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2d – PLATE MODELS OBTAINED FROM 3d – ELASTICITY MODELS

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1. STATEMENT OF THE PROBLEM ; NOTATION

Summation convention ; dx – symbols omitted in $\int$

Latin indices : $i, j, p, \ldots \in \{1,2,3\}$

Greek indices : $\alpha, \beta, \gamma, \ldots \in \{1,2\}$

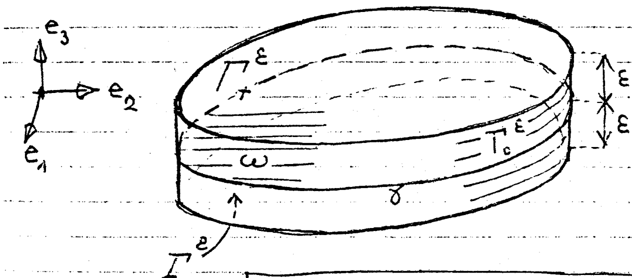
$\partial_i v = \frac{\partial}{\partial x_i}$, $\partial_{ij} v = \frac{\partial^2}{\partial x_i \partial x_j}$.

1.1. The clamped plate problem ; the linear case.

$\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$; Applied forces: $f = (f_i)$ in Ω^ε

$g = (g_i)$ on $\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon$

$u = (u_i) = 0$ on Γ_0^ε .



(1)
$$\boxed{\begin{aligned} &J(u) = \inf_{v \in V^\varepsilon} J(v) \quad , \quad V^\varepsilon = \{ v \in (H^1(\Omega^\varepsilon))^3 \, ; \, v = 0 \text{ on } \Gamma_0^\varepsilon \} , \\ &J(v) = \frac{1}{2} \int_{\Omega^\varepsilon} (A^{-1} \gamma(v))_{ij} \, \gamma_{ij}(v) - \left\{ \int_{\Omega^\varepsilon} f_i v_i + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i v_i \right\} , \end{aligned}}$$

$$\boxed{\gamma_{ij}(v) = \frac{1}{2} (\partial_i v_j + \partial_j v_i)} ,$$

$$\boxed{(AX)_{ij} = \left(\frac{1+\nu}{E}\right) X_{ij} - \frac{\nu}{E} X_{pp} \delta_{ij}} \qquad (E > 0, \; 0 < \nu < \tfrac{1}{2}) ,$$

Young's modulus, Poisson's coef.

$$(A^{-1} Y)_{ij} = \left(\frac{E}{1+\nu}\right) Y_{ij} + \frac{E\nu}{(1+\nu)(1-2\nu)} Y_{pp} \delta_{ij} \; . \quad \text{(Lamé's constants)}$$

Equivalent <u>system</u> (obtained from the variational equations $J'(u)v = 0$ for all $v \in V^\varepsilon$)

$$(2)\qquad \boxed{\begin{aligned} -\partial_j (A^{-1}\gamma(u))_{ij} &= f_i \text{ in } \Omega^\varepsilon \\ u &= 0 \text{ on } \Gamma_0^\varepsilon \\ (A^{-1}\gamma(u))_{i3} &= \pm g_i \text{ on } \Gamma_\pm^\varepsilon \ (1) \end{aligned}}$$

When ε is "small", people solve instead the well-known <u>biharmonic problem</u> (assuming $f_\alpha = g_\alpha = 0$ for convenience):

$$(3)\qquad \boxed{\begin{aligned} \frac{2E\varepsilon^3}{3(1-\nu^2)}\Delta^2 u_3 &= f \text{ in } \omega \ \left(f \overset{\text{def}}{=} g_3^+ + g_3^- + \int_{-\varepsilon}^{\varepsilon} f_3 dx_3\right) \\ u_3 = \partial_\nu u_3 &= 0 \text{ on } \gamma \end{aligned}}$$

<u>Questions</u>: - How do we go from (2) to (3)? (In books of Mechanics, e.g. Landau & Lifchitz ∗, this is achieved through <u>a priori</u> assumptions, geometrical or mechanical in nature).

- In particular, how a <u>system</u> "degenerates" in a <u>single</u> equation?; how a <u>2nd-order</u> problem becomes a <u>4th-order</u> problem?; how do we obtain the <u>boundary conditions</u> $u_3 = \partial_\nu u_3 = 0$ (the "clamped" plate problem)?

(1) Special case of the general b.c. $(A^{-1}\gamma(u))_{ij}\nu_j = g_i$.

One mathematical way to answer these questions is the following: (1)

(i) The problem is written in the *mixed* form

(4)
$$\boxed{\begin{aligned} (A\sigma)_{ij} &= \gamma_{ij}(u) \\ -\partial_j \sigma_{ij} &= f_i \\ u &= 0 \text{ on } \Gamma_0^\varepsilon \\ \sigma_{i3} &= \pm g_i \text{ on } \Gamma_\pm^\varepsilon, \end{aligned}} \qquad (\Leftrightarrow \sigma_{ij} = (A^{-1}\gamma(u))_{ij})$$

i.e, the unknowns are not only the u_i's but also the σ_{ij}'s ($\sigma = (\sigma_{ij})$ = stress tensor). In variational form, these equations represent the *Hellinger-Reissner variational principle*.

Remark: Using the stress-displacement formulation rather than the displacement formulation is crucial for the success of the method.

