This forces gaps to exist in the spectrum $\sigma(L)$ (unless $|b(\sqrt{E})| = 0$). For a reasonably sized period P, the oscillation of $\Delta(E)$ vs. real $E > 0$ arises from the $\cos(\cdot)$, not its amplitude $2|a(\sqrt{E})|$. As $E \to +\infty$, $|a(\sqrt{E})|$ gets very close to 1 and the gaps reduce in size. Their spacing becomes very regular, approaching $(\sqrt{E_j} - \sqrt{E_{j+1}})P \simeq \pi$ for large E. Thus, $\Delta(E)$ vs. $E > 0$ appears as in Figure I.3. Finally, as the period $P \to \infty$, the oscillations of $\Delta(E)$ densely fill the entire positive real axis and yield the radiation spectrum of whole line scattering theory.

Now consider $E < 0$. In this regime,

$$\Delta(E) \sim a(i\sqrt{-E})e^{\sqrt{-E}P} + O(e^{-\sqrt{-E}P});$$

the discriminant has no oscillations due to the exponential. However, the amplitude $a(i\sqrt{-E})$ can have a finite number of zeros which occur at the bound state energies of the whole line problem (for one period of $\overset{\circ}{u}(x)$). For $E < 0$, $|\Delta(E)| \leq 2$ only near these bound state energies. For other values of E, $|\Delta(E)|$ is exponentially large. Thus, for $E < 0$, $\Delta(E)$ appears as sketched in Figure I.3.

To summarize, from representation (I.1) of $\Delta(E)$, we can easily understand the qualitative behavior of $\Delta(E)$, and thus of the spectrum $\sigma(L)$, from our knowledge of whole line scattering theory. We find two rather distinct classes of spectrum, and therefore two rather distinct classes of excitations in the KdV field. From a practical viewpoint, the periodic problem tends to have radiation and solitons, although the borderline between the two excitations is not as distinct as in the whole line case. Somewhat differently stated, if one begins with the spectrum of the whole line problem as depicted in Figure I.4, together with the formula (I.1), one can get a rather concrete feeling for the structure of the periodic spectrum and for the types of excitations present in the periodic wave. In particular, the continuous spectrum in the whole line problem (radiation) develops gaps while the bound states (solitons) spread into narrow bands of spectrum.

This method of understanding the band-gap structure of the spectrum is really unnecessary in this well-known case of Hill's equation. However, we've found the method very useful when studying less familiar eigenvalue problems. Consider, for example, the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0$$

$$u(x,t=0) = \overset{\circ}{u} \quad \text{periodic} \quad (\text{mod } 2\pi)$$

$$u_t(x,t=0) = \overset{\circ}{\pi} \quad \text{periodic.}$$

This evolution equation is rendered completely integrable by the unorthodox eigenvalue problem

$$\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D + \frac{iw}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{16\sqrt{E}}\begin{pmatrix} e^{iu} & 0 \\ 0 & e^{-iu} \end{pmatrix}\right]\vec{\psi} = \sqrt{E}\,\vec{\psi}, \qquad \text{(I.2)}$$

where $w = u_x + u_t$. What is the spectrum of this eigenvalue problem when the potentials e^{iu} and w are periodic?

Just as in the case of Hill's equation, one defines a discriminant $\Delta(E)$ which is analytic in E except at $E = 0$ and $E = \infty$, where it has essential singularities. The spectrum consists of

$$\{E \in \mathbb{C} \,|\, \Delta(E) \quad \text{is real and} \quad |\Delta(E)| \le 2\}.$$

Since the eigenvalue problem is not self-adjoint, this spectrum need not be real.

Tahtajan and Faddeev [L. A. Tahtajan and L. D. Faddeev, Theor. Math. Phys. 21 (1974), 1046] have investigated the whole line problem and its connection with the sine-Gordon equation. Their results are well known. In particular, the spectrum is as depicted in Figure I.5.

If u and w have compact support within a period, we again find

$$\Delta(E) = a(\sqrt{E})e^{-ik(\sqrt{E})P} + \bar{a}(\sqrt{E})e^{ik(\sqrt{E})P}$$

where $a^{-1}(\sqrt{E})$ is the transmission coefficient as defined by Tahtajan and Faddeev, and $k(\sqrt{E}) = [\sqrt{E} - 1/(16\sqrt{E})]$.

In this case,

$$|a(\sqrt{E})|^2 = 1 - |b(\sqrt{E})|^2, \quad \sqrt{E} > 0,$$

and $|a(\sqrt{E})| \leq 1$. Thus, on the positive real axis,

$$\Delta(E) = 2|a(\sqrt{E})| \cos(-k(\sqrt{E})P + \text{ph } a(\sqrt{E})), \quad |a(\sqrt{E})| \leq 1;$$

$|\Delta(E)|$ never exceeds 2; no gaps develop on the positive real axis; the entire positive real axis consists of spectrum.

Moreover, since $\Delta(E)$ is analytic, its real part cannot have a maximum in the E plane. Thus, a maximum E_0 of $\Delta(E)$ on the positive real axis is actually a saddle point in the complex E plane. At this saddle point, a curve of $\text{Im}(\Delta(E)) = 0$ crosses the real axis (which is also a curve of $\text{Im}\,\Delta(E) = 0$). On this curve, $\Delta(E)$ is real, increasing away from E_0, and less than 2 for a small distance into the complex E plane. There is a small spike of spectrum (Figure I.6). In addition, the bound states of the whole line problem (kinks, antikinks, and breathers) spread into small bands of spectrum. Thus, the spectrum of (I.2), when the potentials e^{iu} and w are periodic, takes the generic form of Figure I.6. The periodic sine-Gordon field consists of kink trains, antikink trains, breather trains, and radiation. The spectrum associated with the radiation has spikes.

In this manner, the whole line problem can give considerable insight into the periodic problem. Let us close this section by briefly considering one more example of this principle. Return to the Schrödinger-KdV case,

$$\Delta(E) = a(\sqrt{E})e^{-i\sqrt{E}P} + \bar{a}(\sqrt{E})e^{i\sqrt{E}P}.$$

From the whole line problem, $a(\sqrt{E})$ provides one-half the degrees of freedom and is constant in t under KdV. Since $\Delta(E)$ is fixed by $a(\sqrt{E})$ alone, we expect the same to hold for $\Delta(E)$. Indeed, $\Delta(E)$ does provide one-half the degrees of freedom and is constant in t. Periodic theorists tell us that the vanishing boundary condition eigenvalues μ_j provide the remaining half the degrees of freedom. This fact can be made plausible by considering the case of compact support within a period, for then the μ_j can be computed rather explicitly, and are seen to depend upon phase of $b(\sqrt{E})$. But this phase of $b(\sqrt{E})$ is exactly the remaining half of the information needed for the inversion in the whole line problem. Since μ_j depends upon this phase, it seems less surprising that the map

$$u(x) \to \{\Delta(E), (\mu_i, \pm)\}$$

is invertible.

II. DISCRETE PROBLEMS

A. Outline

We now show how the considerations of Part I can yield qualitative information about the solutions of completely integrable periodic evolution equations. The idea is to find the Floquet discriminant $\Delta(\lambda)$ from the initial conditions, to identify soliton-like and radiation-like spectrum from $\Delta(\lambda)$, and then to predict the general behavior of these components of the solution as time increases. It is quite time-consuming to find the discriminant $\Delta(\lambda)$ for the Lax operators associated with partial differential equations such as PKdV or sine-Gordon; to get $\Delta(\lambda)$ one would have to solve an o.d.e. numerically for many values of λ. Therefore our work has been restricted to integrable ordinary differential equations, primarily the Toda lattice. We explain, in the case of the Toda lattice, the connection between $\Delta(\lambda)$ and the action variables. We then show how one might distinguish "solitons" from "radiation" in the periodic lattice (which is not always possible). In favorable cases, one can even predict soliton speeds and the speeds of radiation wave-packets from $\Delta(\lambda)$.

Another application of this spectral analysis is found in perturbation problems. The unequal mass Toda lattice, for example, is not integrable. $\Delta(\lambda)$ is therefore not constant in time. By observing the change of $\Delta(\lambda)$ in time, one can estimate the interaction of solitons and radiation due to the perturbation.

B. Basic Formulas

The Toda Hamiltonian is

$$H = \sum_n \{\frac{1}{2} p_n^2 + e^{-(q_n - q_{n-1})} + (q_n - q_{n-1}) - 1\}.$$

We consider the sum $n = 1$ to N, with periodicity conditions $q_{n+N} = q_n$, $p_{n+N} = p_n$. For motivation one sometimes looks at the infinite lattice, where the sum goes from $-\infty$ to ∞. Set

$$a_n = \frac{1}{2} e^{-(q_n - q_{n-1})} \tag{2.1}$$

$$b_n = -\frac{1}{2} p_{n-1}. \tag{2.1}$$

Then the Toda equations become

$$\dot{a}_n = a_n(b_{n+1} - b_n)$$

$$\dot{b}_n = 2(a_n^2 - a_{n-1}^2).$$

This system is precisely $\dot{L} = [B,L]$ for the Lax pair

$$L = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & a_N \\ a_1 & b_2 & a_2 & & \\ 0 & a_2 & & & \\ \vdots & & & & \\ a_N & & & & b_N \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a_1 & 0 & \cdots & a_N \\ -a_1 & 0 & a_2 & & \\ & -a_2 & 0 & & \\ \vdots & & & & \\ -a_N & & & & 0 \end{bmatrix}$$

The eigenvalues of L are constant in time. Hence $\det(\lambda I - L)$ is constant in time. Define

$$\Delta(\lambda) = 2^N \det(\lambda I - L) + 2. \tag{2.2}$$

This is really quite analogous to the $\Delta(\lambda)$ defined for the Schrödinger equation in (I.C), although perhaps it does not appear so. One can see the analogy in two ways.

(1) Write the eigenvalue equation $Lu = \lambda u$ as a second-order difference equation

$$a_{n-1}u_{n-1} + a_n u_{n+1} + b_n u_n = \lambda u_n. \tag{2.3}$$

There is a discretized Floquet theory; one has a 2×2 transfer matrix $T(\lambda)$, and $\Delta(\lambda) = \text{trace } T(\lambda)$, just as in Part I.

(2) One can think of $(-D^2 + q)y = \lambda y$ with periodic B.C.'s as an infinite-dimensional operator equation. The periodic eigenvalues are then the roots of

$$\det(-D^2 + q - \lambda) = 0;$$

this can be made quite rigorous with Fredholm theory, and one sees that

$$\det(-D^2 + q - \lambda) = \Delta(\lambda) - 2,$$

up to a constant factor. This is just like (2.2).

The following properties of $\Delta(\lambda)$ can be established:

$\Delta(\lambda)$ is a polynomial of degree N.

$\Delta(\lambda) = 2$ and $\Delta(\lambda) = -2$ have N real roots each. Hence, the graph of $\Delta(\lambda)$ must intersect $\lambda = \pm 2$ N times each; a tangency counts as two intersections.

One can predict some characteristics of $\Delta(\lambda)$ by using the scattering theory appropriate for the infinite lattice. This works just as in Part I.

The infinite lattice is solved by inverse scattering for

$$a_{n-1}u_{n-1} + a_n u_{n+1} + b_n u_n = \lambda u_n, \qquad -\infty < n < \infty.$$

There is continuous spectrum, corresponding to radiation, on $-1 < \lambda < 1$, and bound states corresponding to solitons for $\lambda > 1$ and/or $\lambda < -1$. There are scattering coefficients $\alpha(\lambda)$, $\beta(\lambda)$ satisfying $|\alpha|^2 - |\beta|^2 = 1$ for $-1 < \lambda < 1$, and for a localized potential, repeated periodically with period N,

$$\Delta(\lambda) = \alpha(\lambda)(\lambda + \sqrt{\lambda^2 - 1})^N + \overline{\alpha(\lambda)}(\lambda - \sqrt{\lambda^2 - 1})^N.$$

(The funny $\lambda + \sqrt{\lambda^2 - 1}$ arises because in the scattering theory it is convenient to set $\lambda = (z + z^{-1})/2$; this is the analog of $\lambda = k^2$ in the Schrödinger equation. $(\lambda \pm \sqrt{\lambda^2 - 1})^N$ is then $z^{\pm N}$, which is the analog of $e^{\pm ikx}$.) The $\lambda \pm \sqrt{\lambda^2 - 1}$ reminds one of Cebysev polynomials. When there is no potential, $a(\lambda) \equiv 0$, $\Delta(\lambda)$ is the Cebysev polynomial. In general, it is a modulated Cebysev polynomial. Our pictures of $\Delta(\lambda)$ show some evidence of this, e.g., the oscillatory nature of $\Delta(\lambda)$ and the more rapid oscillations near $\lambda = \pm 1$.

The connection of $\Delta(\lambda)$ with the scattering theory suggests the following rule of thumb:

"Oscillations of $\Delta(\lambda)$ in $-1 < \lambda < 1$ are related to radiation.

Oscillations of $\Delta(\lambda)$ in $\lambda > 1$ or $\lambda < -1$ are related to solitons."

This is of course quite accurate for disturbances localized inside a period. It is a very poor rule for short lattices $(N \leq 10)$ or for some large, wide disturbances. Surprisingly, it is quite a good rule for <u>certain non-localized disturbances in long $(N \gtrsim 25)$ lattices</u>. We show some pictures below.

A very important feature of $\Delta(\lambda)$ is this: The N action variables of the periodic Toda lattice can be read off immediately from the graph of $\Delta(\lambda)$.

The following result has been established. Let

$$m(\lambda) = \begin{cases} \frac{1}{\pi} (\text{sign } \Delta) \cosh^{-1} (|\Delta|/2) & \text{if} \quad |\Delta| \geq 2 \\ 0 \quad \text{if} \quad |\Delta| < 2. \end{cases}$$

$m(\lambda)$ has "bumps" whenever $|\Delta(\lambda)| > 2$; there are at most $N - 1$ bumps.

Theorem: The $N - 1$ action variables are numerically equal to the areas under the $N - 1$ bumps of $m(\lambda)$.

Remarks: (1) If $\Delta(\lambda)$ has a tangency at $\lambda = \pm 2$, we think of this as a bump of area = 0. Then the corresponding action variable = 0.

(2) Why are there only $N - 1$ actions, not N?

Physically, all the degrees of freedom are oscillatory except one which is translation of the center of mass of the entire lattice. Action variables are only associated with oscillatory degrees of freedom. These variables are defined as follows in classical mechanics texts. A given phase point $q_1, \ldots, p_n$ lies on certain surfaces $I_1 = c_1, \ldots, I_N = c_N$, where $I_1, \ldots, I_N$ are the N independent constants of motion. Let γ be any closed curve lying simultaneously on all these surfaces, and define

$$J(\gamma) = \oint_\gamma \sum_1^N p_n dq_n. \tag{2.4}$$

In the periodic Toda lattice, $I_1 = p_1 + \ldots + p_N$, and $I_1 = c_1$ is an infinite plane in phase space. The intersection of the remaining $I_j = c_j$ leads to an $N - 1$ dimensional torus, which has $N - 1$ "independent" closed curves γ_i, for each of which we define $J(\gamma_i)$. It is the presence of I_1 which causes this system to have only $N - 1$ action variables.

(3) In most integrable Hamiltonian systems the loop integrals (2.4) are difficult to evaluate. It is very interesting that for the Toda lattice, these integrals can be done "graphically" in a few seconds of computer time by plotting $m(\lambda)$.

C. Practice Problems

We illustrate the above facts by a couple of simple examples.

Example 1 (Near-linear disturbance): N = 25. Set

$$q_n = .05 \sin \kappa_j n$$

$$p_n = .05\, \omega_j \cos \kappa_j n$$

where

$$\kappa_j = 2\pi j/25$$

$$\omega_j = 2 \sin \kappa_j/2.$$

These are harmonic normal modes. They have low amplitude, so the Toda lattice may be expected to respond linearly. We compute a_n, b_n (see (2.1)) and graph $\Delta(\lambda)$, for $j = 1, 2, 3, 6, 12$.

