数理解析研究所講究録
第 342 巻 1979 年 39-48

Embedding theorems in shape theory

by

Jin Ono

Department for Liberal Arts

Shizuoka University

Let X be a compactum (a compact metric space) and let E^n be an n-dimensional Euclidean space. Borsuk raised several problems concerning to embed X into E^n up to shape (see [1] and [2]). Partial answers of them were given by Trybulec and many other persons. In section (1) we shall trace their works again.

The second topic is the Chapman's complementary theorem. After [3] several attempts to modify it have been done. We shall recall them in section (2).

§ 1

Borsuk defined the index $e(X)$ for every compactum X. But the trouble is that, between [1] and [2], definitions of $e(X)$ are different. So in this note, we use two simbols $e_1(X)$, $e_2(X)$ if we want to distinguish them.
In [1], $e_1(X)$ is defined by

$$e_1(X) = \min (k \mid E^k \supset \exists Y ; Sh(X) = Sh(Y)),$$

and in [2], $e_2(X)$ is defined by

$$e_2(X) = \min (k \mid E^k \supset \exists Y ; Sh(X) \leqq Sh(Y)).$$

Borsuk's problems are followings :

(1) To find a pure definition of $e(X)$.

(2) $Sh(X) \leqq Sh(Y) \Longrightarrow e_1(X) \leqq e_1(Y)$?

(3) Dose there exist for every n a compactum X such that $Fd(X) = n$ and $e(X) = 2n+1$?

(4) Is it true that for every n-dimensional movable continuum X the number $e(X) \leqq 2n$?

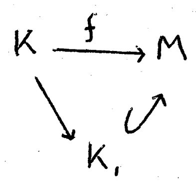
Borsuk pointed out that for $n = 1$ Problem (3) has a positive answer. A solenoid is its example (see [2]).

Trybulec [15] proved the following theorem. It gives an affirmative answer of Problem (4) for the case $n = 1$.

Theorem 1 ([15] Th.3.6). The shape of every movable curve X is plane.

Recentry Ivanšić and Husch investigated them. Their works are essentially based on the embedding theorem up to simple homotopy which was proved by Stallings.

Theorem 2 ([14]). If K is a polyhedron of dimension k , M a manifold of dimension m, and $f: K \longrightarrow M$ a map which is $(2k-m+1)$-connected, then there is a procedure which, when $k \leqq m-3$, yields a k-dimensional subpolyhedron $K_1 \subset M$ and a simple homotopy equivalence $K \longrightarrow K_1$ for which the diagram



is consistent up to homotopy.

Their theorems are following. All of them are pointed case.

Theorem 3 ([10] Cor.3). Every continuum X which is pointed 1-movable and $Fd(X,x) = n \geqq 3$ can be embedded up to pointed shape in E^{2n}.

A pointed space X is said to be r-shape connected provided X is connected and has trivial homotopy pro-groups for $1 \leq k \leq r$. A system map $\{f_i\} : \{X_i\} \longrightarrow \{Y_i\}$ is said to be shape r-connected if it induces an isomorphism of homotopy pro-groups of $\{X_i\}$ and $\{Y_i\}$, denoted by $\pi_j(\underline{X})$ and $\pi_j(\underline{Y})$ for each $1 \leq j < r$ and epimorphism for $j = r$ in the category of pro-groups.

A pointed compactum X has shape finite r-skeleton ($r \geqq 1$) if there exists a finite connected pointed CW-complex K and a tower of pointed CW-complexes $\{X_i\}$ such that $X = \varprojlim \{X_i\}$, and a system map $\{f_i\} : \{K\} \longrightarrow \{X_i\}$ such that $\{f_i\}$ is shape r-connected.

Theorem 4 ([10] Th.5). If X is a pointed compactum, $Fd(X,x) = n$, which is r-shape connected, $n-r \geqq 2$, then (X,x) can be embedded up to shape into E^{2n-r+1}.

Theorem 5 ([10] Th.6). Let X be a pointed compactum which is pointed shape dominated by a polyhedron and let $Fd(X,x) = n \geqq 3$. If (X,x) has trivial shape groups for $1 \leq i \leq r$, $n-r \geqq 3$, then (X,x) can be embeded up to shape into E^{2n-r}.

Theorem 6 ([8] Th.7). Let M be a PL-manifold of dimension q and let X be a continuum which has fundamental dimension n, $q-n \geqq 3$, has stable pro-π_1 which is pro-isomorphic to a finitely presented group and has a shape finite (2n-q+1)-skeleton. If there exists a shape map $\{f_i\} : X \longrightarrow M$ which is shape (2n-q+1)-connected, then there exists a compactum $Z \subseteq M$ such that Sh(X) = Sh(Z).

Theorem 7 ([8] Th.11). Let $Y \subseteq E^q$ be an n-dimensional continuum which has stable pro-π_1 pro-isomorphic to a finitely presented group. Let X be a continuum such that fundamental dimension of $X \leq n$, X has a shape finite (2n-q+1)-skelton and $Sh(X) \leq Sh(Y)$. If $3n < 2q-2$ and $n \geq 3$, then there exists a compactum $Z \subseteq E^q$ such that Sh(Z) = Sh(X).