(ii) Pose the problem over a set Ω ($= \omega \times]-1,1[$) *independent of* ε, and apply the *asymptotic expansion method* (2) | cf. especially J.L. LIONS, Lecture Notes in Math. vol. 323, Springer, for problems posed in variational form

$$\begin{cases} u^\varepsilon = \varepsilon^p u^p + \varepsilon^{p+1} u^{p+1} + \ldots & (\text{for an appropriate } p \in \mathbb{Z}) \\ \sigma^\varepsilon = \varepsilon^p \sigma^p + \varepsilon^{p+1} \sigma^{p+1} + \ldots \end{cases}$$

(iii) Then: – we find that u_3^p *is precisely the solution of* (3) (after returning to the set Ω^ε);

– we can estimate $\| u^\varepsilon - \varepsilon^p u^p \|$ in *appropriate norms* (cf. a forthcoming joint paper and

(2) See K.C. FRIEDRICHS, A.L. GOL'DENVEIZER for the application of the a.e.m. to *equations* (rather than var. eqns), with *simplifying* assumpt. and *no* error analysis

(1) See P.G. CIARLET and P. DESTUYNDER: "A justification of the two-dimensional linear plate model" (to appear).

Destuynder's thesis).

<u>Comments</u>: The computation of $u^{\Gamma+1}$ involves a <u>boundary layer phenomenon</u> (in this sense, it is a <u>singular perturbation problem</u>); cf. Destuynder's thesis.

We can analyze similarly the eigenvalue problem (1), and shell problems (cf. Destuynder's thesis).

The above considerations will be made more specific in the nonlinear case (cf. Sect. 3-4).

1.2 • <u>The nonlinear case</u>(2) The 3d-model will be described in a moment; the 2d-model we have in mind is the famed <u>von Kármán equations</u>:

(5)
$$\boxed{\begin{aligned} a\,\Delta^2 u_3 &= [\psi, u_3] + f, \\ b\,\Delta^2\psi &= -[u_3, u_3], \end{aligned}}$$

← as found for instance in LIONS' book; cf. BREZZI, MIYOSHI

where $a, b > 0$,

$$[f, g] = \partial_{11}f\,\partial_{22}g + \partial_{22}f\,\partial_{11}g - 2\partial_{12}f\,\partial_{12}g,$$

ψ is the <u>Airy stress function</u>, from which one may computes the functions $\sigma^0_{\alpha\beta} = \sigma_{\alpha\beta}(\cdot,\cdot,0)$.

<u>Remark</u>: instead of the form $a\Delta^2 u_3 = [\psi, u_3] + f$, one finds also

(5')
$$\boxed{a\,\Delta^2 u_3 = [\psi, u_3] + [\Phi_0, u_3]},$$

(2) cf. P.G. CIARLET and P. DESTUYNDER: A justification of a nonlinear model in plate theory; to appear in Computer Methods in Applied Mechanics and Engineering (Proc. FENOMECH'78, Stuttgart).

(1) cf. P.G. CIARLET and S. KESAVAN (to appear).

for instance in (¹)

This difference is one of the points we wish to clarify (among others things)

Boundary conditions:

(6)	$u_3 = \partial_\gamma u_3 = 0$ on γ	("clamped" plate)
(7)	$\psi = \partial_\gamma \psi = 0$ on γ	

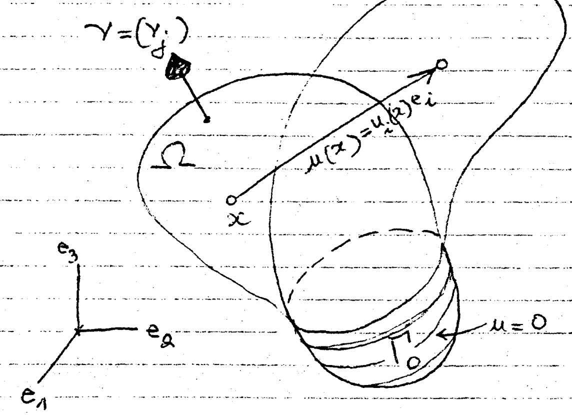
If (6) is acceptable, (7) is much more questionable, as we shall show. We hope also to clarify this point.

Remark. It is perfectly admissible that we do not introduce an Airy function; then we obtain 2d-models in $(u_3, \sigma^0_{\alpha\beta})$ or in (u_i), as we shall do here.

In the following work, we answer in particular a question raised by C. TRUESDEL. It seems that no justification of nonlinear plate models existed so far! (even with a priori assumptions).

(¹) M.S. BERGER "Nonlinearity and Functional Analysis", Academic Press, 1977.

2. THE THREE-DIMENSIONAL NONLINEAR GENERAL MODEL[1]



$$\boxed{\bar\gamma_{ij}(v) = \gamma_{ij}(v) + \frac{1}{2}\partial_i u_\ell \, \partial_j u_\ell}$$

$$\Gamma_1 = \Gamma - \Gamma_0 \quad , \quad \Gamma = \partial\Omega$$

2.1. • <u>The model</u>. It corresponds to the <u>energy</u> (compare with (1)):

(8) $$\boxed{\bar J(v) = \frac{1}{2}\int_\Omega \left(A^{-1}\bar\gamma(v)\right)_{ij} \bar\gamma_{ij}(v) - \left\{ \int_\Omega f_i v_i + \int_{\Gamma_1} g_i v_i \right\}}$$

(functional space will be defined later). To write the equivalent system[2], it is convenient to introduce right now the unknowns σ_{ij} s.t. $(A\sigma)_{ij} = \bar\gamma_{ij}(u)$; $\sigma = (\sigma_{ij})$ is the (second) <u>Piola-Kirchoff stress tensor</u>.

(9)
$$(A\sigma)_{ij} = \bar\gamma_{ij}(u)$$
← (<u>linear</u> stress-strain relation) <u>but</u> "full" strain tensor $\bar\gamma$

$$-\partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) = f_i$$

$$u = 0 \text{ on } \Gamma_0$$

$$(\sigma_{ij} + \sigma_{kj}\partial_k u_i)\nu_j = g_i \text{ on } \Gamma_1$$

Cauchy's law expressed in the reference configuration whence "<u>large displacement</u>" model

(2) As follows from applications of Green's formula.
(1) cf. C. TRUESDELL and W. NOLL: The Nonlinear Field Theories of <u>Mechanics</u>, in <u>Handbuch der Physik</u>, Vol. III/3, Springer, Berlin, 1965.