The amplitudes of these linear modes are essentially the action variables of the harmonic lattice. We therefore expect that we are exciting, successively, actions $J_1, J_2, J_3, J_6, J_{12}$ of the Toda lattice. Figure 1 shows that indeed bumps 1, 2, 3, 6, 12 are made nonzero, all other bumps being zero to plotter accuracy. Note also that the bumps are inside $-1 < \lambda < 1$; these initial data lead to radiation in the Toda lattice.

Example 2 (Pure solitons): $N = 25$. The initial condition is a very narrow cnoidal wave, essentially an infinite lattice soliton with

$$e^{-(q_n - q_{n-1})} - 1 = (\sinh^2 \omega)\ \mathrm{sech}^2\ (n\omega - t \sinh \omega + n_0).$$

Figure 2 shows one very large bump at $\lambda < -1$. The bump is so large that we use a log scale where $|\Delta| > 2$.

Compare Figures 1a and 2. In both cases, action J_1 is nonzero; the other actions are zero. The location of the bump gives additional information about the physical motion-radiation vs. soliton.

Example 3 (Two solitons and radiation): The initial condition is: two infinite-lattice solitons on a 25-particle periodic lattice. The exponential tails are cut off at the ends. This gives rise to two large bumps in $\Delta(\lambda)$, at $\lambda < -1$, and some small bumps inside $-1 < \lambda < 1$. The latter correspond to radiation which is caused by the truncation of the exact soliton (Figure 3a).

Figure 3b shows the graphs of a_n vs. t for $n = 1, \ldots, 25$. The solitons would be very tall at this scale, so their tops have been cut off. The radiation is very evident.

D. Zabusky's Experiment

In Comp. Phys. Comm. 5 (1973), p. 1, Zabusky solved numerically a 200-particle lattice with cubic force law, subject to a large amplitude, long-wave initial condition. The initial displacement broke up into 46 narrow pulses, of which 20 had super-acoustic speeds and 26 had sub-acoustic speeds.

We plot the Toda discriminant, $\Delta(\lambda)$. For the given initial conditions, the two force-laws don't differ very much. At $N = 200$, $\Delta(\lambda)$ is so large that a log log scale is used outside $|\Delta| < 2$. The picture is cut off just inside $\lambda = -1$.

Notice now that there are 19 bumps at $\lambda < -1$ (solitons) and 26 bumps inside $-1 < \lambda < 1$. Zabusky's subacoustic spikes (he calls them L-solitons) therefore correspond to excited radiation-type action variables.

Zabusky measured the speed of the pulses. We can reproduce his measured speeds from $\Delta(\lambda)$ by using the infinite-lattice soliton speed bound state relation

$$\text{speed} = \frac{\sqrt{\lambda^2 - 1}}{\ell n\,[\lambda + \sqrt{\lambda^2 - 1}]}$$

at $\lambda < -1$, and the harmonic formula

$$\text{speed} = \frac{\sin k}{k}$$

inside $-1 < \lambda < 1$.

From this point of view, the subacoustic pulses are essentially harmonic wave packets.

E. When the Method Does Not Work Well

One can plot $\Delta(\lambda)$ for any initial condition of the Toda lattice, and one can read off the action variables as explained earlier. It is not always possible, however, to predict the features of the resulting solution in a useful fashion.

Figure 5 shows the p_n vs. t plot for a four-particle lattice. It is clear that one cannot make a distinction between "solitons" or "radiation" here. One could, using our formulas, compute the oscillation frequencies (we have done so).

Figure 6 shows another extreme case: a 25 particle lattice subject to a large-amplitude mode 12 (= highest frequency) initial condition. One can see the huge middle bumps in $\Delta(\lambda)$, as expected, but it is hard to correlate this with any pattern in the a_n vs. t plot.

In some sense, the motion has to be of "long-wave type" on a "long" lattice for the division into solitons and radiation to be meaningful. The concept of "long-wave type" is not very well defined, and quite subtle. Zabusky's initial condition, for example, was long-wave, but yet broke up into narrow pulses only 4 lattice points wide; it does not remain a long-wave disturbance as far as one can tell from the physical shape of the lattice. What can be said about it, however, is that there is no significant excitation of the short-wave action variables, and that appears to be a crucial requirement for the applicability of our method.

F. A Light Mass Impurity

We take a periodic, $N = 25$, Toda lattice with a light mass at $n = 1$. A narrow soliton is sent in. Figure 7 shows $\Delta(\lambda)$ before the interaction, at several stages during the interaction, and after the first interaction. Figure 8 shows a_n vs. t. Observe the oscillation of the light mass (bottom curve).

The big bump at $\lambda > 1$ is the incoming soliton's action variable. It decreases during the interaction, then grows back a little. At $\lambda < -1$, one sees a growing bump representing the reflected soliton. Just inside $\lambda = -1$ are some radiation actions.

Particularly interesting and unexpected is the growth - and later decay - of a second soliton bump just outside $\lambda = +1$. This bump comes and goes away within one second, and could never be seen on the time line plot of Figure 8.

This suggests a possible analytical approach to soliton-scattering off impurities. To a first approximation, there is only the transmitted and reflected soliton. To a second approximation, there is a three-mode interaction, in which a part of the incident energy is transferred to the reflected soliton by a coupling to a third soliton-type action variable. At Kyoto, we learned of related work on the impurity problem by Yajima and by Nakumura and Takeno. This work must still be related to our preliminary modal analysis.

G. Diffusion of Action

One very useful aspect of the above theory is that action variables can be computed quickly and efficiently. This makes possible a variety of other experiments. The unequal-mass Toda lattice was shown by Casati and Ford to be non-integrable; there are stochastic regions in phase space. We ask: How do the (equal mass) action variables change along a stochastic orbit? On the basis of a different model, Chirikov has argued that they execute a diffusion process. We are in a position to investigate this problem for lattices of several particles. Such a study is currently in progress (in collaboration with J. Ford and F. Vivaldi).

H. Discrete Nonlinear Schrödinger Equations (DNLS)

The analysis of $\Delta(\lambda)$ can, in principle, be carried out for other integrable systems. With some carefully designed computer experiments, one could hope to get insight into the response of integrable systems to perturbations. We have carried out some computations on the system

$$i\dot{Q}_n = Q_{n+1} - 2Q_n + Q_{n-1} + |Q_n|^2(Q_{n+1} + Q_{n-1}),$$

Q_n complex, $Q_{n+N} = Q_n$. This was shown to be completely integrable by Ablowitz and Ladik. The associated eigenvalue problem is

$$\begin{aligned} v_{n+1} &= zv_n + Q_n w_n \\ w_{n+1} &= \frac{1}{z} w_n - Q_n^* v_n . \end{aligned} \tag{2.5}$$

We write (2.5) in vector form:

$$\begin{pmatrix} v \\ w \end{pmatrix}_{n+1} = \begin{pmatrix} z & Q_n \\ -Q_n^* & z^{-1} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_n ,$$

or, with $\phi = (v,w)^T$ as

$$\phi_{n+1} = M_n \phi_n .$$

The transfer matrix across one period N is then

$$T(z) = M_N M_{N-1} \cdots M_1, \tag{2.6}$$

and the discriminant $\Delta(z)$ is again defined as Trace $T(z)$. It follows immediately from (2.6) and from the form of M_n that $\Delta(z) = \Delta(1/z^*)^*$, and because of this symmetry, it suffices to study Δ for $|z| \leq 1$.

The behavior of $\Delta(z)$, and the solutions of the DNLS, appear to be quite complicated for an integrable system. We describe a few preliminary observations.

1. Hamiltonian structure

The DNLS is a Hamiltonian system: Set $Q_n = x_n + iy_n$,

$$D = \begin{pmatrix} 1 + x_1^2 + y_1^2 & & & 0 \\ & 1 + x_2^2 + y_2^2 & & \\ & & \ddots & \\ 0 & & & 1 + x_N^2 + y_N^2 \end{pmatrix}$$

$J = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$. Then with

$$H = \sum_{n=1}^{N} (y_{n+1}y_n + x_{n+1}x_n - \log(1 + x_n^2 + y_n^2))$$

one has

$$\frac{d}{dt}\binom{x}{y} = J\begin{pmatrix} \partial H/\partial x \\ \partial H/\partial y \end{pmatrix}.$$

One can introduce ordinary conjugate variables q_n, p_n, e.g., by

$$x_n = \sqrt{e^{2p_n} - 1}\ \sin q_n$$

$$y_n = \sqrt{e^{2p_n} - 1}\ \cos q_n;$$

in terms of these $\dot{q}_n = \partial H/\partial p_n$, $\dot{p}_n = -\partial H/\partial q_n$, but the Hamiltonian becomes very complicated.

2. Action variables

Since the DNLS is a Hamiltonian system, one expects the eigenvalues of (2.5) to be a complete set of integrals in involution. By analogy with our earlier results for the Toda lattice, there ought to be a direct relation between the function $\Delta(z)$ and the action variables. This has not been investigated. Graphical studies of $\Delta(z)$ are not very convenient because one needs plots for complex $|z| \leq 1$, which are quite time-consuming on the computer. Theory is needed to tell us what to look for in the graph.

3. Stationary solutions

We now consider stationary solutions of DNLS. It will be seen that these are quite complicated already, which indicates that the full phase-space structure of DNLS is probably even more involved.

A stationary solution satisfies the nonlinear difference equations

$$Q_{n+1} - 2Q_n + Q_{n-1} + |Q_n|^2(Q_{n+1} + Q_{n-1}) = 0; \qquad (2.7)$$

if N is the period, then $Q_0 = Q_N$, $Q_1 = Q_{N+1}$. (2.7) can be rewritten as a mapping of a discrete phase plane. Let $P_n = Q_{n+1} - Q_n$, then

$$\begin{aligned} Q_{n+1} &= Q_n + P_n \\ P_{n+1} &= P_n - 2\frac{|Q_n + P_n|^2(Q_n + P_n)}{1 + (Q_n + P_n)^2} . \end{aligned} \qquad (2.8)$$

If we define the mapping T of the Q, P plane (Q, P complex) by

$$T(Q,P) = \left(Q + P,\ P - 2\,\frac{|Q + P|^2 (Q + P)}{1 + (Q + P)^2} \right),$$

we may rewrite (2.8) as

$$(Q_{n+1}, P_{n+1}) = T(Q_n, P_n).$$

In general, points (Q_1,P_1), $(Q_2,P_2) = T(Q_1,P_1)$, $(Q_3,P_3) = T^2(Q_1,P_1)$, ... will not satisfy the periodicity condition $Q_{n+N} = Q_n$. To get a stationary period-N solution of DNLS, we need to find fixed points of the N^{th} iterate, T^N.

Example: $(1,0) \to (1,-1) \to (0,-1) \to (-1,0) \to (-1,1) \to (0,1) \to (1,0)$ is one orbit of T. So $Q_1 = 1$, $Q_2 = 1$, $Q_3 = 0$, $Q_4 = -1$, $Q_5 = -1$, $Q_6 = 0$ is a stationary (real) solution of period 6 for DNLS.

The mapping T is symplectic; if we write $Q = x + iy$, $P = \sigma + i\tau$, then T leaves the form $dx \wedge d\sigma + dy \wedge d\tau$ invariant. In that sense, it is a discrete-time Hamiltonian system with 2 degrees of freedom. The mapping T is also integrable - it leaves invariant the function

$$F(Q,P) = |P|^2 + QP^* - PQ^* + |Q + P|^2 |Q|^2.$$

All iterates $T^N(\mathring{Q},\mathring{P})$ will therefore lie on the complex curve

$$F(Q,P) = F(\mathring{Q},\mathring{P}).$$

The existence of F can of course be verified by direct computation, but it is instructive to derive it a little differently. This leads us to a short digression.

Whenever an evolution equation possesses local conservation laws, e.g.,

$$T_t + X_x = 0$$

then the <u>stationary</u> solutions will satisfy $X_x = 0$, i.e., the expression X is an integral for the o.d.e. defining the stationary solution. Consider, for example, a higher KdV equation,

$$u_t = \left(\frac{\delta H_{n+1}}{\delta q} + c_n \frac{\delta H_n}{\delta q} + \dots + c_1 \frac{\delta H_1}{\delta q} \right)_x , \tag{2.9}$$

where $\delta H_j/\delta q$ is the gradient of the j^{th} conserved functional, and c_j's are fixed constants. The stationary solutions $(u_t = 0)$ are known to be n-gap (= periodic or quasi-periodic "n-soliton") potentials. (2.9) possesses infinitely many local conservation laws,

$$(T_j)_t + (X_j)_x = 0, \qquad j = 1, 2, \dots .$$

Hence, the stationary equation (2.9), viewed as a nonlinear o.d.e. in x, of order $2n + 1$, has n integrals X_j. In fact, it is known to be a completely integrable Hamiltonian system <u>in x</u>. (The other X_j, $j \geq n + 1$, depend on the first n for stationary solutions.)

Returning now to DNLS, we check that

$$\frac{d}{dt} Q_n Q^*_{n-1} = S_n - S_{n-1} \tag{2.10}$$

where $S_n = |Q_n|^2 - Q_{n+1}Q^*_{n-1}(1 + |Q_n|^2)$. In general, with period N, we have an integral of DNLS:

$$\frac{d}{dt}\sum_{1}^{N} Q_n Q^*_{n-1} = 0.$$

For stationary solutions, it follows from (2.10) that S_n = const, and some simple algebra converts this into

$$|P_n|^2 + Q_n P^*_n - P_n Q^*_n + |Q_n + P_n|^2 |Q_n|^2 = \text{constant}.$$

If we restrict our attention to stationary solutions for which all Q_n are <u>real</u>, then the mapping T becomes an <u>area-preserving</u> map of the Q,P plane to itself. All orbits under T lie on the invariant curves

$$P^2 + Q^2(Q + P)^2 = \text{const}.$$

In the complex case, one can compute that the stationary points of DNLS lie on a compact surface

$$(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2 + (x_n x_{n-1} + y_n y_{n-1})^2 = c, \quad n = 1, \ldots, N$$

in phase space $(Q_n = x_n + iy_n)$. For a given period N, c must of course have a very particular value.

In this connection, it is interesting to note that another discretization of the nonlinear Schrödinger equation,

$$i\dot{Q}_n = Q_{n+1} - 2Q_n + Q_{n-1} + 2Q_n|Q_n|^2 \tag{2.11}$$

leads to an area-preserving phase-plane map, say S, when stationary real solutions are sought. The computer suggests right away that S is not integrable (not every orbit lies on an invariant curve). It is then highly probable that (2.11) is a non-integrable system of o.d.e.'s. We have here a computer test for integrability of certain kinds of large-dimensional systems. It should be possible to adapt these ideas to some p.d.e.'s as well, and we are currently investigating a problem of this type. At any rate, the stationary solutions of DNLS, as the above discussion shows, are the simplest analogs of the n-gap solutions familiar from the better studied systems.

Consider now, for definiteness, the period 6 stationary real solution 1, 1, 0, -1, -1, 0. For it,

$$\Delta(z) = (z^6 + z^{-6}) - 2(z^4 + z^{-4}) + 3(z^2 + z^{-2}) + 4. \qquad (2.12)$$

The corresponding phase points Q_n, $P_n = Q_{n+1} - Q_n$ lie on the curve

$$P^2 + Q^2(Q + P)^2 = 1. \qquad (2.13)$$

It is a well-known fact about area-preserving maps that any other point (Q_1,P_1) on this curve also has period 6: $T^6(Q_1,P_1) = (Q_1,P_1)$. The curve (2.13) therefore defines a one-parameter family of distinct, stationary real solutions of the period 6 DNLS. It turns out that $\Delta(z)$ is given by (2.12) for all of these solutions.

If we recall that $\Delta(z)$ determines all the integrals of DNLS, we conclude that there is a one-dimensional invariant torus in the phase space, each point of which is a real stationary point for DNLS.