Theorem 8 ([8] Cor.12). Let $Y \subseteq E^q$, $q \geq 5$, be an n-dimensional continuum which satisfies cellularity criterion (see section 2). If X satisfies the same conditions as in Theorem 7, then X can be embedded up to shape into E^q.

Theorem 9 ([9] Th.1). Let X be a continuum of fundamental dimension $Fd(X) = k \geq 3$ which is pointed (2k-q+1)-movable. If there exists a shape (2k-q+1)-connected shape map $\underline{f} : X \longrightarrow M$ of X into a q-dimensional PL-manifold which is either closed or is open and dominated by a finite complex and if $q-k \geq 3$, then there exists a k-dimensional continuum $Y \subseteq M$ and a shape equivalence $\underline{g} : X \longrightarrow Y$ such that $\underline{i}\underline{g} = \underline{f}$ where $\underline{i}$ is the shape map induced by the inclusion $Y \hookrightarrow M$.

Theorem 10 ([9] Cor.2). Let X be a shape r-connected continuum of fundamental dimension $k \geq 3$ which is (r+1)-pointed movable, then there exists a continuum $Y \subseteq E^{2k-r}$ which has the same shape as X.

They showed that in Theorem 9 the conditions of movability and shape connectivity are essential. In section 4 of [9], counter examples are given. But the condition of (2k-q+1)-movability can be replaced with S^{2k-q+1}-movability.

For Problem (3), Duvall and Husch constructed such spaces for the case of $n = 2^k$ ($k > 1$) (see [6]).

§ 2

Before discussing complementary theorems, we must see some definitions of nice embeddings. Let X be a compactum in a space M.

(1) X is a Z-set in M if for every nonempty homotopically trivial open set U in M, $U \setminus X$ is nonempty and homotopically trivial.

(2) X is a Z_k-set (k an integer ≥ 0) in M if for every nonempty k-connected open set U in M, $U \setminus X$ is nonempty and k-connected.

(3) X is a strong Z_k-set ($k \geq 0$) in E^n if for each compact subpolyhedron P of E^n having dimension $\leq k+1$ and each $\varepsilon > 0$, there is an ε-push h of (E^n, X) such that $h(X) \cap P = \phi$.

(4) X satisfies the cellularity criterion (CC) if given a neighborhood U of X, there is a neighborhood $V \subset U$ of X such that every loop in $V \setminus X$ is null-homotopic in $U \setminus X$.

(5) X is globally 1-alg in M if given a neighborhood

U of X, there is a neighborhood $V \subset U$ of X such that every loop in $V \setminus X$ which is null-homologous in $V \setminus X$ is null-homotopic in $U \setminus X$.

(6) X satisfies the small loops condition (SLC) if for any neighborhood U of X there is a neighborhood $V \subset U$ of X and an $\varepsilon > 0$ such that each loop in $V \setminus X$ of diameter less than ε is null-homotopic in $U \setminus X$.

(7) X satisfies the inessential loops condition (ILC) if for every neighborhood U of X there is a neighborhood $V \subset U$ of X such that each loop in $V \setminus X$ which is null-homotopic in V is also null-homotopic in $U \setminus X$.

Chapman's complementary theorem is following.

Theorem 11 ([3] Th.2). If X and Y are compacta Z-embedded in the Hilbert cube Q, then X and Y have the same shape iff $Q \setminus X$ and $Q \setminus Y$ are homeomorphic.

In this theorem, if we replace Q with E^n or S^n what kinds of conditions are needed ? Chapman showed the next theorem, and Geoghegan and Summerhill improved it.

Theorem 12 ([4] Th.1). Let X, Y be compacta such that dim X, dim $Y \leq m$.

(a) For any integer $n \geq 2m+2$ there is copies $X', Y' \subset E^n$ (of X, Y respectively) such that if Sh(X) = Sh(Y), then $E^n \setminus X'$ and $E^n \setminus Y'$ are homeomorphic.

(b) For any integer $n \geq 3m+3$ there is copies $X', Y' \subset E^n$ (of

X, Y respectivery) such that if $E^n \setminus X'$ and $E^n \setminus Y'$ are homeomorphic, then $Sh(X) = Sh(Y)$.

Theorem 13 ([7] Th.1.1). Let X and Y be nonempty compact strong Z_{n-k-2}-sets in E^n ($k \geqq 0$, $n \geqq 2k+2$). Then X and Y have the same shape iff $E^n \setminus X$ and $E^n \setminus Y$ are homeomorphic.

Proposition ([7] Prop.1.3). Every compactum of dimension $\leqq k$ can be embedded in E^n ($n \geq 2k+1$) as a strong Z_{n-k-2}-set.

On the other hand, Venema proved the following theorems.

Theorem 14 ([17] Th.1). Let X and Y be compacta in E^n, $n \geqq 5$, satisfying ILC and having shape dimension in the trivial range with respect to n ($2Fd(X)+2 \leqq n$, $2Fd(Y)+2 \leqq n$), then $E^n \setminus X$ and $E^n \setminus Y$ are homeomorphic iff $Sh(X) = Sh(Y)$.