The linear stress - strain relation corresponds to an energy of the form (8). It can be shown [1] that in a general energy

$$J(v) = \int_\Omega F(\gamma(v)),$$

this corresponds to the first term in the Taylor expansion of F $\underbrace{}_{\text{around } \gamma = 0}$, whence our model corresponds to "small" strains γ.

Remark. Whereas in the linear case, the energy was ~~a~~ quadratic, here we have tri- and quadri- linear terms in J.

2.2. ● Choice of function spaces for a variational formulation of (9). We multiply eqns in (9) by test functions and integrate by parts. Formally:

$(A\sigma)_{ij} = \bar{\gamma}_{ij}(u) \Leftrightarrow$

$$(10)\quad \boxed{\begin{aligned} &\forall \tau \in \Sigma, \ \int_\Omega (A\sigma)_{ij}\tau_{ij} - \int_\Omega \tau_{ij}\gamma_{ij}(u) - \frac{1}{2}\int_\Omega \tau_{ij}\partial_i u_\ell \partial_j u_\ell = 0, \\ &\forall v \in V, \ \int_\Omega \sigma_{ij}\gamma_{ij}(v) + \int_\Omega \underbrace{\sigma_{ij}}_{\in L^2} \underbrace{\partial_i u_\ell}_{\in L^4} \underbrace{\partial_j v_\ell}_{\in L^4} = \int_\Omega f_i v_i + \int_{\Gamma_1} g_i v_i, \end{aligned}}$$

$\left.\begin{aligned} -\partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) &= f_i \\ (\sigma_{ij} + \sigma_{kj}\partial_k u_i)\nu_j &= g_i \end{aligned}\right\} \Leftrightarrow$

($u = 0$ contained in def. of V)

$$(11)\quad \boxed{\begin{aligned} V &= \{ v = (v_i) \in (W^{1,4}(\Omega))^3 \, ; \, v = 0 \text{ on } \Gamma_0 \}, \\ \Sigma &= \{ \tau = (\tau_{ij}) \in (L^2(\Omega))^9 \, ; \, \tau_{ij} = \tau_{ji} \}. \end{aligned}}$$

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[1] cf. e.g. R. VALID: "La Mécanique des Milieux Continus et le Calcul des Structures", Eyrolles, Paris, 1977.

2.3. • <u>Existence of a solution</u>. We only obtain a <u>partial</u> result for:

– the <u>pure</u> Dirichlet problem ($u = 0$ on $\Gamma_0 = \Gamma$)(1),

– <u>sufficiently small applied forces</u>.

<u>Principle of the proof</u>: We eliminate the unknowns σ_{ij} and after integrating by parts $\left(\int_\Omega (\dots)\,\partial_j v_i = \int_\Omega (\mathcal{A}u)\,v_i\right)$ we obtain :

$$\mathcal{A}(u) = f \quad (\text{in the distribution sense at least})$$

with (writing now $(A^{-1}Y)_{ij} = a_{ijkl}\,Y_{kl}$) (2)

$$(\mathcal{A}(u))_i = -\partial_j \Big(a_{ijkl}\,\gamma_{kl}(u) \cancel{\}} + \frac{1}{2} a_{ijkl}\,\partial_k u_m\,\partial_l u_m + $$

$$+ a_{ljki}\,\gamma_{ki}(u)\,\partial_l u_i + \frac{1}{2} a_{ljkp} \underbrace{\overbrace{\partial_k u_m}^{\in W^{1,4}}\,\overbrace{\partial_p u_m}^{\in W^{1,4}}\,\overbrace{\partial_l u_i}^{\in W^{1,4}}}_{\in W^{1,4}} \Big).$$

Because $W^{1,4}(\Omega)$, $\Omega \subset \mathbb{R}^3$, is an <u>algebra</u> (cf. ADAMS' book), $\mathcal{A}$ maps $(W^{2,4}(\Omega))^3$ into $(L^4(\Omega))^3$ and is of class $\mathcal{C}^1$ (sum of k-linear continuous mappings; $W^{1,4}(\Omega)$ is an algebra).

Now $\mathcal{A}'(0)$ <u>is nothing</u> but the <u>linear elasticity system</u>!

Consequently, if we can prove that

$$\mathcal{A}'(0) : (W^{2,4}(\Omega))^3 \to (L^4(\Omega))^3$$

is an <u>isomorphism</u>, existence around the origin will follow from the <u>implicit function theorem</u>.

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(1) The extension to $u = u_0$ on $\Gamma_0 = \Gamma$ is possible.

(2) It is simply shorter to use here the coefficients a_{ijkl} rather than the Lamé's constants introduced p. 1.

In other words, we need a regularity result: for all $f \in L^4(\Omega)$, there exists a solution in $W^{2,4}(\Omega)$. This follows from:

(i) $H^2(\Omega)$-regularity for $f \in L^2(\Omega)$ for the elasticity system (cf. NEČAS' book, p. 260).

(ii) the index of the mapping

$$\mathcal{A}'(0) : (W^{2,p}(\Omega))^3 \to (L^p(\Omega))^3$$

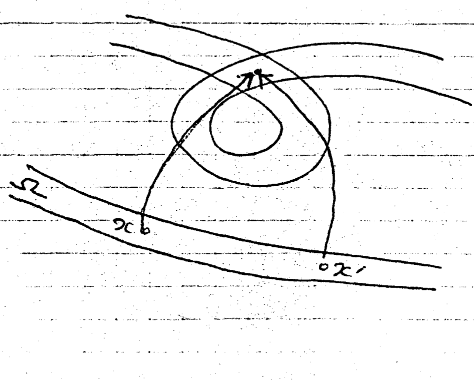
is independent of $p \in]1,\infty[$ (1) ($\mathcal{A}'(0)$ is injective)

Remark. Contrary to a common belief, this does not follow from AGMON-DOUGLIS-NIRENBERG; who rather prove: If we have the $W^{2,p}$-regularity, then $f \in W^{m,p} \Rightarrow u \in W^{m+2,p}$ for any $m \geq 1$.