To prove the assertion about $\Delta(z)$, one considers the Hamiltonian

$$H_1 = \frac{1}{2}\sum_1^N y_{n+1}x_n - x_{n+1}y_n .$$

It is in involution with the DNLS Hamiltonian. One can show that the set of real stationary DNLS solutions of period N is a closed solution curve for the Hamiltonian system H_1. Since H_1 leaves the integrals of DNLS invariant, $\Delta(z)$ must be the same for all real stationary DNLS solutions of a given period. $\Delta(z)$ is actually constant for the complex stationary solutions as well, but from now on we only consider the real ones.

One quickly sees that these real stationary points are unstable for the time-dependent DNLS. For, they are real, and the $i\dot{Q}_n$ term drives the complex part of Q_n away from zero. There should be separatrices emanating from these unstable stationary points. Where do they go? They go into the complex-Q_n domain, but it is not implausible that at least some of them end at real Q_n. Because all the integrals of DNLS must be constant along such a separatrix, $\Delta(z)$ cannot change, and keeps the value (2.12). It is then very likely that the other end of the separatrix is again a phase-point arising from the invariant curve (2.13).

We have done a lot of "supposing" here, but the picture we propose has been supported to some degree by a few computer experiments carried out in 1976. At that time, we were trying to understand the work of Bogoyavlenskii (Comm. Math. Phys. 51 (1976), 201) on perturbations of the Toda lattice. He discovers, for the Toda lattice, a phase-space portrait related to the one we propose for DNLS; however, for the Toda lattice, the unstable stationary points are out at infinity of phase space; they are reached when one spring is stretched infinitely far. An unphysical change of variables is necessary to bring them in to a finite region. For DNLS, the unstable real stationary points are there from the beginning (we have not tried to determine the stability of the complex stationary solutions). On the other hand, DNLS has far fewer such points than do the models studied by Bogoyavlenskii.

Certain conjectures and problems pose themselves quite naturally after a reading of Bogoyavlenskii's paper. A solution of DNLS which starts near an unstable stationary point will follow the separatrix to another one. Then it will move near a possibly different separatrix to the vicinity of perhaps a third unstable stationary point. Is there a simple law describing this reflection amongst stationary points? In Bogoyavlenskii's work, the reflections are associated with the Weyl subgroups of certain simple Lie groups. What happens to this whole picture when a perturbation is added to DNLS? One would hope perhaps for a qualitative description of the orbits in a stochastic region of phase space.

All this is conjecture, and some of the ideas may not be applicable, but yet the whole problem, understanding the action variables, $\Delta(z)$, the reflection of separatrices, and the role of the area-preserving map T in the dynamical theory, seems to be interesting and not altogether easy. We plan to resume a study of these problems some day, but as we have not had time to pursue the matter for the past two years, this sketch of our ideas was included here in the hope that a reader might become interested in one or another aspect of the DNLS system.

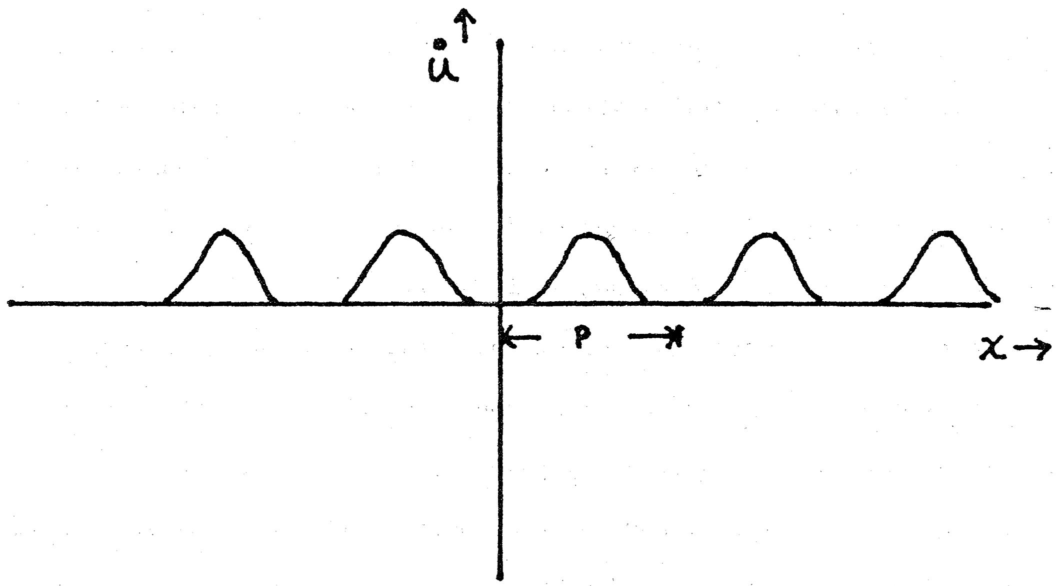


Figure I.1. Periodic potential $\mathring{u}(x)$ with compact support within period P.

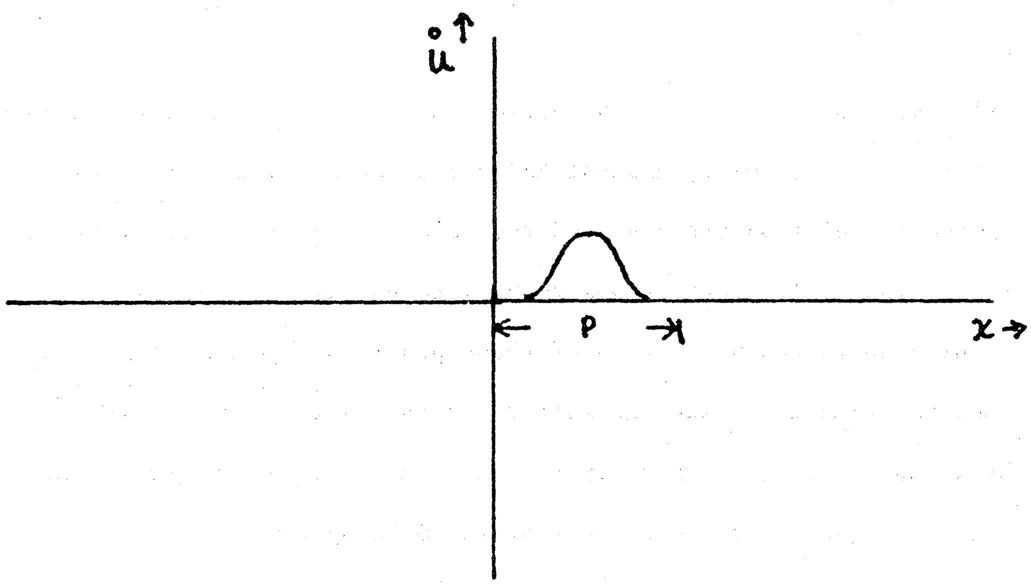


Figure I.2. One period of $\mathring{u}(x)$.

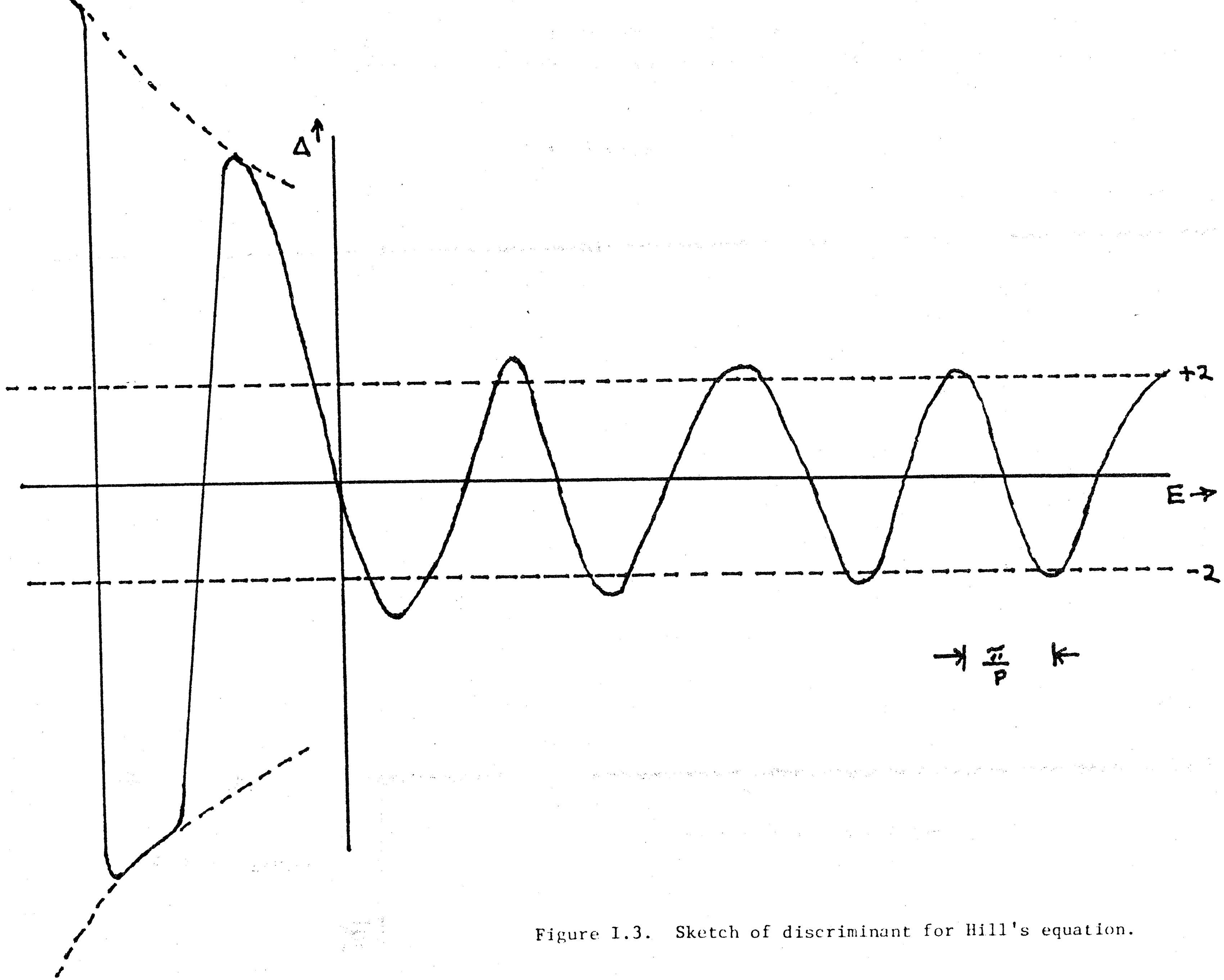


Figure I.3. Sketch of discriminant for Hill's equation.

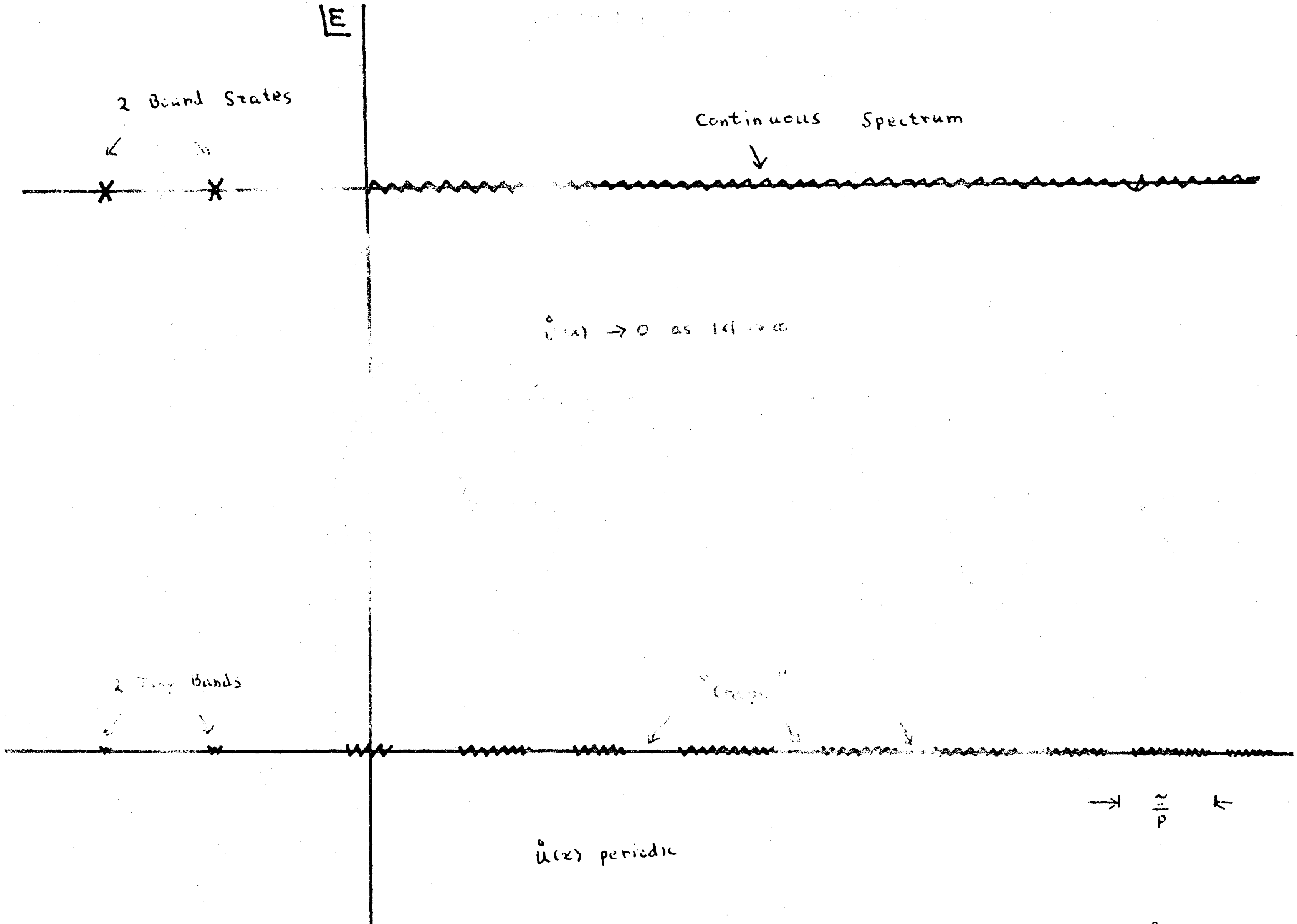


Figure I.4. Spectrum for Schrödinger's equation with potential $\overset{\circ}{u}(x)$ which a) vanishes as $|x| \to \infty$ or b) is periodic in x.

