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Billiard problem between two eccentric circles

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In 1961 Ja.G.Sinai proved a theorem which states that a system of hard pellets in a two-dimensional torus is ergodic, and in particular it is a K-system. The hard pellets referred to above are convex bodies impenetrable and bouncing elastically. This suggests us that a billiard problem which is not applicable to this theorem is worth studying. The system considered here is a billiard problem in a domain enclosed by two eccentric circles. This system is a simplest example which denies Sinai's presumption.

We introduce the coordinate system (l,s), where l is the length of the arc from the point P on the outer circle, divided by the total circumference, $|l| \leqq 0.5$ and $s=\sin\alpha$, α being the reflected angle at the point of collision with the outer circle $|\alpha| < \frac{\pi}{2}$ (see Fig.1). The particle at $P_1(l_1 ,s_1)$ collides with the outer circle, directly or after a collision with the inner circle, and gives the next point $P_2(l_2 ,s_2)$. The mapping T which transforms $P_1(l_1,s_1)$ to $P_2(l_2,s_2)$ on (l,s) plane is shown area-preserving. In what follows we shall be interested in the sets of points obtained by repeated application of the mapping T.

The geometry of the present system is conveniently described by two parameters $r' = r/R$ and $\delta' = \delta/R$, where R and r are the radii of the outer and inner circles respectively and δ is the distance between the centers of two circles. The dashed circle of the radius $r+\delta$ centered at O in Fig.1 shows the boundary, in the outer domain of which any concentric circle is a caustic, i.e., a trajectory tangent to it at any point remains tangent to it after reflection. In the case of concentric circles $\delta = 0$, a mass point collides to the outer circle with the reflection angle α kept constant. Then the mapping in the (l,s) plane lies also on a straight line parallel to the l-axis. This billiard problem is thus said completely integrable.

In the case of eccentric circles, some typical mappings are shown for $r' = 1/2$, $\delta' = 1/4$ in Fig.2. We can distinguish four regions A,B,C and D. In the region A there exist the invariant curves surrounding the stable fixed point of T at the origin (0,0). The other fixed point (1/2,0) is unstable. This invariant curves are elliptic near the origin and become more and more distorted on increasing their sizes. These invariant curves are supposed to be expressed by some complicated function but yet unknown. The point on the boundaries between the two regions B and C corresponds to such a set of α and l that yields an orbit just tangent to the inner circle. These boundaries make contact with the region A, or in other words, every orbit on the region A collides with the two circles alternately, and the orbit corresponding to the outermost invariant curve of A is tangent once at least to the inner circle.

In the region C, any point on the outer circle goes to the outer circle without colliding with the inner circle, keeping α constant and therefore the mapped point lies on a line parallel to the l-axis. The mapping confined in the region B lies on an invariant curve just like in the region A, but in the region B, when mapping proceeds further, the mapped point escapes the region B and enters into the region C, and re-enters into the region B, but, this time, the mapped points in the region B lie on other invariant curve different from the initial one before leaving the region B and entering into the region C. The invariant curves in the regions A and B are to be determined by the following relations,

$$\sin \alpha_1 + \delta' \sin(\alpha_1 - \omega_1) = r' \sin \beta$$

$$\sin \alpha_2 + \delta' \sin(\alpha_2 + \omega_2) = r' \sin \beta$$

$$2\beta = \alpha_1 - \omega_1 + \alpha_2 + \omega_2 .$$

The points appearing in the regions B and C in Fig.2 are generated by repeated mappings T from one initial point. The transition from one invariant curve to another by traversing the region C makes the outlook of the mapping stochastic.

Fig.2 also shows that six fixed points of T^6 exist and that they are stable and give rise to rather large islands around them. The region D has a continuous family of caustics.

On the other hand, in the case where r' is equal to or smaller than δ', the fixed point at the origin becomes hyperbolic and unstable. No regular region just like the region A in Fig.2 does appear. Thus the mapped points easily escape the region B and enter into the region C. The stochastic region extends over all the region except the region D.

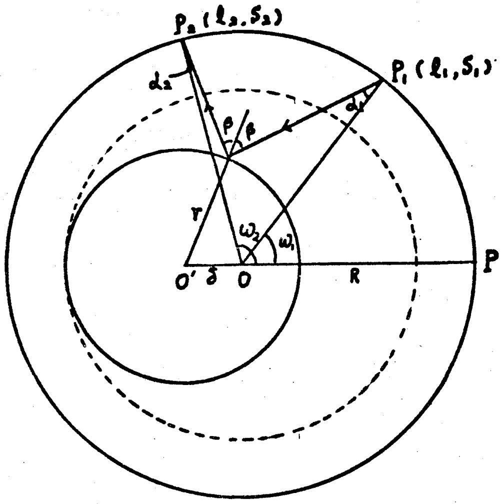

Fig.1, the coordinate system (l,s) in the two eccentric circles centered at O and O', where s = sin α .

Fig.2, mappings for $r' = 0.5$, $\delta' = 0.25$.