2.4. • 1-1 character of the mapping

$$\phi : x \in \Omega \to \phi(x) = x + u(x).$$

Of course, it is desirable to avoid the following situation:



(1) cf. G. GEYMONAT: Sui problemi ai limiti per i sistemi lineari ellitici, Ann. Mat. Pura Appl. LXIX (1965), 207-284.

One has

Jacobian of ϕ at $x = \boxed{J_\phi(x) = \det(I + (\partial_j u_i))}$,

hence if $|u|_{1,\infty,\Omega}$ is <u>small enough</u>,

$$\forall x \in \bar{\Omega}, \; J_\phi(x) \neq 0 .$$

But this follows from the previous result and

$$\boxed{W^{2,4}(\Omega) \hookrightarrow \mathcal{C}^1(\bar{\Omega})}$$

Using (1), we know that

$$\left.\begin{array}{r} \phi : \mathcal{O} \supset \Omega \to \mathbb{R}^3 \text{ of class } C^1 \\ \forall x \in \bar{\Omega}, \; J_\phi(x) \neq 0 \; (2) \\ \phi|_\Gamma \text{ is 1-1} \end{array}\right\} \Rightarrow \phi : \bar{\Omega} \to \mathbb{R}^3 \text{ is 1-1.}$$

whence the conclusion follows.

<u>Remark</u> (in passing): Application to isoparametric f.e. !

2.5. ● <u>Open problems</u>. (i) Existence by other means (elsewhere than around 0). Results of Ball ?

(ii) Even with the implicit function thm, corresponding regularity result for the 3d-clamped plate problem ? (only hope is because cylindrical domain; otherwise even H^2-regularity does not hold for Dirichlet <u>and</u> Neumann b.c.).

(iii) Numerical analysis of f.e.m. for this 3d-problem ? Any reference ?

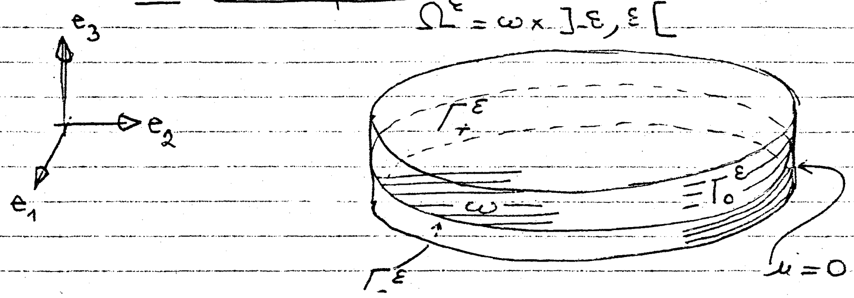
(2) This condition may be relaxed to $\Omega - \{\text{finite set}\}$ and $\Gamma - \{\text{non-empty}\ldots\}$

(1) G.H. MEISTERS and C. OLECH, "Locally one-to-one mappings and a classical theorem on Schlicht functions, Duke Math. J. <u>30</u> (1963), 63-80.

3. THE PLATE PROBLEM; APPLICATION OF THE ASYMPTOTIC EXPANSION METHOD

3.1. The 3d-problem.

$\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$



Applied forces:

$f = (f_i)$ ~~in~~ Ω^ε

$g = (g_i)$ on Γ_0^ε

$u = 0$ on Γ_0^ε ("clamped" plate)

$$\boxed{\begin{array}{l} \Sigma^\varepsilon = \{\tau = (\tau_{ij}) \in (L^2(\Omega^\varepsilon))^9 \,;\ \tau_{ij} = \tau_{ji}\}. \\ V^\varepsilon = \{v = (v_i) \in (W^{1,4}(\Omega^\varepsilon))^3 \,;\ v = 0 \text{ on } \Gamma_0^\varepsilon\}, \end{array}}$$

$$\boxed{\begin{array}{l} \forall \tau \in \Sigma^\varepsilon, \ \int_{\Omega^\varepsilon} (A\sigma)_{ij}\tau_{ij} - \int_{\Omega^\varepsilon} \tau_{ij}\gamma_{ij}(u) - \frac{1}{2}\int_{\Omega^\varepsilon} \tau_{ij}\partial_i u_\ell \partial_j u_\ell = 0, \\ \forall v \in V^\varepsilon, \ \int_{\Omega^\varepsilon} \sigma_{ij}\gamma_{ij}(v) + \int_{\Omega^\varepsilon} \sigma_{ij}\partial_i u_\ell \partial_j v_\ell = \int_{\Omega^\varepsilon} f_i v_i + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i v_i \end{array}}$$

Remark. The functions f_i and g_i are assumed smooth enough for all subsequent purposes

3.2. Transformation into a problem posed over a domain independent of ε.

Objective: To make as simple as possible the dependence on ε. We let

$\Omega = \omega \times]-1, 1[= \Omega^1$

$\Gamma_0 = \Gamma_0^1, \quad \Gamma_\pm = \Gamma_\pm^1,$

$V = V^1, \ \Sigma = \Sigma^1.$

We make the following changes of variables and functions

$$X = (x_1, x_2, x_3) \in \overline{\Omega} \to X^\varepsilon = (x_1, x_2, \varepsilon x_3) \in \overline{\Omega}^\varepsilon$$