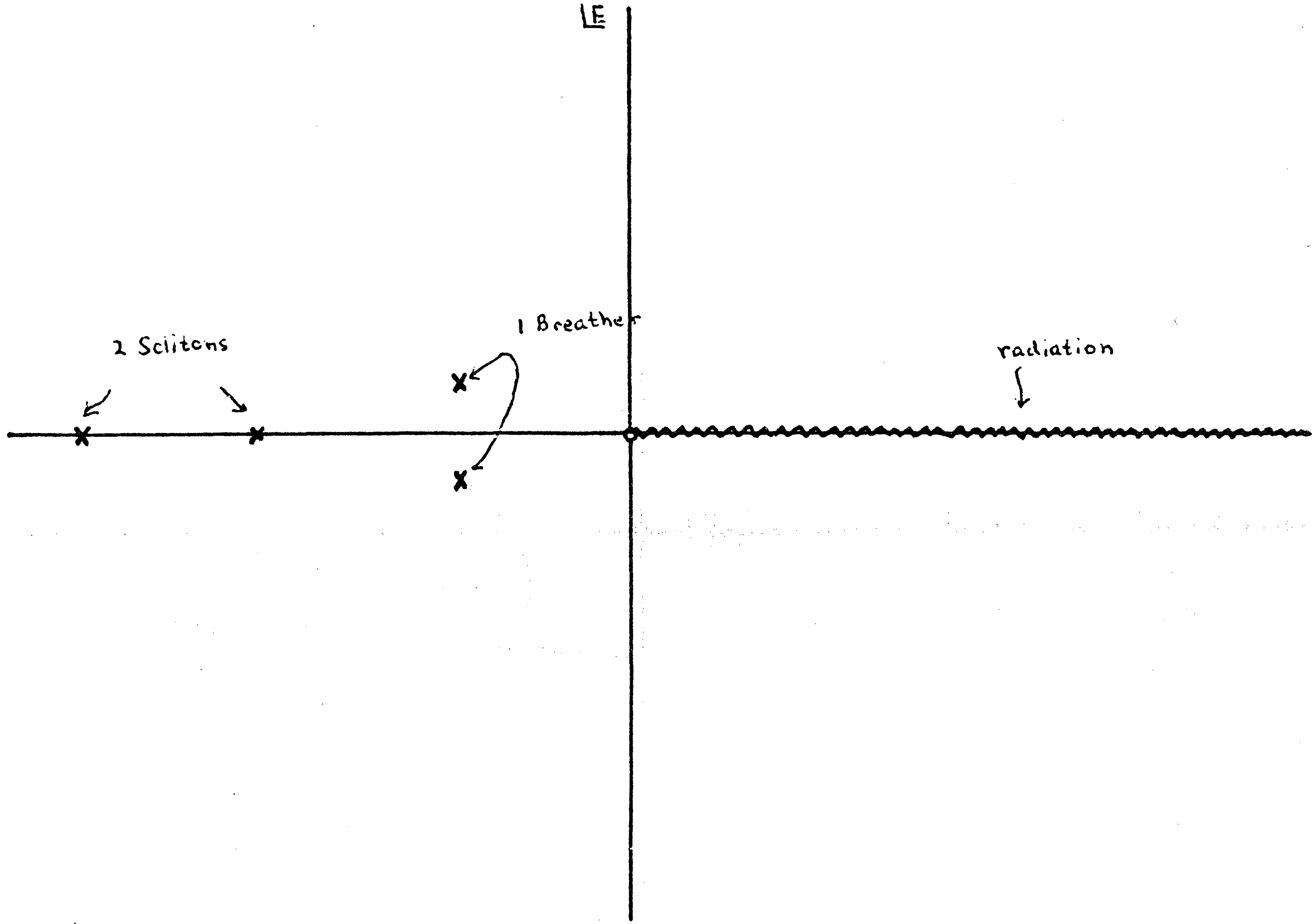


Figure I.5. Spectrum for the eigenvalue problem (I.2) when $u \pmod{\pi}$ and w vanish as $|x| \to \infty$.

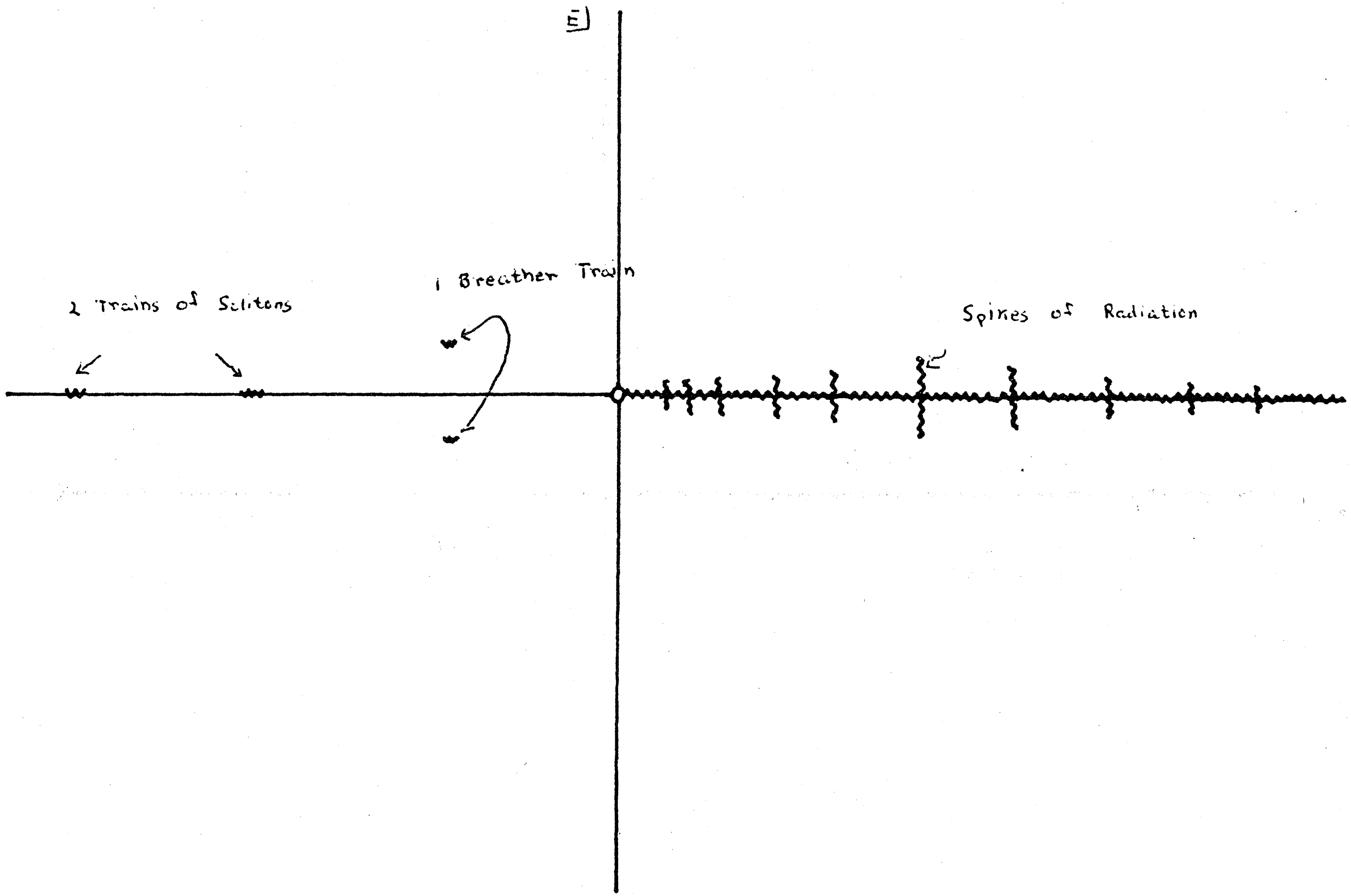
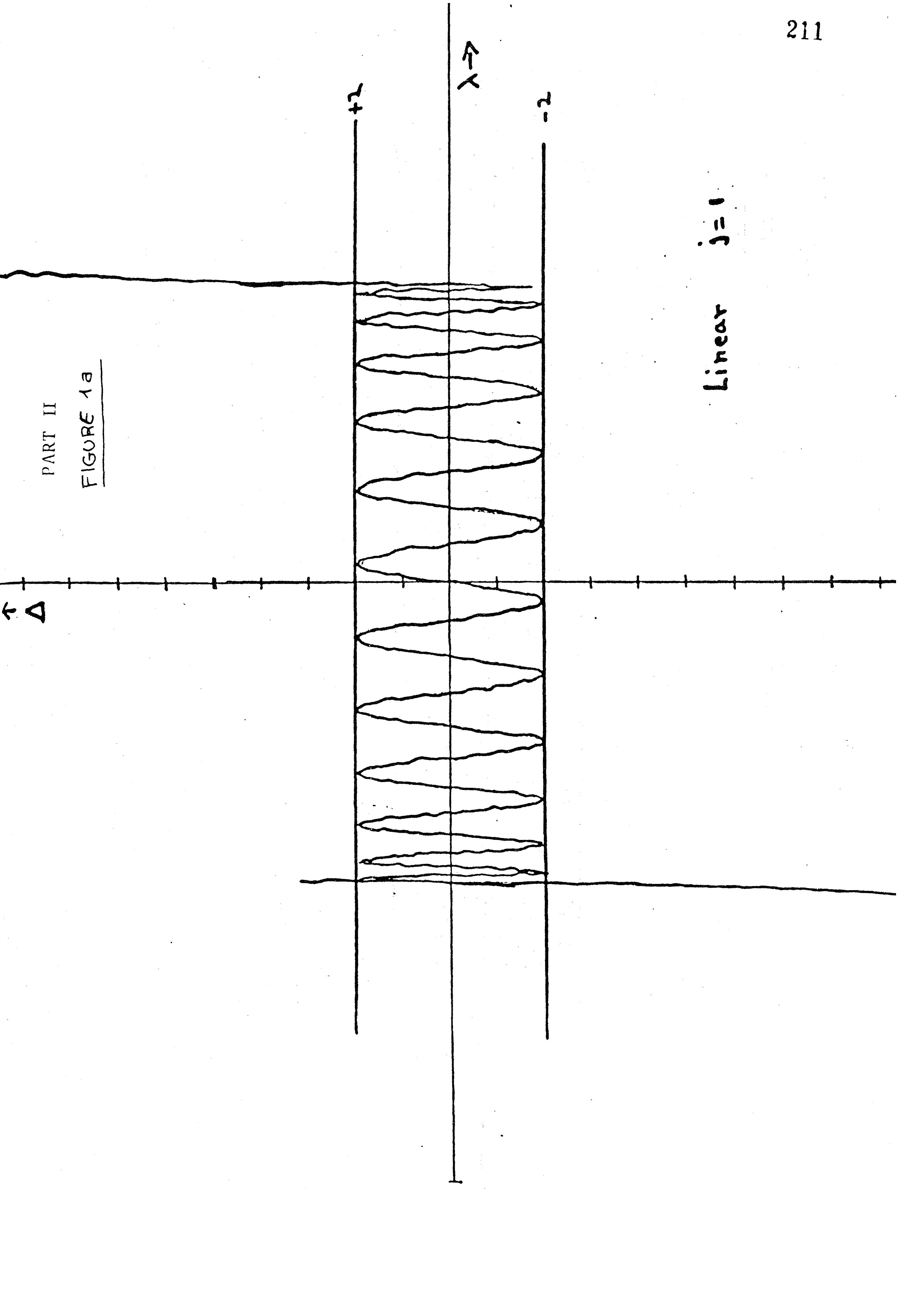
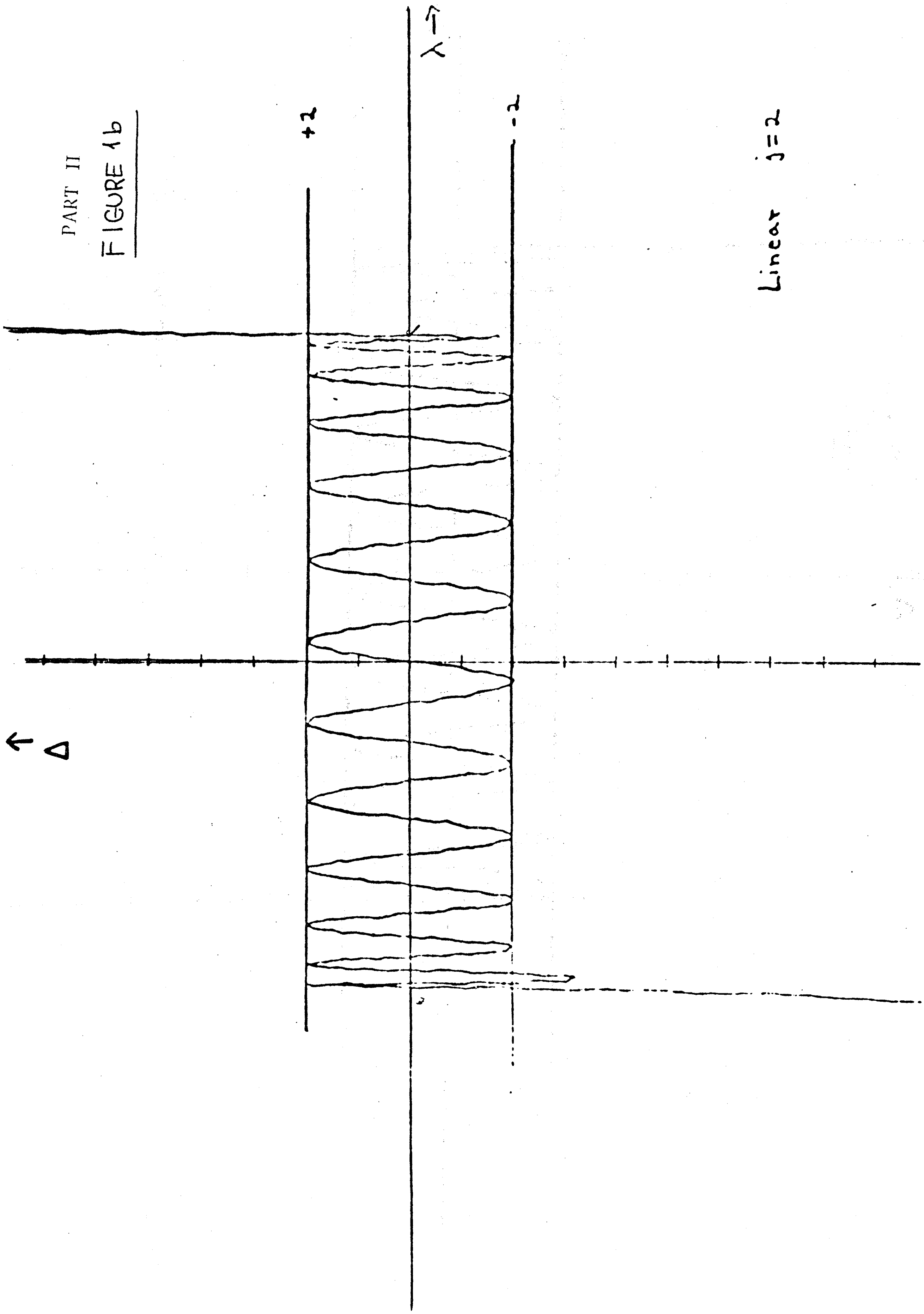


Figure I.6. Spectrum for the eigenvalue problem (I.2) when u(mod π) and w are periodic.



