A THEOREM ON BINARY DIGITS

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Let \( B(n) \) denote the number of digits 1 in the representation of a natural number \( n \) in the binary scale. It is well known that for most \( n \), the number \( B(n) \) is about half the total number of digits, so that \( B(n) \) is roughly equal to \( \frac{1}{2} n \), where \( n = \nu(n) = \log_2 n \) with \( \log_2 \) denoting the logarithm to the base 2. In fact it follows from the Central Limit Theorem of probability theory that the number \( n \) with

\[
\frac{B(n) - \frac{1}{2} n}{\sqrt{n}} \leq \frac{1}{2}
\]

have density

\[
p(\frac{1}{2}) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-2t^2} dt.
\]

Here we say that a set \( S \) of natural numbers has density \( p \) if the number \( S(x) \) of elements \( n \in S \), \( n \leq x \) satisfies the asymptotic relation \( S(x) \sim px \) as \( x \to \infty \).

Stolarsky was the first to compare \( B(n) \) and \( B(kn) \), where \( k > 1 \), in a fixed odd integer. He called a number \( k \)-sturdy if \( B(n) \equiv B(kn) \),
and similarly if it is $k$-sturdy for every $k$. Stolarsky proved that the $3$-sturdy numbers have density $\frac{1}{2}$. Here we are going to sketch a proof that for any odd $k > 1$, the $k$-sturdy numbers have density $\frac{1}{2}$.

The interest in the proof lies in the fact that it uses Markov Chains. The main result is as follows.

**Theorem.** Let $k_1, \ldots, k_s$ be distinct odd integers. The matrix $M = (\psi_{j,j})$ with entries $\psi_{j,j} = k_i^{-1} (\gcd(k_1,k_2))^2$ has an inverse $Q = (q_{j,j})$, and the quadratic form $Q(t_1, \ldots, t_s) = \sum_{i,j=1}^{s} q_{j,j} t_i t_j$ is positive definite. Hence

$$Q(t_1, \ldots, t_s) = \left(\frac{\pi}{2}\right)^{s/2} \left(\det M\right)^{-1/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_s$$

is well defined for $(t_1, \ldots, t_s) \in \mathbb{R}^s$. The main assertion now is that the natural numbers having simultaneously

$$\frac{B(k_1,n) - \frac{1}{2}n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

have density $Q(t_1, \ldots, t_s)$.

A corollary is that for distinct odd integers $k_1, k_2$, the numbers $n$ with $B(k_1, n) - B(k_2, n) \leq \eta \sqrt{n}$ have density

$$\varphi(\eta) = \left(\pi (1 - \alpha)^{-1/2}\right) \int_{-\infty}^{\infty} e^{-t^2/(1 - \alpha)} dt$$

with $\alpha = k_1^{-1} k_2^{-1} (\gcd(k_1, k_2))^2$. In particular, numbers $n$ with $B(k, n) \leq B(k_2, n)$ have density $\varphi(0) = \frac{1}{2}$. Another corollary is that the sturdy numbers have density $0$.

Let $\mathcal{O}$ be the ring of $2$-adic integers.
\[ N = a_1 + 2a_2 + 2^2a_3 + \ldots \]

with each digit \( a_k \) either 0 or 1. The triple \( (\Omega, \mathcal{F}, P) \), where \( P \) is the Hausdorff measure on \( \Omega \), and \( \mathcal{F} \) consists of \( P \)-measurable subsets of \( \Omega \), is a probability triple. Write \( B_h(N) \) for the number of
digits 1 among \( a_1, \ldots, a_h \). Given distinct odd \( k_1, \ldots, k_s \), put
\[ S_h^{(i)} = B_h(k_i; N) - \frac{1}{2} h \quad (i = 1, \ldots, s), \]
and write
\[ R_h = \left\{ h^{-1/2} S_h^{(i)} \leq \frac{1}{2} \quad (i = 1, \ldots, s) \right\}. \]

The theorem can be shown to be a consequence of the

**Proposition.** \( \lim_{h \to \infty} R_h = g \left( f_1, \ldots, f_s \right) \).

Write \( k_i N = b_i^{(i)} + 2 b_2^{(i)} + \ldots \), and put \( X_i^{(i)} = \left\{ \begin{array}{ll}
\frac{1}{2} & \text{if } b_i^{(i)} = 1 \\
-\frac{1}{2} & \text{if } b_i^{(i)} = 0
\end{array} \right. \).

Then \( S_h^{(i)} = X_1^{(i)} + \ldots + X_h^{(i)} \). For given \( i \), the random variables
\[ X_1^{(i)}, X_2^{(i)}, \ldots \]
are independent. But the random vectors \( \vec{x}, \vec{y}, \ldots \)
where \( \vec{x} = (X_1^{(i)}, \ldots, X_h^{(i)}) \), are not independent. Hence we cannot use
the Central Limit Theorem for sums of independent random variables.

Call a vector \( \vec{x} = (x_1, \ldots, x_s) \) admissible, if there is a real
number \( \alpha \) in \( 0 \leq \alpha < 2 \) with \( x_i = \left[ \frac{x_i}{2^\alpha} \right] \quad (i = 1, \ldots, s) \), where \( \left[ \cdot \right] \)
denotes the integer part. There are finitely many admissible vectors.
The vector \( \vec{0} = (0, 0) \) is admissible, and if \( \vec{x} \) is admissible, then
both \( \vec{x}^0 = (\left[ x_1 / 2^\alpha \right], \ldots, \left[ x_s / 2^\alpha \right]) \) and \( \vec{x}^1 = (\left[ x_1 / 2^\alpha \right] + k_1, \ldots, \left[ x_s / 2^\alpha \right] + k_s) \)
are admissible.

Given \( N \) as above, put \( N^t = a_1 + 2a_2 + \ldots + 2^{t-1}a_t \). Put \( \delta_0 = 0 \).
and 
\[ x_t = z_t(N) = \left( 2^{-t+1} k, \eta_t, \ldots, 2^{-t+1} k, \eta_t \right) \quad (t = 1, 2, \ldots) \]

Then \( z_0, z_1, \ldots \) are (vector valued) random variables. Clearly, \( z_t(N) \) is always admissible, and conversely if \( z \) is admissible, then there is an \( N \) and a \( t \) with \( z_t(N) = z \). It is now easy to prove the

**Lemma.** The random variables \( z_0, z_1, \ldots \) form a Markov Chain.

The transition probabilities are given by the rule that for given \( z_t \), we have \( x_{t+1} \) either equal to \( z_t \), or to \( z_t' \), each with probability \( \frac{1}{2} \).

We now observe that the random variable \( x_t \) is a "functional" of \( z_t \): We have \( f(k) = f^{(2)}(z_t) \), where \( f^{(2)}(z) = \frac{1}{2} \) if \( x_t \) is odd, \( f^{(2)}(z) = -\frac{1}{2} \) if \( x_t \) is even. Hence the machinery of Markov Chains can be used to complete the proof of our theorem.

In the special case when \( s = 2, k_1 = 1, k_2 = 3 \), there are 6 admissible vectors: \( a = (0, 0), b = (0, 1), c = (0, 2), d = (1, 3), e = (1, 4), f = (1, 5) \). The transition probabilities are given by the following diagram, where each arrow represents a probability of \( \frac{1}{2} \):

References:


K.B. Stolarski. Integers, whose multiples have anomalous digit frequencies. *Acta Arith.* (To appear)