(12) $$\boxed{\begin{array}{l} \sigma_{\alpha\beta}(X^\varepsilon) = \sigma^\varepsilon_{\alpha\beta}(X), \ \sigma_{\alpha 3}(X^\varepsilon) = \varepsilon \sigma^\varepsilon_{\alpha 3}(X), \ \sigma_{33}(X^\varepsilon) = \varepsilon^2 \sigma^\varepsilon_{33}(X \\ v_\alpha(X^\varepsilon) = v^\varepsilon_\alpha(X), \quad v_3(X^\varepsilon) = \varepsilon^{-1} v^\varepsilon_3(X), \end{array}}$$

(as a result: $\varepsilon \int_\Omega \sigma^\varepsilon_{ij} \gamma_{ij}(v^\varepsilon) = \int_{\Omega^\varepsilon} \sigma_{ij} \gamma_{ij}(v)$),

(13) $$\boxed{\begin{array}{l} f_\alpha(X^\varepsilon) = \varepsilon^2 f^\varepsilon_\alpha(X), \quad f_3(X^\varepsilon) = \varepsilon^3 f^\varepsilon_3(X), \\ g_\alpha(X^\varepsilon) = \varepsilon^3 g^\varepsilon_\alpha(X), \quad g_3(X^\varepsilon) = \varepsilon^4 g^\varepsilon_3(X). \end{array}}$$

(as a result: $\int_{\Omega^\varepsilon} f_i v_i + \int_{\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-} g_i v_i = \varepsilon^3 \left(\int_\Omega f^\varepsilon_i v^\varepsilon_i + \int_{\Gamma_+ \cup \Gamma_-} g^\varepsilon_i v^\varepsilon_i \right)$).

Proposition. The element $(\sigma^\varepsilon, u^\varepsilon) \in \Sigma \times V$ obtained from $(\sigma, u) \in \Sigma^\varepsilon \times V^\varepsilon$ through (12), satisfies:

$$\boxed{\begin{array}{ll} (14) & \forall \tau \in \Sigma, \ \mathcal{A}_0(\sigma^\varepsilon, \tau) + \varepsilon^2 \mathcal{A}_2(\sigma^\varepsilon, \tau) + \varepsilon^4 \mathcal{A}_4(\sigma^\varepsilon, \tau) + \\ & \quad + \mathcal{B}(\tau, u^\varepsilon) + \mathcal{C}_0(\tau, u^\varepsilon, u^\varepsilon) + \varepsilon^{-2} \mathcal{C}_{-2}(\tau, u^\varepsilon, u^\varepsilon) = 0, \\ (15) & \forall v \in V, \ \mathcal{B}(\sigma^\varepsilon, v) + 2\mathcal{C}_0(\sigma^\varepsilon, u^\varepsilon, v) + 2\varepsilon^{-2} \mathcal{C}_{-2}(\sigma^\varepsilon, u^\varepsilon, v) = \varepsilon^2 \mathcal{F}(v) \end{array}}$$

where in particular (we record only the expressions useful in the sequel):

...ue that all ...s in $\mathcal{A}_0$ are in $\{1,2\}$)

(16) $$\boxed{\begin{array}{l} \mathcal{A}_0(\sigma, \tau) = \int_\Omega \left\{ \frac{(1+\nu)}{E} \sigma_{\alpha\beta} - \frac{\nu}{E} \sigma_{\mu\mu} \delta_{\alpha\beta} \right\} \tau_{\alpha\beta}, \\ \mathcal{B}(\tau, v) = -\int_\Omega \tau_{ij} \gamma_{ij}(v); \quad \mathcal{C}_{-2}(\tau, u, v) = -\frac{1}{2} \int_\Omega \tau_{ij} \partial_i u_3 \partial_j v_3, \\ \mathcal{F}(v) = \int_\Omega f^\varepsilon_i v_i + \int_{\Gamma_+ \cup \Gamma_-} g^\varepsilon_i v_i. \end{array}}$$

3.3. Formal expansion of $(\sigma^\varepsilon, u^\varepsilon)$

Equations (14)-(15) suggest that we let

$$(17)\qquad \boxed{(\sigma^\varepsilon, u^\varepsilon) = \varepsilon^2(\sigma^2, u^2) + \varepsilon^3(\sigma^3, u^3) + \cdots}\,.$$

Then we plug this formal expansion into (14)-(15) and we equate to zero the factors of the successive powers of ε. In this fashion, we obtain

i) equations to be satisfied by (σ^2, u^2),

ii) recurrence relations satisfied by the next terms.

Remarks. – At this stage this is completely formal; nothing guarantees that such (σ^p, u^p) exist in $\Sigma \times V$ or even in a larger space.

– If we had started by ε^p, $p \leq 1$, then the resulting eqns for (σ^p, u^p) correspond to $u_3^p = 0$ (an unwanted property for what is supposed to be an approximation of the 3d-problem. Besides, it does not "contain" the linear case. □

By inspection we find that (σ^2, u^2) should satisfy

Add^l terms wrt the linear case

$$(18)\qquad \forall \tau \in \Sigma,\quad \mathcal{A}_0(\sigma^2, \tau) + \mathcal{B}(\tau, u^2) + \mathcal{C}_{-2}(\tau, u^2, u^2) = 0,$$

$$(19)\qquad \forall v \in V,\quad \mathcal{B}(\sigma^2, v) + 2\,\mathcal{C}_{-2}(\sigma^2, u^2, v) = \mathcal{F}(v).$$

(consider the factors of ε^2 = the smallest power of ε).

4. MAIN RESULTS

• Theorem. If the forces f_α, g_α are sufficiently small(1), problem (18)-(19) has (at least) one solution in the space $\Sigma \times V$, which coincides with the solution of a known nonlinear 2d-plate model.

Idea of the proof. From now on, we let $(\tau^2, u^2) = (\tau, u)$ for notational brevity.

Step 1. (u_i) is a Kirchoff-Love displacement field.