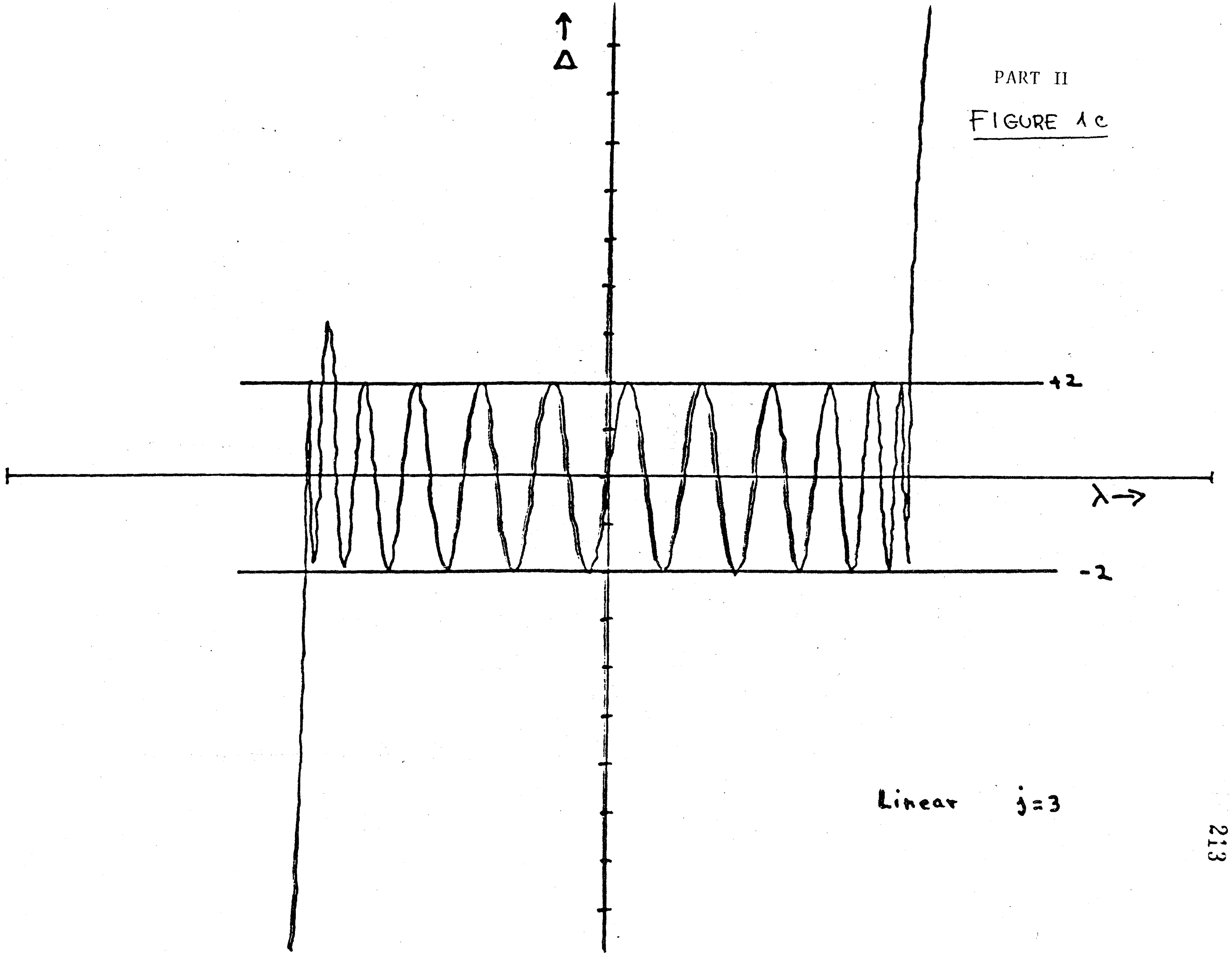
PART II
FIGURE 1a

PART II
FIGURE 1b



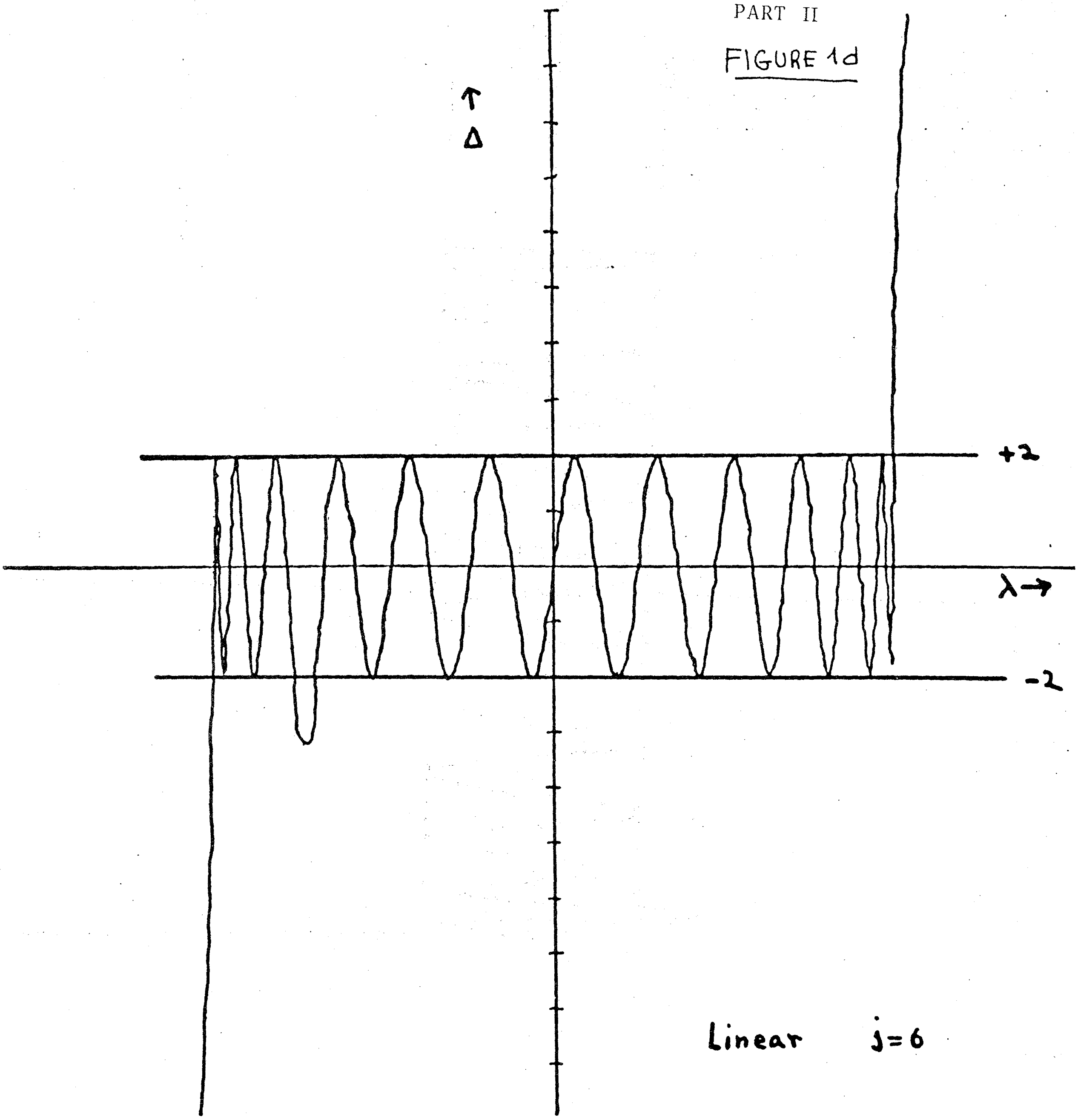
PART II

FIGURE 1c

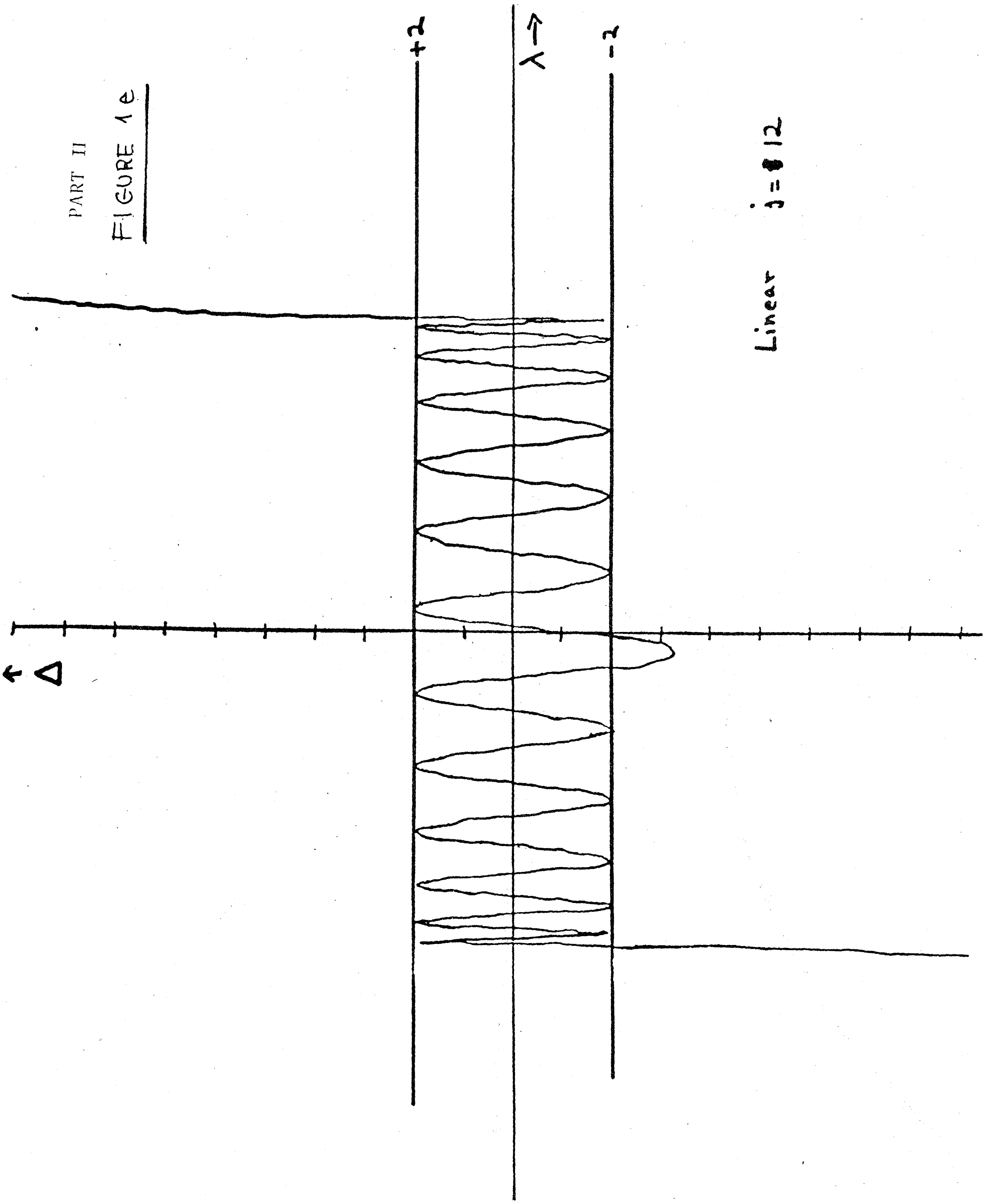


PART II

FIGURE 1d



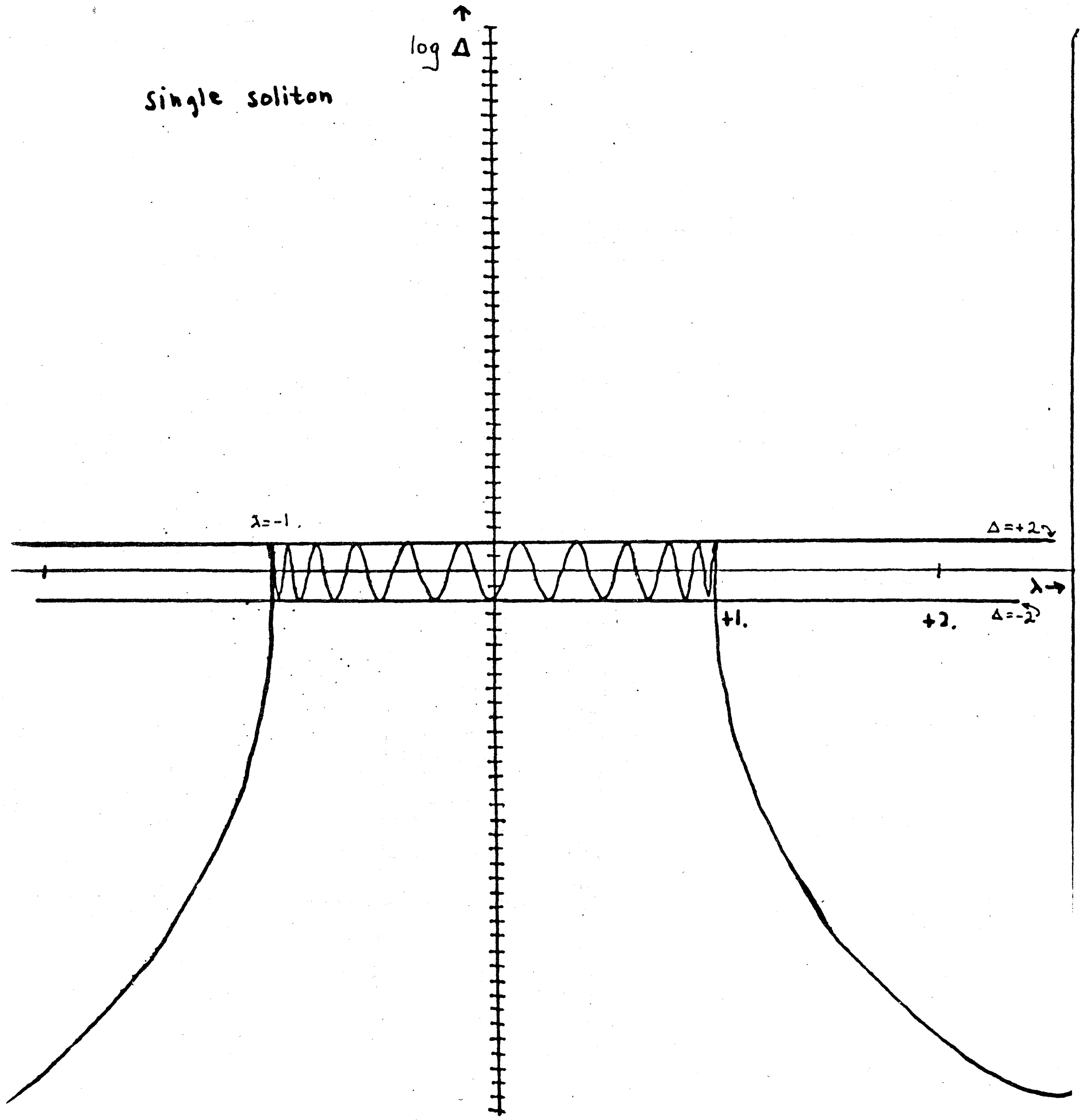
Linear j=6



PART II
FIGURE 1e