Theorem 15 ([17] Th.2). Let X and Y be globally 1-alg compacta in E^n, $n \geqq 5$, and let A, B be compact connected abelian topolgical groups with $2\dim(A)+2 \leqq n$. If $Sh(X) = Sh(A)$ and $Sh(Y) = Sh(B)$ then the following are equivalent:
(a) $E^n \setminus X$ and $E^n \setminus Y$ are homeomorphic,
(b) $Sh(X) = Sh(Y)$, and
(c) A and B are topologically isomorphic.

In the case that X and Y are compacta in S^n, Coram and Duvall proved the following theorem.

Theorem 16 ([5] Th.3.3). Let X, Y be S^k-like continua in S^n such that X and Y satisfy SLC, $1 \leq k \leq n-4$. Then $S^n \setminus X$ and $S^n \setminus Y$ are homeomorphic iff $Sh(X) = Sh(Y)$.

More generally, there are Rushing and his students' works.

Theorem 17 ([13] Th.1). Let $X \subset S^n$, $n \geq 5$, be compact. Then, for $k \neq 1$ $Sh(X) = Sh(S^k)$ is equivalent to $S^n \setminus X \cong S^n \setminus S^k$ if X is globally 1-alg (and if $S^n \setminus X$ has the homotopy type of S^1 when $k = n-2$).

Theorem 18 ([16] Th.1). Let $X \subset S^n$, $n \geq 5$, be a globally 1-alg compactum, and let S_p^k be the (k-1)-fold suspension of S_p^1 (a solenoid). Then $S^n \setminus X \cong S^n \setminus S_p^k$, $k \neq 1$, if and only if $Sh(X) = Sh(S_p^k)$ and in case $k = n-2$, $\pi_1(S^n \setminus X)$ is abelian and $\pi_i(S^n \setminus X) = 0$, $i \geq 2$.

Theorem 19 ([11] Th.2). Let X, Y be globally 1-alg continua in S^n, $n \geq 5$, having the shape of finite complexes K, L (respectively) in trivial range such that $\pi_1(K)$, $\pi_1(L)$ are abelian. If either $\pi_1(K) = \pi_1(L) = 0$ or $\pi_2(K) = \pi_2(L) = 0$, then $Sh(X) = Sh(Y)$ iff $S^n \setminus X \cong S^n \setminus Y$.

Theorem 20 ([12] Th.2). Let X_1 and X_2 be globally 1-alg continua in S^n ($n \geq 6$) having the shape of codimension 3, closed, $0 <(2m_i-n+1)$-connected topological manifolds $M_i^{m_i}$, $i = 1,2$ (respectively). Then, $S^n \setminus X_1 \cong S^n \setminus X_2$ iff $Sh(X_1) = Sh(X_2)$.

References

[1]. K.Borsuk, On several probrems in the theory of shape. Studies in topology, Academic Press 1975, 67-79.

[2]. ________, Theory of shape. Math. Monographs, Vol.59. Warsaw 1975.

[3]. T.A.Chapman, On some application of infinite-dimensional manifolds to the theory of shape. Fund. Math. 76 (1972), 181-193.

[4]. ___________, Shape of finite-dimensional compacta. Fund. Math. 76 (1972), 261-276.

[5]. D.Coram & P.F.Duvall,Jr, Neighborhood of sphere-like continua. Gen. Top. and its Appl. 6 (1976) 191-198.

[6]. P.F.Duvall & L.S.Husch, A continuum of dimension n which does not embed up to shape in 2n-space.(to appear)

[7]. R.Geoghegan & R.R.Summerhill, Concerning the shape of finite dimentional compacta. Trans.A.M.S. 179 (1973) 281-292.

[8]. L.S.Husch & I.Ivanšić, Shape domination and embedding up to shape. (submitted to Compositio Math.)

[9]. ________ & _________, Embeddings and concordances of embeddings up to shape. (submitted to Acta Math.)

[10]. I.Ivanšić, Embedding compacta up to shape. Bull. Acad. Polon. Sci. 25 (1977) 471-475.

[11]. V.T.Liem, Certain continua in S^n of the same shape have homeomorphic complement. Trans.A.M.S. 218 (1976) 207-217.

[12]. V,T.Liem, Certain continua in S^n with homeomorphic complements have the same shape. Fund. Math. 97 (1977) 221-228.

[13]. T.B.Rushing, The compacta X in S^n for which $Sh(X)=Sh(S^k)$ is equivalent to $S^n \setminus X \approx S^n \setminus S^k$. Fund. Math. 97 (1977) 1-8.

[14]. J.Stallings, The embedding of homotopy types into manifolds. Mimeo note, Princeton University 1965.

[15]. A,Trybulec, On shape of movable curves. Bull. Acad. Polon. Sci. 21 (1973) 727-733.

[16]. G.A.Venema, Weak flatness for shape classes of sphere-like continua. Gen. Top. and its Appl. 7 (1977) 303-319.

[17]. __________, Embeddings of compacta with shape dimensional in the trivial range. Proc. A.M.S. 55 (1976) 443-448.