Let us write eqns (18) for $\tau = \begin{pmatrix} 0 & 0 & \tau_{13} \\ 0 & 0 & \tau_{23} \\ \tau_{31} & \tau_{23} & 0 \end{pmatrix}$ and $\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}$:

$$\forall \tau_{\alpha 3} \in L^2(\Omega), \quad \int_\Omega \tau_{\alpha 3} \left(\partial_\alpha u_3 + \partial_3 u_\alpha + \partial_\alpha u_3 \partial_3 u_3 \right) = 0$$

$$\forall \tau_{33} \in L^2(\Omega), \quad \int_\Omega \tau_{33} \left(\partial_3 u_3 + \tfrac{1}{2} (\partial_3 u_3)^2 \right) = 0,$$

whence

$$\begin{cases} \partial_\alpha u_3 + \partial_3 u_\alpha + \partial_\alpha u_3 \partial_3 u_3 = 0, \\ \partial_3 u_3 + \tfrac{1}{2} (\partial_3 u_3)^2 = 0 \end{cases}$$

extra terms wrt the linear case.

$\longrightarrow$ either $\partial_3 u_3 = 0$ or $\partial_3 u_3 = -2$.

To circumvent the ambiguity, let us henceforth restrict ourselves to those solutions u_3 which are in $W^{2,4}(\Omega) \subsetneq C^1(\bar{\Omega})$(2), whence $\partial_3 u_3 = -2$ ruled out ($u_3 = 0$ on Γ_0).

$$\partial_3 u_3 = 0 \Rightarrow \partial_\alpha u_3 + \partial_3 u_\alpha = 0 \Rightarrow \cancel{\partial_{\alpha 3} u_3} + \partial_{33} u_\alpha = 0$$

$$\Rightarrow \exists\, u_\alpha^0, u_\alpha^1 \in W_0^{1,4}(\omega) \text{ s.t. } u_\alpha = u_\alpha^0 + x_3 u_\alpha^1.$$

$\therefore\ \partial_\alpha u_3 = -\partial_3 u_\alpha = -u_\alpha^1 \quad \therefore\ u_3 \in W_0^{2,4}(\omega)$ (u_3 ind^t of x_3, in $W^{2,4}(\Omega)$ $= 0$ on Γ_0, and $\partial_\alpha u_3 \in W_0^{1,4}(\omega)$)

(1) Therefore, no restriction is imposed upon the functions f_3, g_3.
(2) This is *a posteriori* justified by the fact that we can find ~~...~~ solutions possessing this regularity.

contains the b.c. for u_3

To sum up:

$$\boxed{\begin{array}{l} u_3 \text{ is independent of } x_3 \text{ and is in } W_0^{2,4}(\omega) \\ \exists u_\alpha^0 \in W_0^{1,4}(\omega),\ u_\alpha = u_\alpha^0 - x_3 \partial_\alpha u_3 . \end{array}}$$

<u>Remark</u>. This is also the first step towards the transformation into a 4th-order problem, since $u_3 \in W_0^{2,4}(\omega)$. □

<u>Remark</u>. In the linear case, no need to assume u_3 is in $H^2(\Omega)$; it is automatically found. □

<u>Step 2</u>. <u>Computation of the functions</u> (u_α^0, u_3).

We let successively (all other components are zero)

$$\begin{cases} \tau_{\alpha\beta} = \tau_{\alpha\beta}^0 \in L^2(\omega) \text{ in } (18) \\ v_\alpha = v_\alpha^0 \in W_0^{1,4}(\omega) \text{ in } (19) \end{cases}$$

$$\begin{cases} \tau_{\alpha\beta} = x_3 \hat{\tau}_{\alpha\beta} \in L^2(\omega) \text{ in } (18) \\ \left.\begin{array}{l} v_\alpha = x_3 \partial_\alpha v \\ v_3 = v \end{array}\right\} v \in W_0^{2,4}(\omega) \text{ in } (19) \end{cases}$$

(if (18) and (19) are to be satisfied, then they should be satisfied in particular by such fcns; a remarkable fact is that it is an iff cond.)

Then after elimination of the other unknowns, we find a <u>2d-problem</u> of the form: Find $(u_1^0, u_2^0, u_3) \in (W_0^{1,4}(\omega))^2 \times W_0^{2,4}(\omega)$ s.t.

$$(20) \quad \begin{cases} \forall v_\alpha^0 \in W_0^{1,4}(\omega), \ldots \\ \forall v \in W_0^{2,4}(\omega), \ldots \end{cases}$$

For simplicity <u>only</u>, assume $f_\alpha = g_\alpha = 0$. Then (20) is equivalent to (after returning to the set Ω^ε):

$= 0$ in the linear case

$$\frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta^2 u_3 = \varepsilon \sum_{\alpha,\beta=1}^{2} \sigma_{\alpha\beta}^0 \partial_{\alpha\beta} u_3 + \left(g_3^+ + g_3^- + \int_{-\varepsilon}^{\varepsilon} f_3 \, dx_3\right)$$

$$\partial_\alpha \sigma_{\alpha\beta}^0 = 0, \text{ where } \cancel{\sigma_{\alpha\beta}^0 = \frac{2E}{(1-\nu^2)} \{(1-\nu)}$$

$$u_\alpha^0 = 0 \text{ on } \gamma, \quad u_3 = \partial_\nu u_3 = 0 \text{ on } \gamma$$

equivalent to (after returning to the set Ω^ε):

$$
(21)\quad
\begin{cases}
(a) & \dfrac{2E\varepsilon^3}{3(1-\nu^2)}\,\Delta^2 u_3 = \varepsilon\,\sigma^0_{\alpha\beta}\,\partial_{\alpha\beta} u_3 + \left(g_3^+ + g_3^- + \int_{-\varepsilon}^{\varepsilon} f_3\,dx_3\right) \\
(b) & \partial_\alpha \sigma^0_{\alpha\beta} = 0, \\
(c) & u^0_\alpha = \text{ on } \gamma, \quad u_3 = \partial_\nu u_3 = 0 \text{ on } \gamma,
\end{cases}
$$