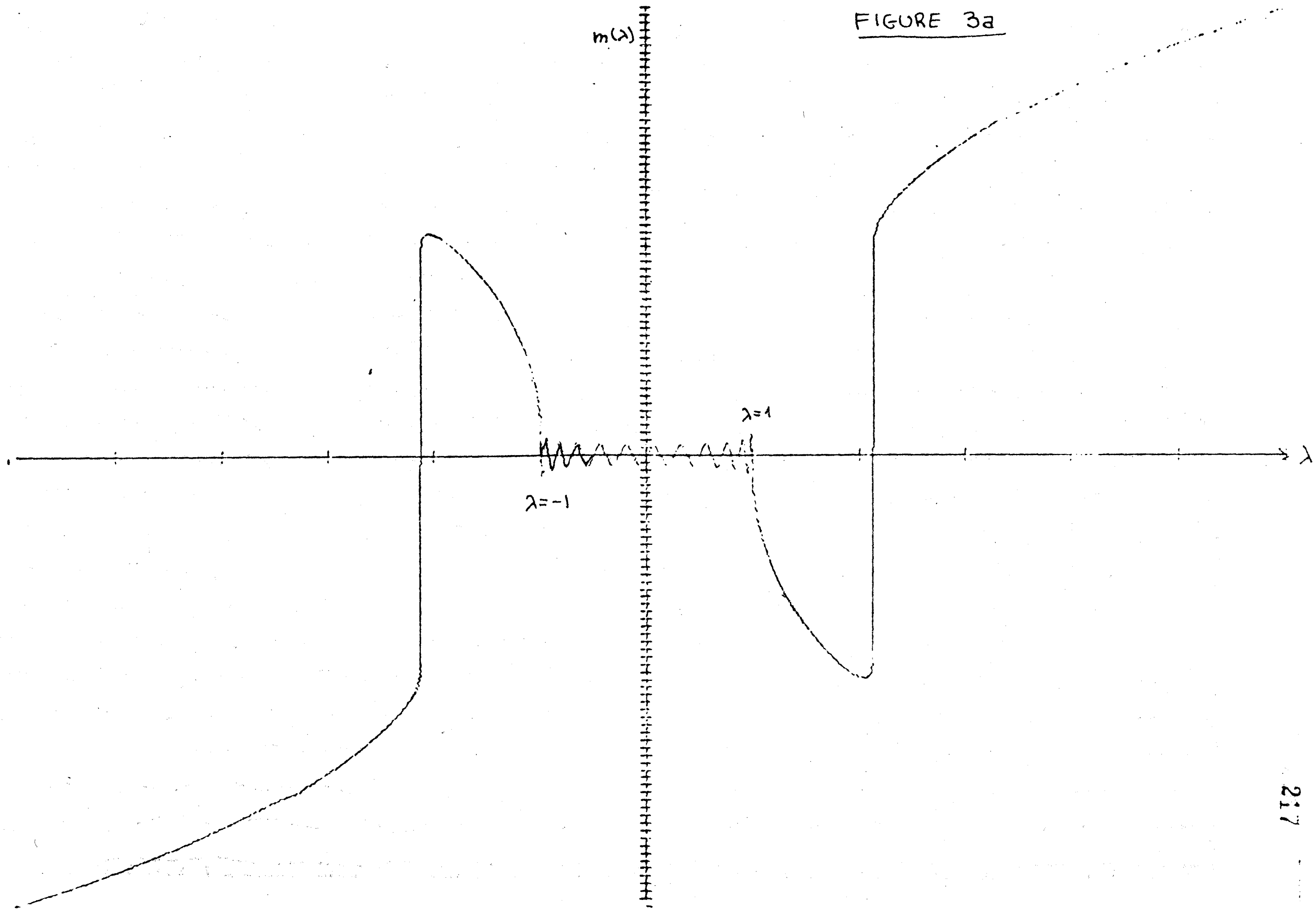
PART II

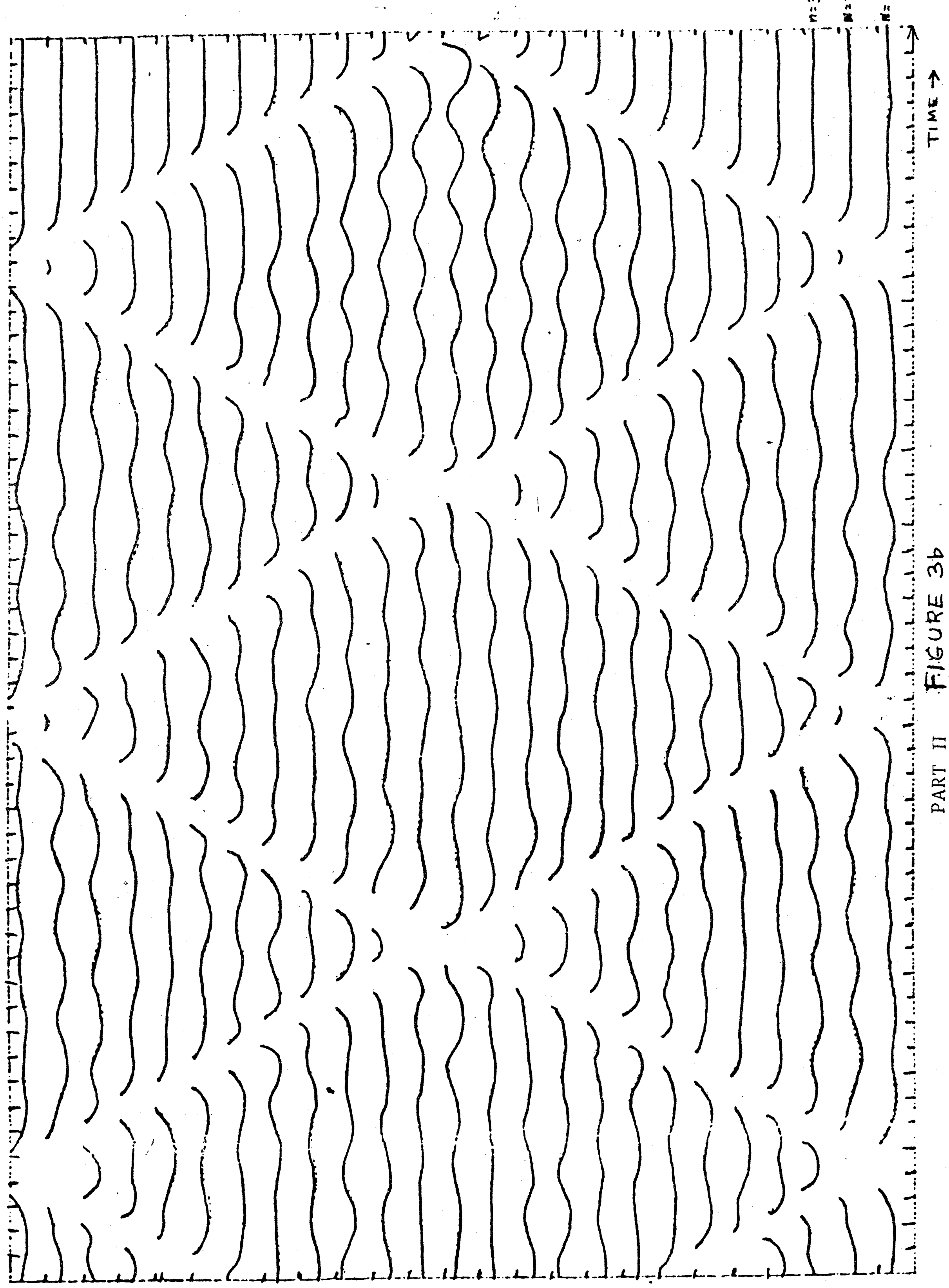
FIGURE 2



PART II

FIGURE 3a

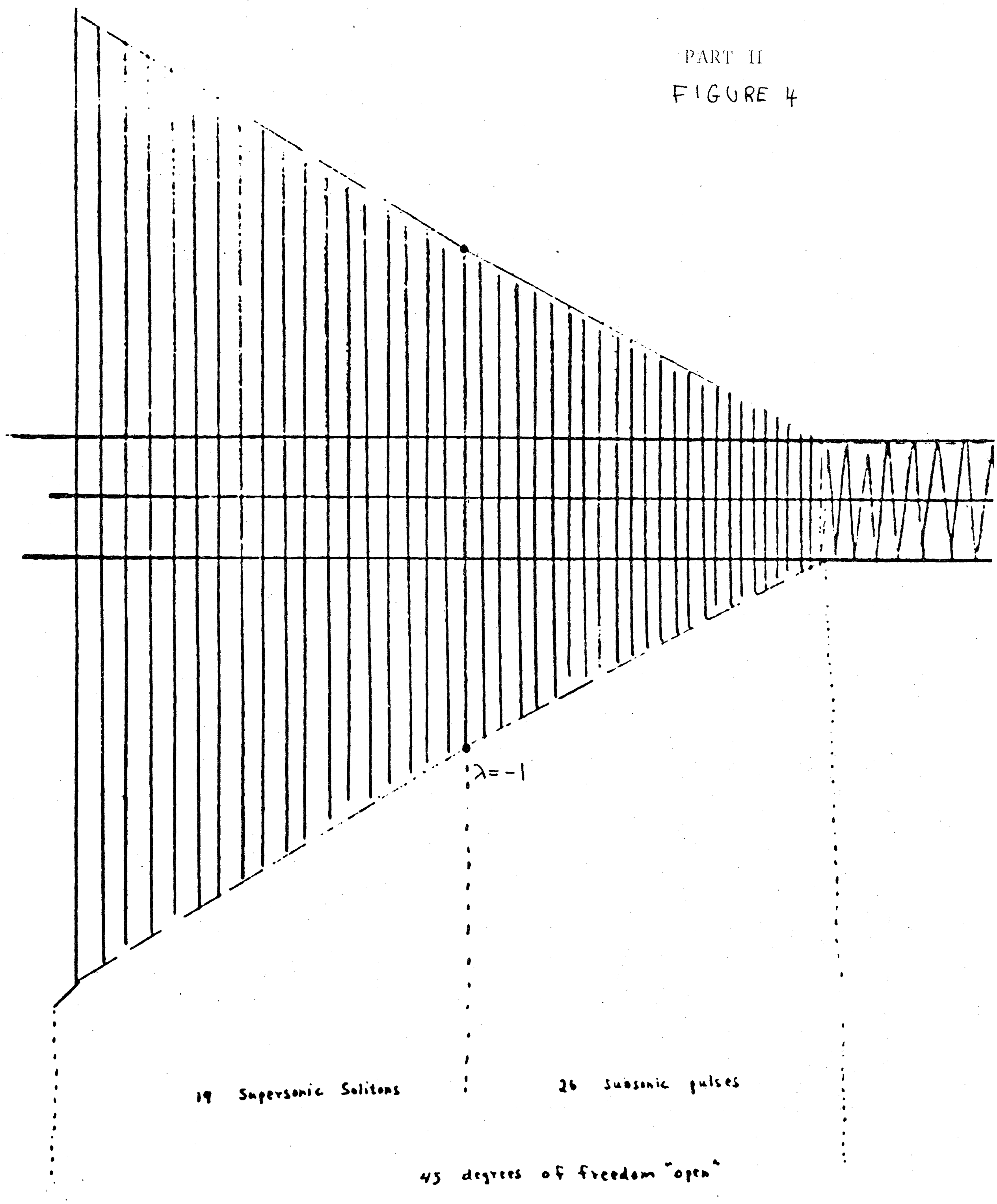




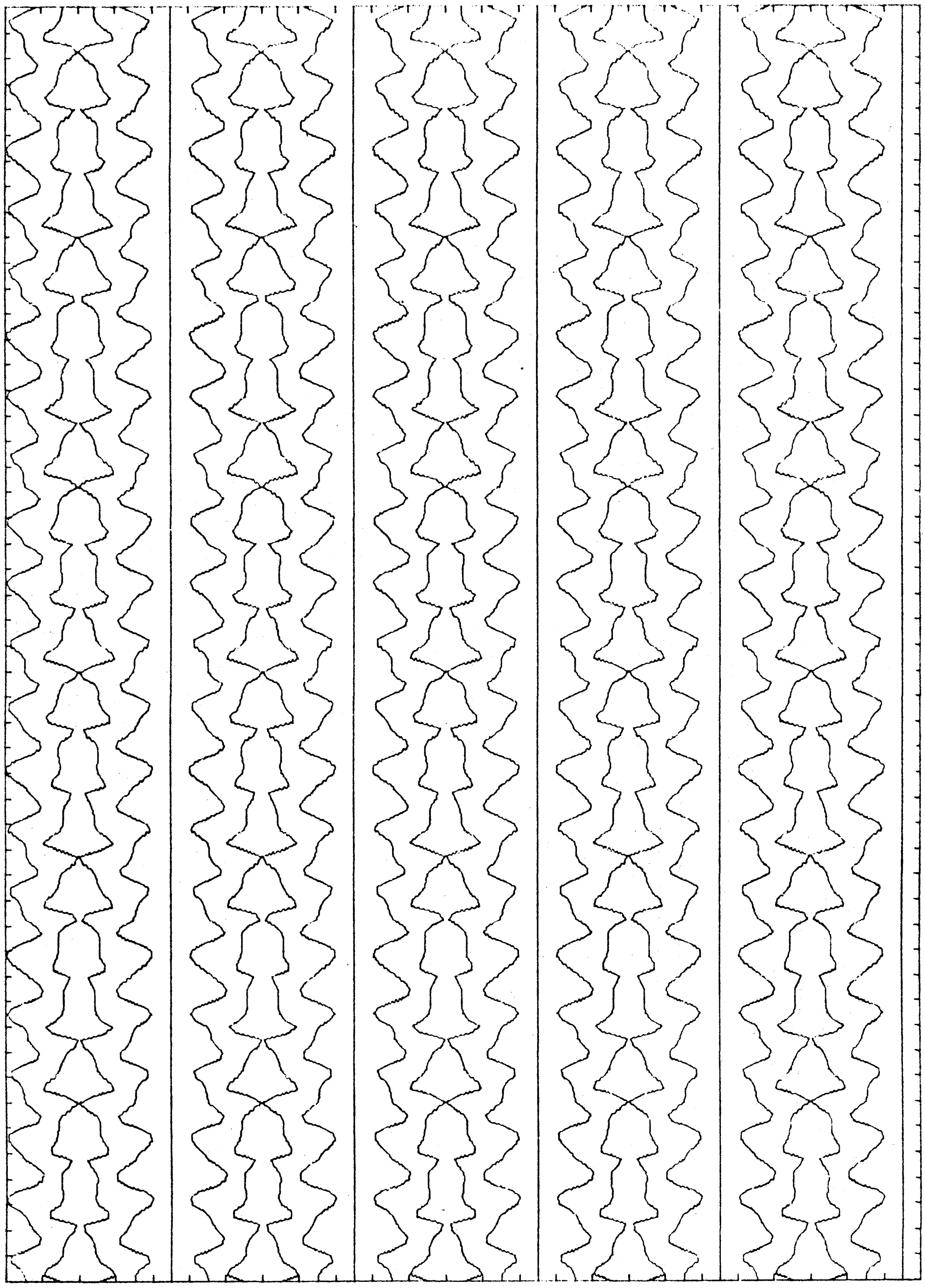
PART II FIGURE 3b

PART II

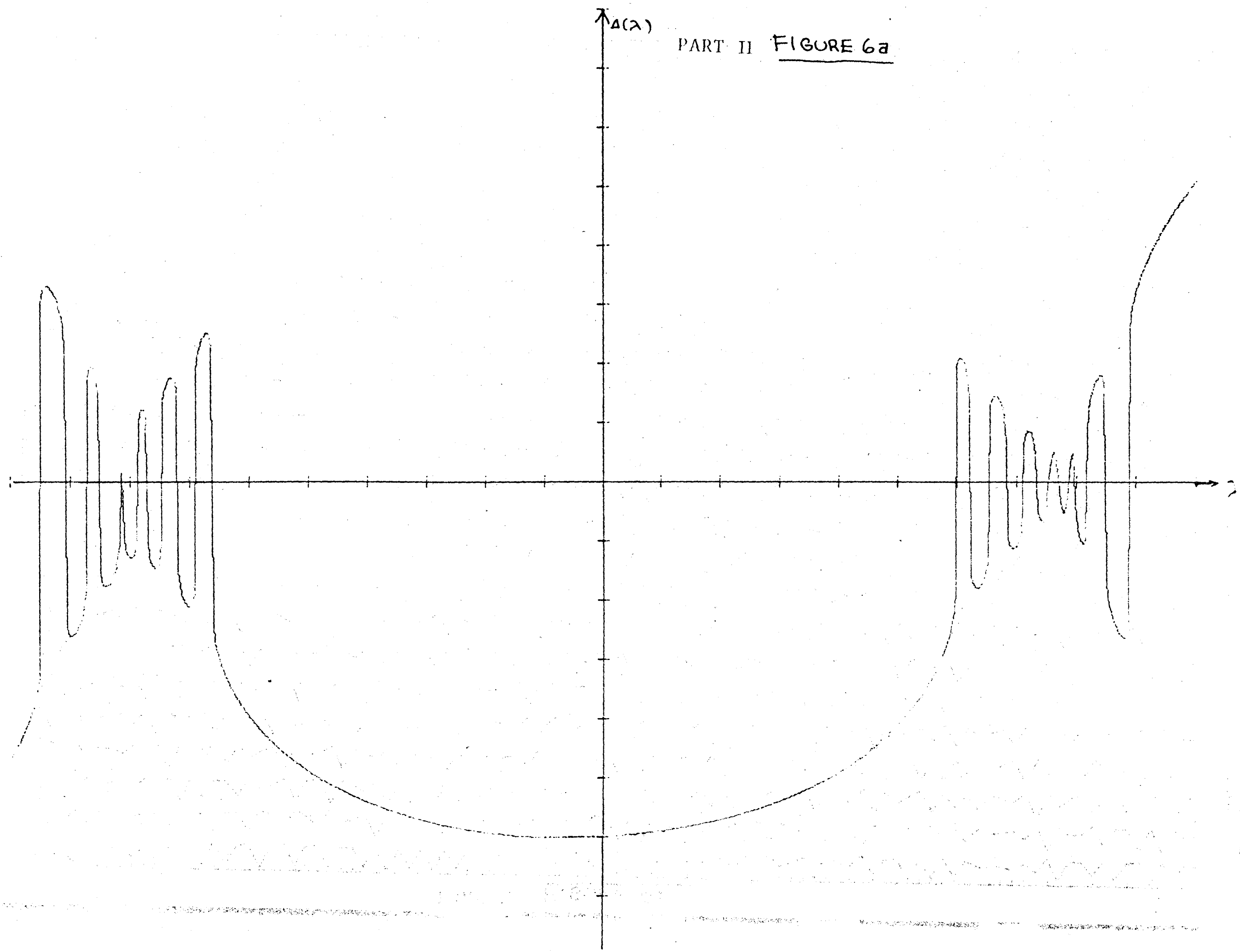
FIGURE 4



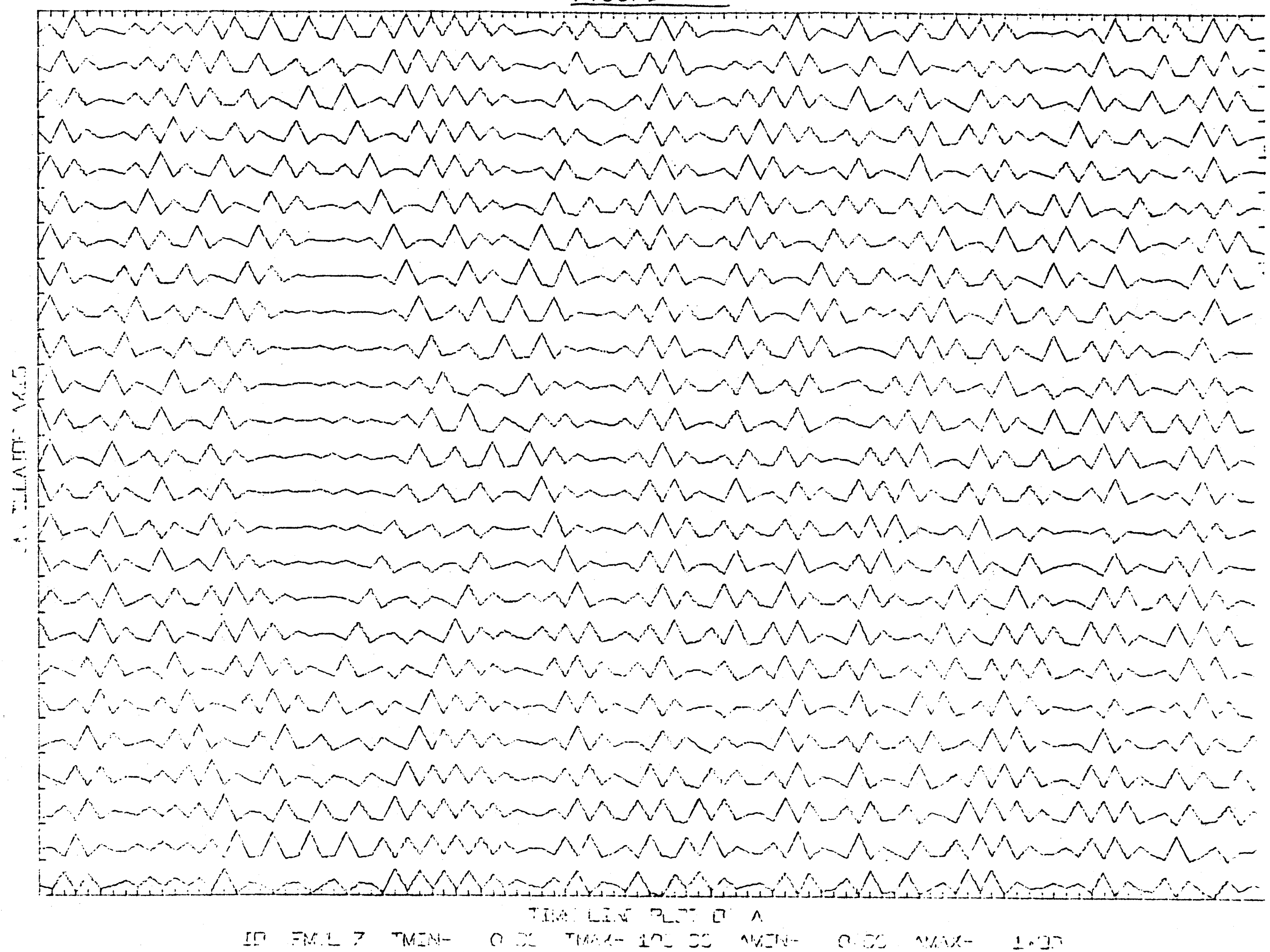