where

$$
(21')\quad
\begin{cases}
\sigma^0_{11} = \dfrac{2E}{(1-\nu^2)}\left\{\partial_1 u^0_1 + \frac{1}{2}(\partial_1 u_3)^2 + \nu\left(\partial_2 u^0_2 + \frac{1}{2}(\partial_2 u_3)^2\right)\right\}, \\
\sigma^0_{12} = \dfrac{2E}{(1+\nu)}\left\{\gamma_{12}(u^0) + \partial_1 u_3\,\partial_2 u_3\right\}, \\
\sigma^0_{22} = \dfrac{2E}{(1-\nu^2)}\left\{\partial_2 u^0_2 + \frac{1}{2}(\partial_2 u_3)^2 + \nu\left(\partial_1 u^0_1 + \frac{1}{2}(\partial_1 u_3)^2\right)\right\}.
\end{cases}
$$

<u>Remarks</u>. The notation $\sigma^0_{\alpha\beta}$ is justified because $\sigma^0_{\alpha\beta} = \sigma_{\alpha\beta}(\cdot,\cdot,0)$[1]. Likewise, observe that $u^0_\alpha = u_\alpha(\cdot,\cdot,0)$. □

UCIAL CONCLUSION: We have therefore obtained a <u>known nonlinear 2d-model for plates</u> (cf. e.g. the books of STOKER and WOINOWSKY-KRIEGER). Notice in particular that the boundary conditions (which involve the functions u^0_α and u_3) have been found without any ambiguity. ▨

(1) ~~cf. Ste~~ this can be seen only in Step 3.

Step 3. If the norms $\|g_\alpha\|_{L^2(\Gamma_+ \cup \Gamma_-)}$ and $\|f_\alpha\|_{L^2(\Omega)}$ are small enough (1), problem (20) (= (21) if $f_\alpha = g_\alpha = 0$) has at least solution, which has the following regularity:

$$u = (u_1^0, u_2^0, u_3) \in \left(W_0^{1,4}(\omega) \cap W^{3,4}(\omega)\right)^2 \times \left(W_0^{2,4}(\omega) \cap W^{4,4}(\omega)\right).$$

Principle: Eqns (20) assert that $j'(u)v = 0$, for an appropriate functional j, already defined over the space $W = (H_0^1(\omega))^2 \times H_0^2(\omega)$. On this space, $j \to \infty$ as $\|v\|_W \to \infty$ (0). Next, although j is not convex, we show it is weakly lower semi-continuous on W (in particular because the injection $H_0^2(\omega) \hookrightarrow W^{1,4}(\omega)$ is compact).

The asserted regularity follows from an argument similar to that used by (2).

Step 4. Computation of the stresses: All the functions σ_{ij} are given by explicit formulas involving the functions u_α^0 and u_i.

Then it is an easy matter to check that we have indeed obtained a solution to (18)-(19). □

Conclusion: Without any a priori assumption, either of a mechanical or geometrical nature, we have found a known nonlinear 2d-plate model.

(0) This is where we need that f_α, g_α be small. Also, this property would not be true on the original space.

(1) We now return to the general case ($f_\alpha \neq 0$, $g_\alpha \neq 0$).

(2) LIONS, J.L.: Quelques Methodes de Resolution de Problèmes aux Limites non Linéaires, Dunod, Paris, 1969.

5. INTRODUCTION OF THE AIRY ~~STRESS~~ FUNCTION

Let us return to the case where $f_\alpha = g_\alpha = 0$; cf. eqns (21).

<u>Lemma 1.</u> $\left.\begin{array}{l} \sigma^0_{\alpha\beta} \in W^{2,4}(\omega)\ (^1) \\ \partial_\alpha \sigma^0_{\alpha\beta} = 0 \\ \sigma^0_{12} = \sigma^0_{21} \end{array}\right\} \Rightarrow$ $\exists!\ \phi \in W^{4,4}(\omega)/P_1(\omega)\ (^2)$ s.t.

$$\boxed{\partial_{11}\phi = \sigma^0_{22},\ \partial_{12}\phi = -\sigma^0_{12} = -\sigma^0_{21},\ \partial_{22}\phi = \sigma^0_{11}.}\quad (^3)$$

<u>Proof.</u> Relies essentially on Poincaré's theorem properly extended to Sobolev's spaces. □

Equations (21a) then become

$$(23)\qquad \boxed{\frac{2E\varepsilon^3}{3(1-\nu^2)}\Delta^2 u_3 = 2\varepsilon\,[\phi, u_3] + \left(g_3^+ + g_3^- + \int_{-\varepsilon}^{\varepsilon} f_3\,dx_3\right)}$$

and we still have, by (21c):

$$(24)\qquad \boxed{u_3 = \partial_\nu u_3 = 0 \text{ on } \gamma.}$$

On the other hand, a straightforward computation shows that

$$(25)\qquad \boxed{\Delta^2\phi = -E\,[u_3, u_3]}$$

<u>Conclusion:</u> (23) and (25) are the <u>von Kármán equations</u>; We have (in (24)) b.c. for u_3.

<u>It remains to find ~~the~~ appropriate b.c. for ϕ.</u>

~~Preliminary~~

$(^1)$ As follows from Step 4 of the previous theorem.
$(^2)$ $P_1(\omega)$ = space of pol. of degree ≤ 1 over ω.
$(^3)$ ϕ is called the AIRY stress function.