PART II FIGURE 5



PART II FIGURE 6a

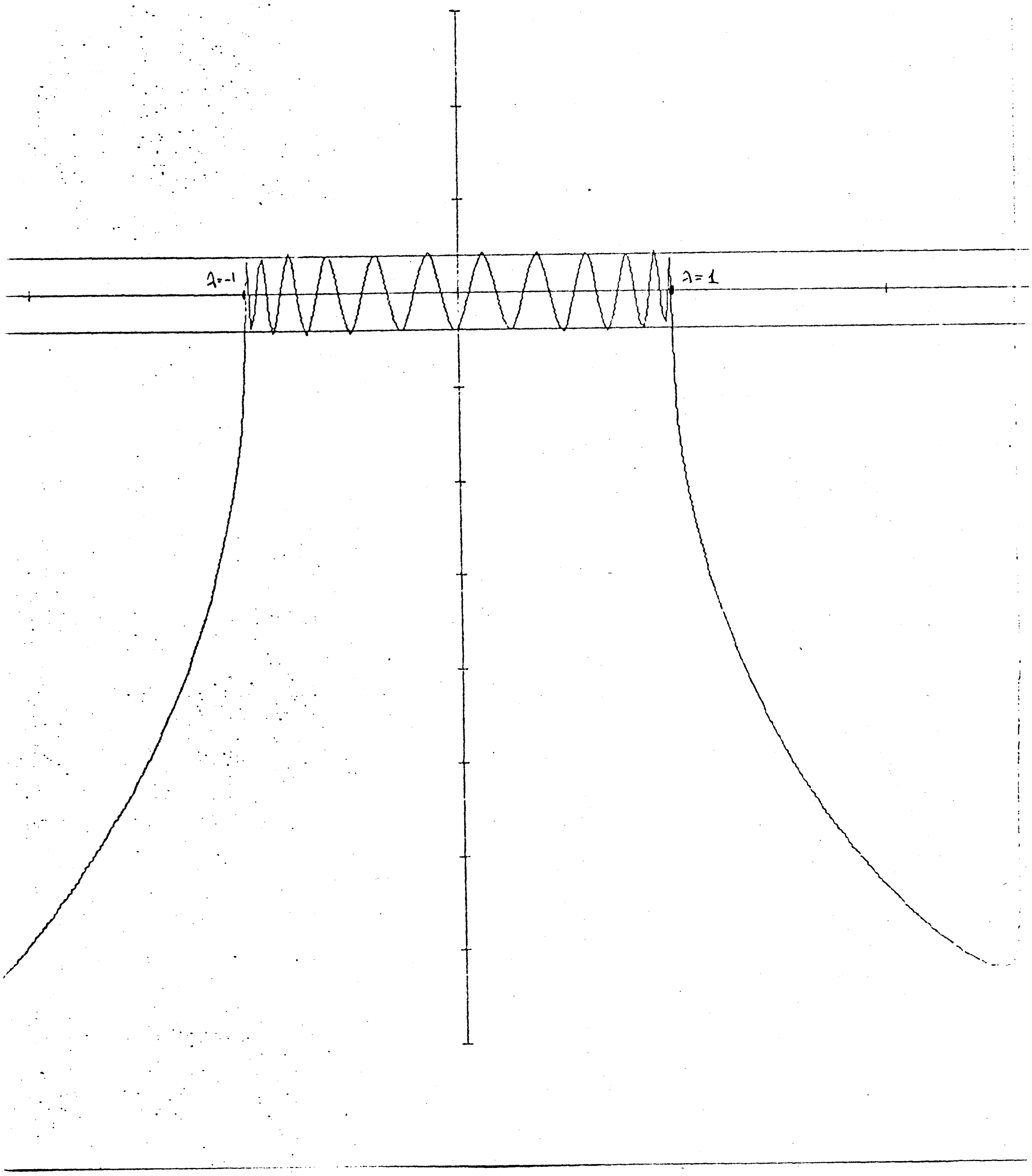


PART II FIGURE 6b

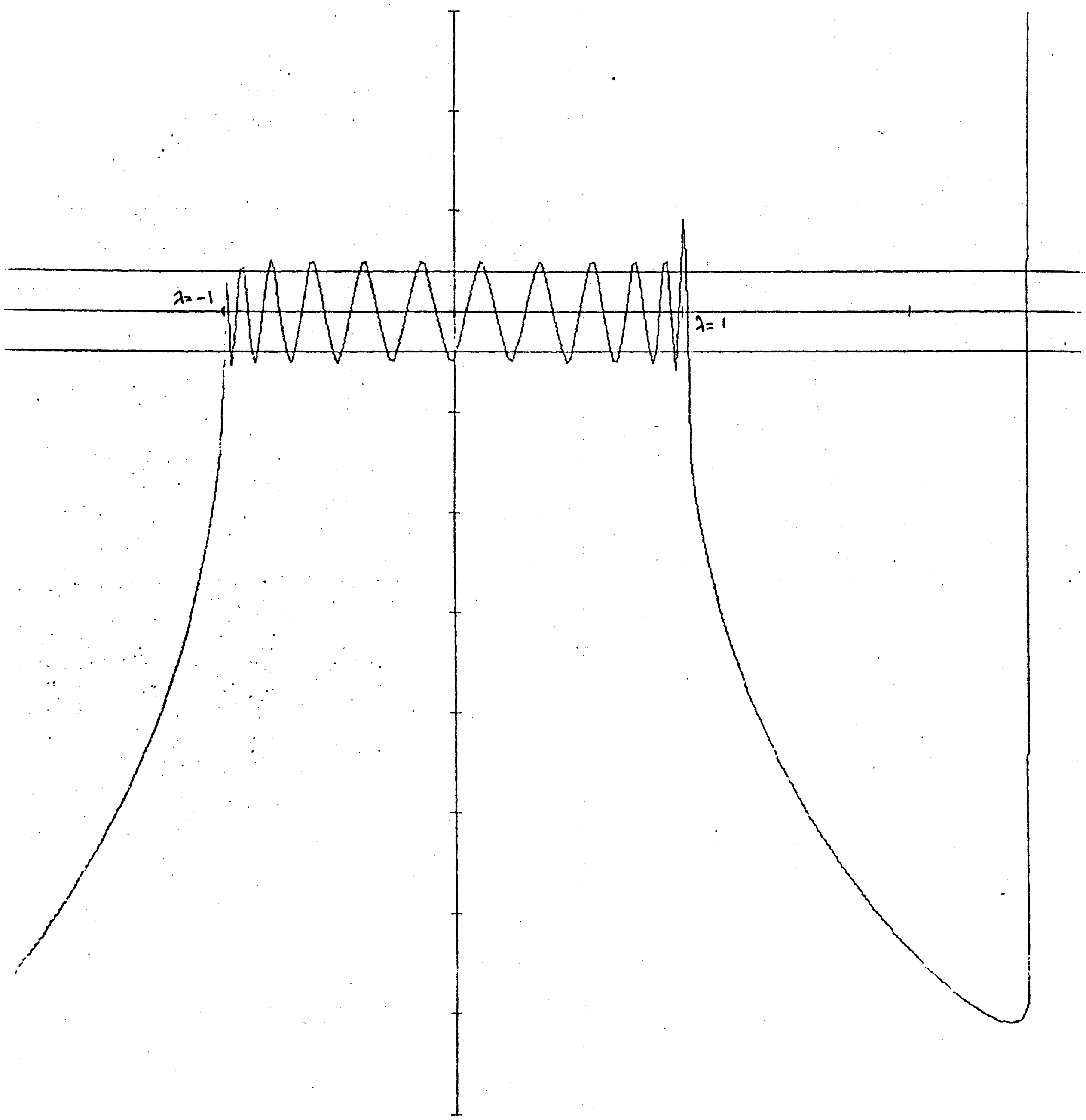


PART II FIGURE 7a

t = 6·8

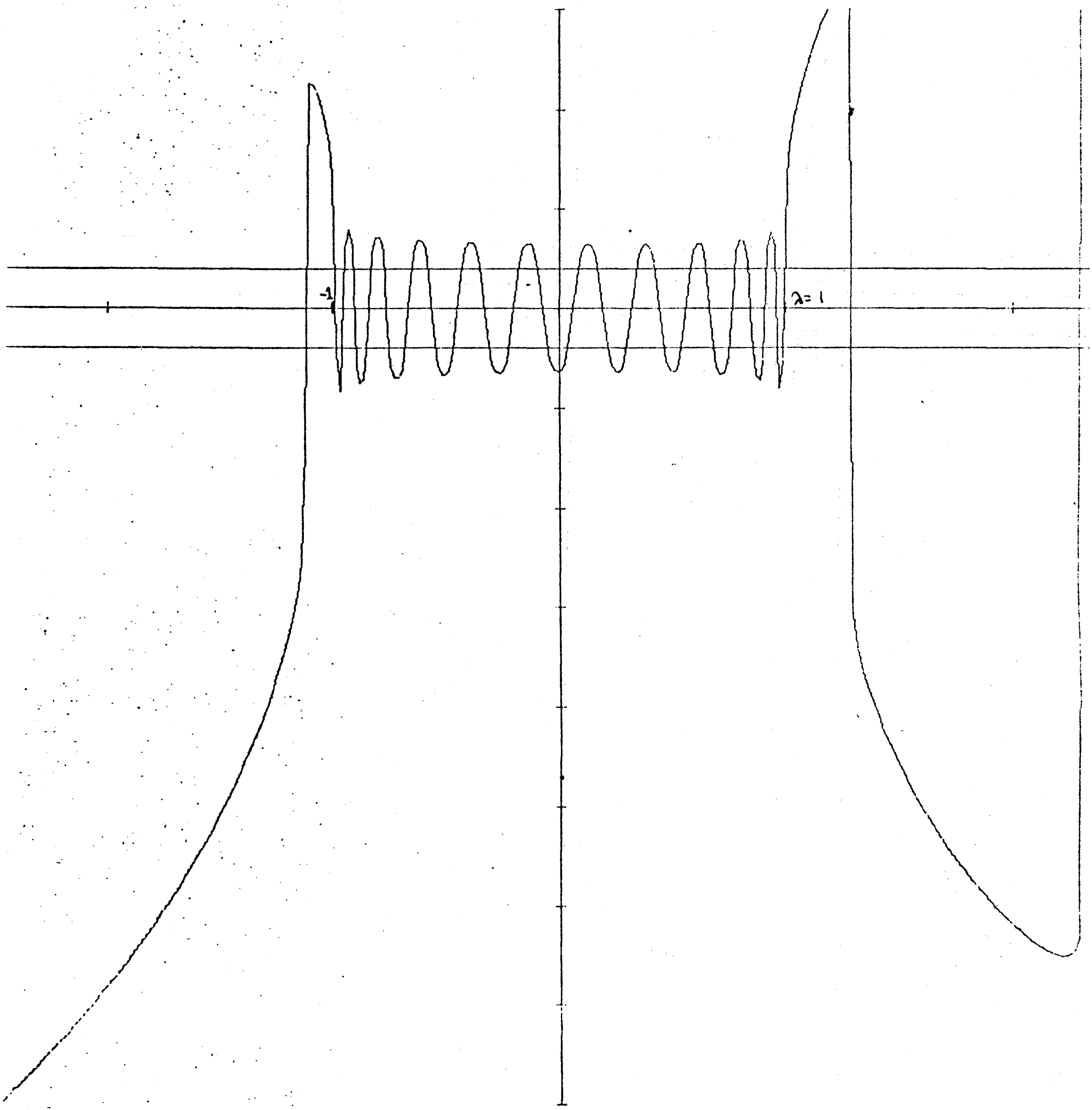


PART II Figure 7b
$t = 7.1$



PART II FIGURE 7c

$t = 7.5$



PART II FIGURE 7d
$t = 7.7$

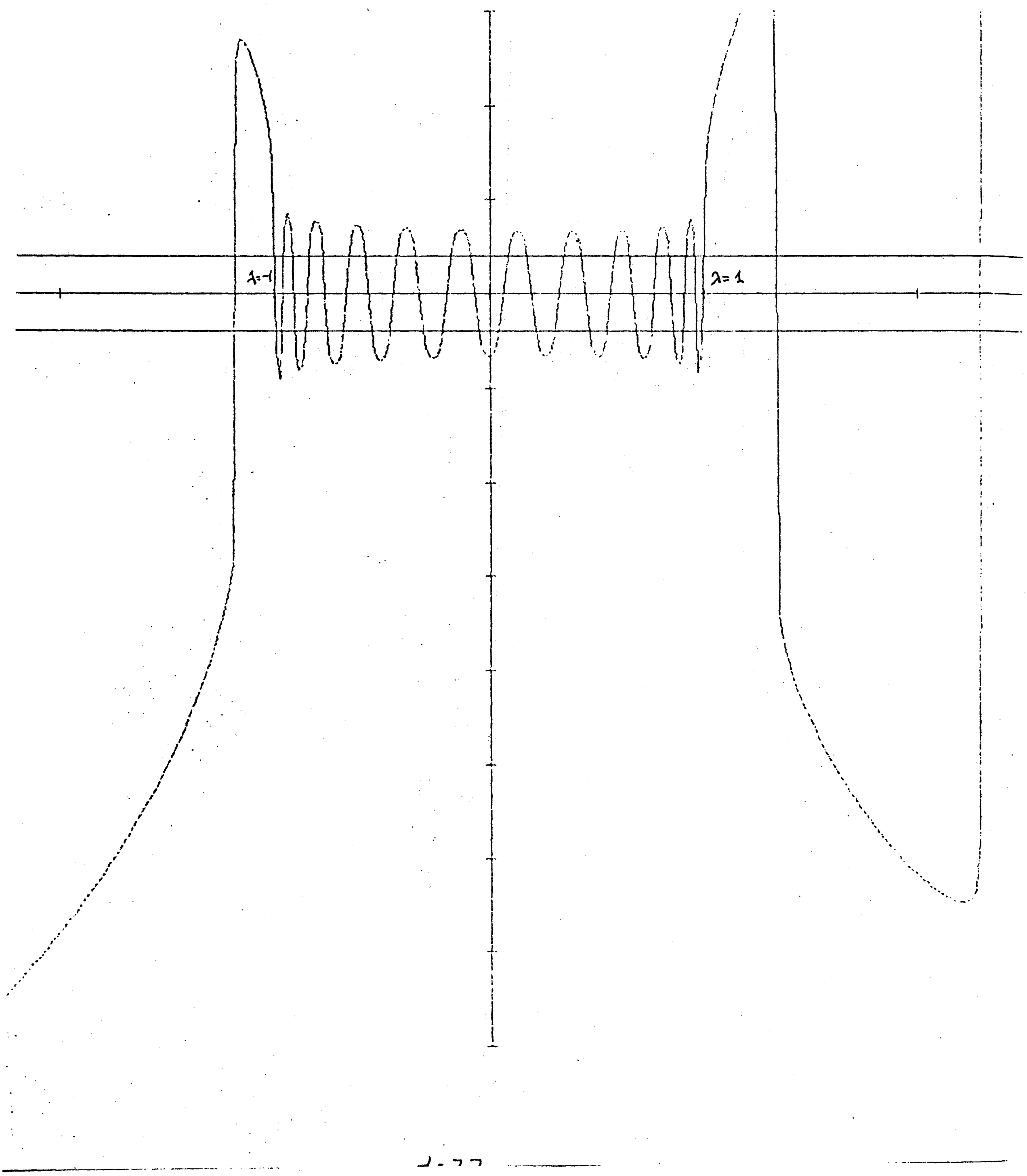
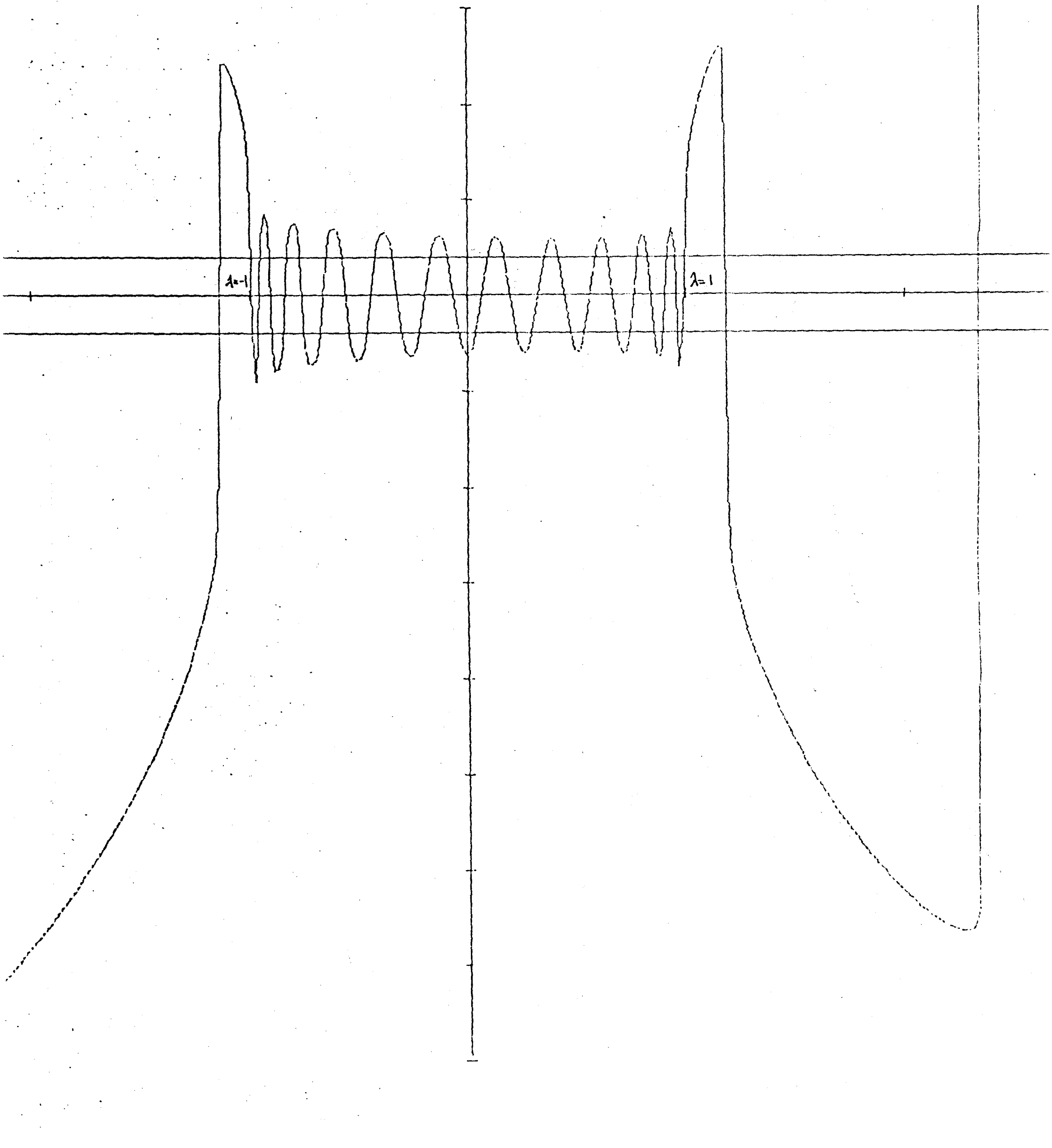


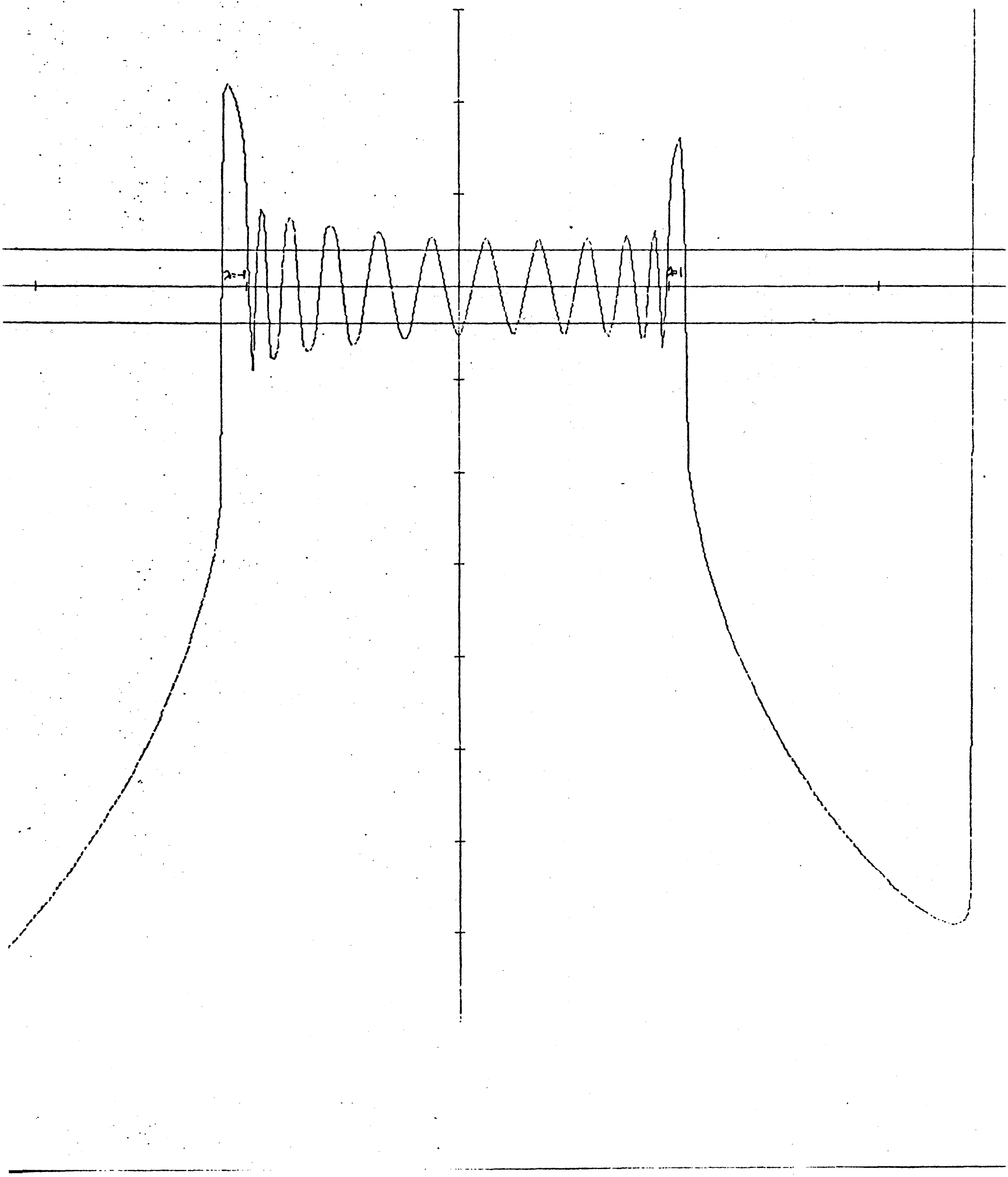
FIGURE 7E

PART II

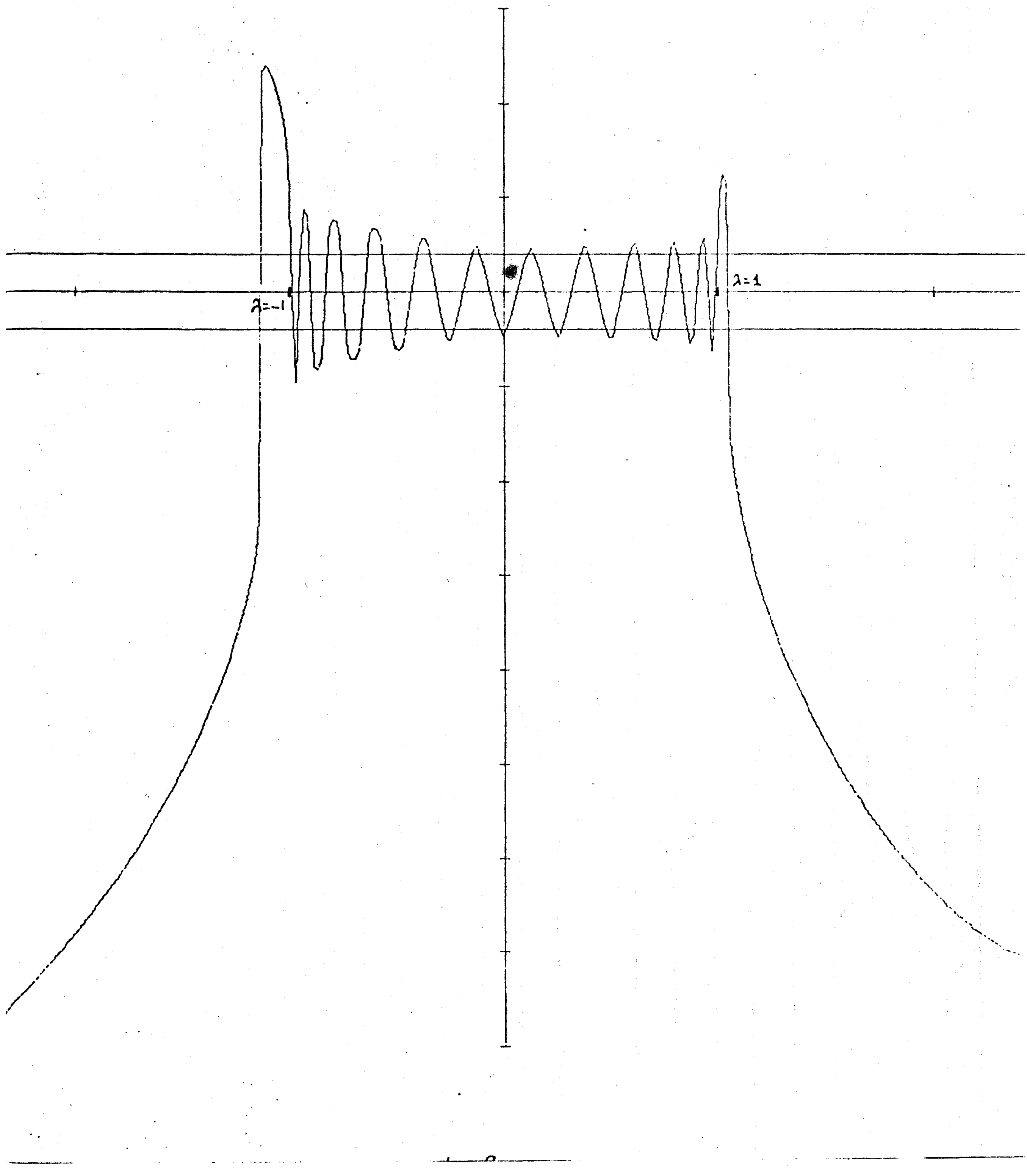
$t = 8.1$



PART II Figure 7f
t = 8.3



PART II Figure 7g
t= 9



PART II FIGURE 8