Let ϕ_0 be the (unique) solution of

$$(26)\qquad \boxed{\begin{array}{l} \Delta^2\phi_0 = 0 \quad \text{in } \omega \\ \left.\begin{array}{l}\phi_0 = \phi \\ \partial_\nu\phi_0 = \partial_\nu\phi\end{array}\right\} \text{on } \gamma \end{array}} \quad (^1)$$

Then the functions u_3 and

$$\boxed{\psi = \phi - \phi_0}$$

satisfy

$$(27)\qquad \boxed{\begin{array}{l} \dfrac{2E\varepsilon^3}{3(1-\nu^2)}\Delta^2 u_3 = 2\varepsilon[\psi, u_3] + 2\varepsilon[\phi_0, u_3] + \left(g_3^+ + g_3^- + \int_{-\varepsilon}^{\varepsilon} f_3\,dx_3\right) \\ \Delta^2\psi = -E[u_3, u_3] \\ u_3 = \partial_\nu u_3 = 0 \quad \text{on } \gamma \\ \psi = \partial_\nu\psi = 0 \quad \text{on } \gamma \end{array}}$$

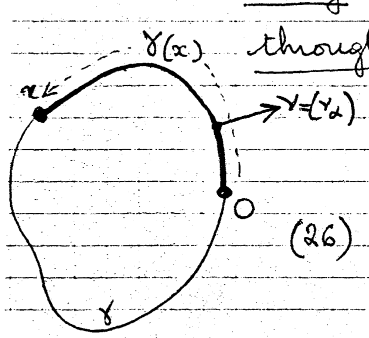
Conclusion: If we want to impose the b.c. $\psi = \partial_\nu\psi = 0$ on γ, this is at the expense of adding the term $[\phi_0, u_3]$ in the first equations. There is no reason to expect ϕ_0 to vanish.

Let us now examine how to compute $\phi, \partial_\alpha\phi$ along g. From Lemma 1, it seems that we can only compute the 2nd partial derivatives $\partial_{\alpha\beta}\phi$ from the knowledge of the $\sigma^0_{\alpha\beta}$. However we have:

---

(1) Once we have solved our 2d-problem as in Sect. 4, ~~it~~ the functions ϕ is known (up to a pol. of degree 1) by Lemma 1,

Lemma 2. Assume wlg that $0 \in \gamma$. We define ϕ uniquely by specifying that $\phi(0) = \partial_1\phi(0) = \partial_2\phi(0)$. Then one can compute the functions $\phi, \partial_1\phi, \partial_2\phi$ along γ as functions of the 4 quantities $\sigma^0_{\alpha\beta}$, through the formulas:



(26)
$$\boxed{\begin{aligned} \partial_1\phi(x) &= -\int_{\gamma(x)} h_2 \\ \partial_2\phi(x) &= \int_{\gamma(x)} h_1 \\ \phi(x) &= \int_{\gamma(x)} (x_1 h_2 - x_2 h_1) - x_1 \int_{\gamma(x)} h_2 + x_2 \int_{\gamma(x)} h_1 \end{aligned}}$$

where

(27)
$$\boxed{\begin{aligned} h_1 &= \sigma^0_{11}\nu_1 + \sigma^0_{21}\nu_2 \\ h_2 &= \sigma^0_{12}\nu_1 + \sigma^0_{22}\nu_2 \end{aligned}}$$

Conclusion: This suggests that the original 3d-problem be defined with the following b.c. on Γ_0^ε:

(28)
$$\boxed{\left.\begin{aligned} u_3 &= 0 \\ \sigma_{11}\nu_1 + \sigma_{21}\nu_2 &= h_1 \\ \sigma_{12}\nu_1 + \sigma_{22}\nu_2 &= h_2 \end{aligned}\right\} \text{ on } \Gamma_0^\varepsilon}$$

where h_1, h_2 are given functions. In the linear case at least, this is a perfectly admissible set of b.c. provided the applied forces satisfy a suitable compatibility condition (cf. e.g. DUVAUT & LIONS).

Assuming we can do this (the details remain to be checked)(*), let us examine various special cases. For simplicity, assume we started with

$$\begin{cases} \Delta^2 u_3 = [\psi, u_3] + [\Phi_0, u_3] + f & \text{in } \omega \\ \Delta^2 \psi = -[u_3, u_3] & \text{in } \omega \\ u_3 = \partial_\nu u_3 = 0 & \text{on } \gamma \\ \psi = \partial_\nu \psi = 0 & \text{on } \gamma \end{cases}$$

Uniform pressure, or traction, along γ:

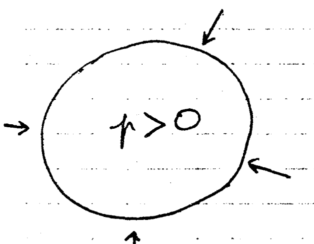
$$\sigma^0_{\alpha\beta} = p \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad p \in \mathbb{R}$$

The unique solution of problem (26) is seen to be (apply Lemma 2):

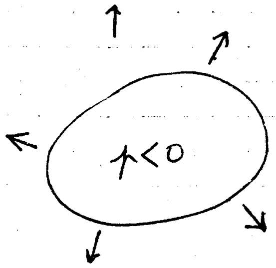
$$\Phi_0 = \left(p \, \frac{x_1^2 + x_2^2}{2} \right).$$

Whence the equation

$$\Delta^2 u_3 = [\psi, u_3] + p \, \Delta u_3 + f$$



$p > 0 \longrightarrow$ bifurcation (around 0 when $f = 0$; y the book of BERGER) 7



$p < 0 \longrightarrow$ membrane theory, $p u_3 \rightarrow$ sol of $\boxed{-\Delta u = f}$

(*) In particular, it seems that we shall not obtain the boundary condition $\partial_\nu u_3 = 0$ on γ. Besides, there remain some problems as regards the nonlinearity.

6. FINAL REMARKS

Open problems. 1) Apply all this to evolution problems

2) Convergence analysis in the nonlinear case.

3) Existence of a 3d-solution around a 2d-solution ?

etc...